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FURTHER EXTENSION OF AN ORDER PRESERVING OPERATOR INEQUALITY

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To the memory of
Professor Masahiro Nakamura
in deep sorrow

(*communicated by M. Fujii*)

Abstract. A capital letter means a bounded linear operator on a Hilbert space H. The celebrated Löwner-Heinz inequality asserts that $A \geqslant B \geqslant 0$ ensures $A^{\alpha} \geqslant B^{\alpha}$ for any $\alpha \in [0,1]$, but $A^p \geqslant B^p$ does not always hold for $p > 1$. From this point of view, we shall prove the following result.

Let $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \ldots, p_{2n} \geqslant 1$ for natural number n. Then the following inequality holds for $r \geqslant t$:

$$A^{1-t+r} \geqslant \left\{ A^{\frac{r}{2}} \Big[\underbrace{A^{\frac{-t}{2}} \{A^{\frac{t}{2}} \ldots [A^{\frac{-t}{2}} \{A^{\frac{t}{2}} (A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}} \ n \text{ times and } A^{\frac{t}{2}} \ n-1 \text{ times by turns}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}}\}^{p_3} A^{\frac{-t}{2}}]^{p_4} A^{\frac{t}{2}} \ldots A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}} \ n \text{ times and } A^{\frac{t}{2}} \ n-1 \text{ times by turns}} \Big]^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{\varphi[2n;r,t]}}$$

$$\text{where } \varphi[2n;r,t] = \underbrace{\Big\{ \ldots [\{[(p_1 - t)p_2 + t]p_3 - t\}p_4 + t]p_5 - \ldots - t \Big\} p_{2n} + r}_{-t \text{ appears } n \text{ times and } t \text{ appears } n-1 \text{ times by turns}}$$

$$= r + \prod_{i=1}^{2n} p_i + \underbrace{(\prod_{i=3}^{2n} p_i + \prod_{i=5}^{2n} p_i + \ldots + \prod_{i=7}^{2n} p_i + \ldots + p_{2n-1}p_{2n})t}_{n-1 \text{ terms}}$$

$$- \underbrace{(\prod_{i=2}^{2n} p_i + \prod_{i=4}^{2n} p_i + \prod_{i=6}^{2n} p_i + \ldots + p_{2(n-1)}p_{2n-1}p_{2n} + p_{2n})t}_{n \text{ terms}}.$$

This result is further extension of the following previous one: *if $A \geqslant B \geqslant 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geqslant 1$, $A^{1-t+r} \geqslant \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$ holds for $r \geqslant t$ and $s \geqslant 1$, in particular, $A^{1+r} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ for $p \geqslant 1$ and $r \geqslant 0$.*

1. Introduction

An operator T is said to be *positive* (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in H$, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

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THEOREM LH. (Löwner-Heinz inequality, denoted by (LH) *briefly).*

$$\text{If } A \geqslant B \geqslant 0 \text{ holds, then } A^{\alpha} \geqslant B^{\alpha} \text{ for any } \alpha \in [0,1]. \tag{LH}$$

This was originally proved in [L] and then in [H]. Many nice proofs of (LH) are known. We mention [P] and [B]. Although (LH) asserts that $A \geqslant B \geqslant 0$ *ensures* $A^{\alpha} \geqslant B^{\alpha}$ *for any* $\alpha \in [0,1]$, unfortunately $A^{\alpha} \geqslant B^{\alpha}$ *does not always hold for* $\alpha > 1$. The following result has been obtained from this point of view.

THEOREM A.
If $A \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$,

(*i*) $$(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^{p}B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(*ii*) $$(A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with* .$(1+r)q \geqslant p+r$

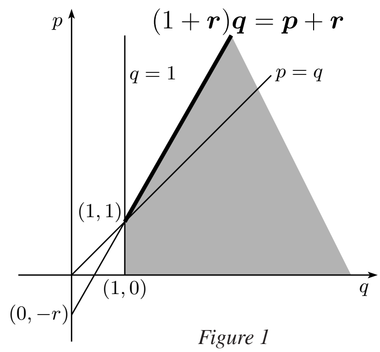

Figure 1

The original proof of Theorem A is shown in [F1], an elementary one-page proof is in [F2] and alternative ones are in [MF], [K1]. It is shown in [T1] that the conditions p, q and r in Figure 1 are best possible.

