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A REFINEMENT OF THE INEQUALITY BETWEEN ARITHMETIC AND GEOMETRIC MEANS

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Abstract. In this note we present a refinement of the AM-GM inequality, and then we estimate in a special case the typical size of the improvement.

THEOREM 1. *For $i=1,\dots,n$, let $x_i \geqslant 0$, suppose that some $x_i > 0$, and let $\alpha_i > 0$ satisfy $\sum_{i=1}^n \alpha_i = 1$. Then*

$$\exp\left(2\left(1-\frac{\sum_{i=1}^n \alpha_i x_i^{1/2}}{\left(\sum_{i=1}^n \alpha_i x_i\right)^{1/2}}\right)\right