THEOREM B. *If* $A \geqslant B \geqslant 0$ *with* $A > 0$, *then for* $t \in [0,1]$ *and* $p \geqslant 1$,

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.1}$$

holds for $r \geqslant t$ *and* $s \geqslant 1$.

The original proof of Theorem B is in [F3], and an alternative one is in [MF-K]. An elementary one-page proof of (1.1) is in [F4]. We mention that further extensions of Theorem B and related results to Theorem A are in [MF-K-N], [F5], [F-W], [F-Y-Y], [K2] and [Y-G]. It is originally shown in [T2] that the exponent value $\frac{1-t+r}{(p-t)s+r}$ of the right hand of (1.1) is best possible and alternative ones are in [MF-M-N], [Y]. It is known that the operator inequality (1.1) interpolates Theorem A and an inequality equivalent to the main result of Ando-Hiai log majorization [A-H] by the parameter $t \in [0,1]$.

2. Definitions of $C_{A,B}[2n]$ and $q[2n]$

Let $A > 0$, $B \geqslant 0$, $t \in [0,1]$ and $p_1, p_2, ..., p_n, ..., p_{2(n-1)}, p_{2n-1}, p_{2n} \geqslant 1$ for a natural number n.

Let $C_{A,B}[2n] = C_{A,B}[2n; p_1, p_2, ..., p_{2(n-1)}, p_{2n-1}, p_{2n}]$ be defined by

$$C_{A,B}[2n] = C_{A,B}[2n; p_1, p_2, ..., p_{2(n-1)}, p_{2n-1}, p_{2n}]$$
$$= \underbrace{A^{\frac{t}{2}}\left\{A^{\frac{-t}{2}}\left[A^{\frac{t}{2}}...\left[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\right]^{p_4}...A^{\frac{t}{2}}\right]^{p_{2n-1}}A^{\frac{-t}{2}}\right\}^{p_{2n}}A^{\frac{t}{2}}}_{\rightarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}}. \tag{2.1}$$

For examples,

$$C_{A,B}[2] = A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}$$

and

$$C_{A,B}[4] = A^{\frac{t}{2}}\left[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\right]^{p_4}A^{\frac{t}{2}}.$$

Next we define

$$q[2n] = q[2n; p_1, p_2, ..., p_n, ..., p_{2(n-1)}, p_{2n-1}, p_{2n}]$$
$$= \underbrace{\left\{...[\{[(p_1-t)p_2+t]p_3-t\}p_4+t]p_5-...-t\right\}p_{2n}+t}_{-t \text{ and } t \text{ alternately } n \text{ times appear}}. \tag{2.2}$$

For examples,

$$q[2] = (p_1-t)p_2+t \quad \text{and} \quad q[4] = \left[\{(p_1-t)p_2+t\}p_3-t\right]p_4+t.$$

The following Lemma is easily shown by (2.1) and (2.2).

LEMMA 2.1. *For $A > 0$ and $B \geqslant 0$ and any natural number n*

(*i*) $C_{A,B}[2(n+1)] = A^{\frac{t}{2}}\left(A^{\frac{-t}{2}}(C_{A,B}[2n])^{p_{2n+1}}A^{\frac{-t}{2}}\right)^{p_{2(n+1)}}A^{\frac{t}{2}}$ (2.3)

(*ii*) $q[2(n+1)] = (q[2n]p_{2n+1}-t)p_{2(n+1)}+t.$ (2.4)

Also we remark that (2.1) and (2.2) easily imply

$$C_{A,A}[2n] = A^{q[2n]} \quad \text{holds for any natural number } n. \tag{2.5}$$

$$q[2n] = \prod_{i=1}^{2n}p_i + \underbrace{(\prod_{i=3}^{2n}p_i + \prod_{i=5}^{2n}p_i + \prod_{i=7}^{2n}p_i + ... + p_{2n-1}p_{2n}+1)}_{\text{n terms}}t$$
$$-\underbrace{(\prod_{i=2}^{2n}p_i + \prod_{i=4}^{2n}p_i + \prod_{i=6}^{2n}p_i + ... + p_{2(n-1)}p_{2n-1}p_{2n}+p_{2n})}_{\text{n terms}}t. \tag{2.6}$$

3. Statement of results

THEOREM 3.1. *Let $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, ..., p_{2n} \geqslant 1$. Then the following inequality holds,*

$$A \geqslant \Big[\underbrace{A^{\frac{t}{2}} \Big\{ A^{\frac{-t}{2}} \Big[A^{\frac{t}{2}} ... \Big[A^{\frac{-t}{2}} \big\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}}}_{\leftarrow\ A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \big\}^{p_3} A^{\frac{-t}{2}} \Big]^{p_4} ... A^{\frac{t}{2}} \Big]^{p_{2n-1}} A^{\frac{-t}{2}} \Big\}^{p_{2n}} A^{\frac{t}{2}}}_{\rightarrow\ A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} \Big]^{\frac{1}{q[2n]}} \tag{3.1}$$

where $q[2n]$ is in (2.2).

COROLLARY 3.2. *If $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, p_3, p_4 \geqslant 1$, then the following inequality holds,*

$$A \geqslant (C_{A,B}[4])^{\frac{1}{q[4]}},$$

that is,

$$A \geqslant \Big\{ A^{\frac{t}{2}} [A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \}^{p_3} A^{\frac{-t}{2}}]^{p_4} A^{\frac{t}{2}} \Big\}^{\frac{1}{q[4]}}.$$

THEOREM 3.3. *Let $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, ..., p_{2n} \geqslant 1$ for natural number n. Then the following inequality holds for $r \geqslant t$,*

$A^{1-t+r} \geqslant$

$$\Bigg\{ A^{\frac{r}{2}} \Big[\underbrace{A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} ... [A^{\frac{-t}{2}} \big\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \big\}^{p_3} A^{\frac{-t}{2}}]^{p_4} A^{\frac{t}{2}} ... A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}} \Big]^{p_{2n}} A^{\frac{r}{2}} \Bigg\}^{\frac{1-t+r}{\varphi[2n;r,t]}} \tag{3.2}$$

where $\varphi[2n;r,t] = q[2n] + r - t$.

COROLLARY 3.4. [F6] *If $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, p_3, p_4 \geqslant 1$,*

$$A^{1-t+r} \geqslant \Big\{ A^{\frac{r}{2}} \Big[A^{\frac{-t}{2}} \big\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \big\}^{p_3} A^{\frac{-t}{2}} \Big]^{p_4} A^{\frac{r}{2}} \Big\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}}$$

holds for $r \geqslant t$.

We need the following lemma.

LEMMA A. [F3, Lemma 1] *Let X be a positive invertible operator and Y be an invertible operator. For any real number λ,*

$$(YXY^*)^{\lambda} = YX^{\frac{1}{2}} (X^{\frac{1}{2}} Y^* Y X^{\frac{1}{2}})^{\lambda-1} X^{\frac{1}{2}} Y^*.$$

Proof of Theorem 3.1.

The case $n = 1$. (3.1) for $n = 1$ is shown by putting $r = t$ in (1.1) of the Theorem B, that is, *if $A \geqslant B \geqslant 0$ with $A > 0$, then for $t \in [0,1]$ and $p_1, p_2 \geqslant 1$*

$$\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \}^{\frac{1}{(p_1-t)p_2+t}} \leqslant A \tag{3.3}$$

holds.

We shall state the following proof of (3.3) because the method of the proof is very useful to give a proof of the general case for natural number n of (3.1) (see Remark 3.1).

First step. The case $2 \geqslant p_2 \geqslant 1$. As $p_2-1, \frac{1}{(p_1-t)p_2+t} \in [0,1]$ and $A^t \geqslant B^t$ by LH since $t \in [0,1]$ and taking inverses of both sides, $A^{-t} \leqslant B^{-t}$, we have

$$\begin{aligned} B_1 &= \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{\frac{1}{(p_1-t)p_2+t}} \\ &= \{B^{\frac{p_1}{2}}(B^{\frac{p_1}{2}}A^{-t}B^{\frac{p_1}{2}})^{p_2-1}B^{\frac{p_1}{2}}\}^{\frac{1}{(p_1-t)p_2+t}} \end{aligned}$$

by Lemma A

$$\leqslant \{B^{\frac{p_1}{2}}(B^{\frac{p_1}{2}}B^{-t}B^{\frac{p_1}{2}})^{p_2-1}B^{\frac{p_1}{2}}\}^{\frac{1}{(p_1-t)p_2+t}} = B \leqslant A = A_1. \tag{3.4}$$

Second step. Repeating (3.4) for $A_1 \geqslant B_1 > 0$, we have

$$A_1 \geqslant \{A_1^{\frac{t}{2}}(A_1^{\frac{-t}{2}}B_1^{p_1'}A_1^{\frac{-t}{2}})^{p_2'}A_1^{\frac{t}{2}}\}^{\frac{1}{(p_1'-t)p_2'+t}} \tag{3.5}$$

holds for $p_1' \geqslant 1$, $2 \geqslant p_2' \geqslant 1$. Taking $p_1' = (p_1-t)p_2+t \geqslant 1$ in (3.5) it yields

$$\begin{aligned} A &\geqslant \{A^{\frac{t}{2}}[A^{\frac{-t}{2}}A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}A^{\frac{-t}{2}}]^{p_2'}A^{\frac{t}{2}}\}^{\frac{1}{(p_1-t)p_2p_2'+t}} \\ &= \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2p_2'}A^{\frac{t}{2}}\}^{\frac{1}{(p_1-t)p_2p_2'+t}}, \quad \text{for } 4 \geqslant p_2p_2' \geqslant 1. \end{aligned} \tag{3.6}$$

Repeating this process from (3.4) to (3.6), we obtain (3.1) for any $p_2 \geqslant 1$, in the case $n=1$.

The general case for some natural number n. Let $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, ..., p_{2n} \geqslant 1$. Assume (3.1) holds for some natural number n. We shall show (3.1) for $n+1$, that is, if $A \geqslant B \geqslant 0$ with $A > 0$, then for $t \in [0,1]$ and $p_1, p_2, ..., p_{2n}, p_{2n+1}, p_{2(n+1)} \geqslant 1$ by (2.3) it follows

$$C_{A,B}[2(n+1)]^{\frac{1}{q[2(n+1)]}} = \left\{A^{\frac{t}{2}}\left(A^{\frac{-t}{2}}(C_{A,B}[2n])^{p_{2n+1}}A^{\frac{-t}{2}}\right)^{p_{2(n+1)}}A^{\frac{t}{2}}\right\}^{\frac{1}{q[2(n+1)]}} \leqslant A. \tag{3.7}$$

For this, we note that $q[2(n+1)] \geqslant 1$ and we have

$$A^{-t} \leqslant [C_{A,B}(2n)]^{\frac{-t}{q[2n]}} \tag{3.8}$$

by the assumption of induction, and LH since $t \in [0,1]$ and taking inverses of both sides.

First step. The case $2 \geqslant p_{2(n+1)} \geqslant 1$. Denote $v = p_{2n+1}$ and $w = p_{2(n+1)}$ briefly. We have

$$\begin{aligned} &C_{A,B}[2(n+1)]^{\frac{1}{q[2(n+1)]}} \\ &= \left\{A^{\frac{t}{2}}\left(A^{\frac{-t}{2}}(C_{A,B}[2n])^{v}A^{\frac{-t}{2}}\right)^{w}A^{\frac{t}{2}}\right\}^{\frac{1}{q[2(n+1)]}} \qquad \text{by (2.3)} \\ &= \left\{(C_{A,B}[2n])^{\frac{v}{2}}\left((C_{A,B}[2n])^{\frac{v}{2}}A^{-t}(C_{A,B}[2n])^{\frac{v}{2}}\right)^{w-1}(C_{A,B}[2n])^{\frac{v}{2}}\right\}^{\frac{1}{q[2(n+1)]}} \end{aligned}$$

(by Lemma A)

$$\leqslant \left\{ (C_{A,B}[2n])^{\frac{v}{2}} \left((C_{A,B}[2n])^{\frac{v}{2}} (C_{A,B}[2n])^{\frac{-t}{q[2n]}} (C_{A,B}[2n])^{\frac{v}{2}} \right)^{w-1} (C_{A,B}[2n])^{\frac{v}{2}} \right\}^{\frac{1}{q[2(n+1)]}}$$

$$= \left((C_{A,B}[2n])^{v+(v-\frac{t}{q[2n]})(w-1)} \right)^{\frac{1}{q[2(n+1)]}}$$

$$= \left((C_{A,B}[2n])^{\frac{(q[2n]v-t)w+t}{q[2n]}} \right)^{\frac{1}{q[2(n+1)]}}$$

$$= (C_{A,B}[2n])^{\frac{1}{q[2n]}} \qquad \text{because } q[2(n+1)] = (q[2n]v-t)w+t \text{ holds by (2.4)}$$

$$\leqslant A \quad \text{by (3.1)} \tag{3.9}$$

The first inequality follows by LH since $w-1 \in [0,1]$. Namely (3.9) ensures that (3.7) holds under the assumption $p_1, p_2, \ldots, p_{2n}, v \geqslant 1$ and $2 \geqslant w \geqslant 1$, that is,

$$C_{A,B}[2(n+1)]^{\frac{1}{q[2(n+1)]}} = \left\{ A^{\frac{t}{2}} \left[A^{\frac{-t}{2}} (C_{A,B}[2n])^v A^{\frac{-t}{2}} \right]^w A^{\frac{t}{2}} \right\}^{\frac{1}{q[2(n+1)]}} \leqslant A \tag{3.10}$$

holds for any $2 \geqslant w \geqslant 1$.

Second step. Put $A_1 = A$ and $B_1 = \left\{ A^{\frac{t}{2}} \left(A^{\frac{-t}{2}} (C_{A,B}[2n])^v A^{\frac{-t}{2}} \right)^w A^{\frac{t}{2}} \right\}^{\frac{1}{q[2(n+1)]}}$ in (3.10).

Since $A_1 \geqslant B_1 \geqslant 0$ with $A_1 > 0$ by (3.10), and (3.3) implies

$$\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B_1^{p_1'} A^{\frac{-t}{2}})^{w'} A^{\frac{t}{2}} \}^{\frac{1}{(p_1'-t)w'+t}} \leqslant A \tag{3.11}$$

for any $t \in [0,1]$, $p_1' \geqslant 1$ and $w' \geqslant 1$.

In (3.11), put $p_1' = q[2(n+1)] = (q[2n]v-t)w+t \geqslant 1$ by (2.4). Then we have

$$\left\{ A^{\frac{t}{2}} \left(A^{\frac{-t}{2}} A^{\frac{t}{2}} (A^{\frac{-t}{2}} (C_{A,B}[2n])^v A^{\frac{-t}{2}})^w A^{\frac{t}{2}} A^{\frac{-t}{2}} \right)^{w'} A^{\frac{t}{2}} \right\}^{\frac{1}{q[2n]v-t)ww'+t}}$$

$$= \left\{ A^{\frac{t}{2}} \left(A^{\frac{-t}{2}} (C_{A,B}[2n])^v A^{\frac{-t}{2}} \right)^{ww'} A^{\frac{t}{2}} \right\}^{\frac{1}{q[2n]v-t)ww'+t}} \leqslant A. \tag{3.12}$$

Since $2 \geqslant w \geqslant 1$ in (3.10), and any $w' \geqslant 1$ in (3.11), (3.12) ensures that (3.7) holds for any $p_{2(n+1)} \geqslant 1$. □

REMARK 3.1. In fact (3.3) itself can be shown by [MF-K, Theorem 2], here we state the proof of [§ 3.2.1, F5] for the sake of readers' convenience.

Proof of Corollary 3.2. We have only to put $n = 2$ in Theorem 3.1. □

Proof of Theorem 3.3. Put $A_1 = A$ and

$$B_1 = (C_{A,B}[2n])^{\frac{1}{q[2n]}}$$

$$= \Bigg[\underbrace{A^{\frac{t}{2}} \Big\{ A^{\frac{-t}{2}} \Big[A^{\frac{t}{2}} \ldots \Big[A^{\frac{-t}{2}} \Big\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} A^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \Big\}^{p_3} A^{\frac{-t}{2}} \Big]^{p_4} \ldots A^{\frac{t}{2}} \Big]^{p_{2n-1}} A^{\frac{-t}{2}} \Big\}^{p_{2n}} A^{\frac{t}{2}}}_{\rightarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} \Bigg]^{\frac{1}{q[2n]}}$$

in (3.1) in Theorem 3.1. Then $A_1 \geqslant B_1$ by (3.1) holds for $t \in [0,1]$ and $p_1, p_2, ...p_{2n} \geqslant 1$, by applying Theorem A,

$$A_1^{1+r_1} \geqslant (A_1^{\frac{r_1}{2}} B_1^{s_1} A_1^{\frac{r_1}{2}})^{\frac{1+r_1}{s_1+r_1}} \qquad \text{holds for } s_1 \geqslant 1 \text{ and } r_1 \geqslant 0. \tag{3.13}$$

In (3.13) we have only to put $r_1 = r - t \geqslant 0$ and $s_1 = q[2n] \geqslant 1$ to obtain (3.2) since $s_1 + r_1 = q[2n] + r - t = \varphi[2n, r, t]$. □

Proof of Corollary 3.4. Put $n = 2$ in Theorem 3.3. □

REMARK 3.2. Corollary 3.4 yields Theorem B by putting $p_2 = p_3 = 1$.

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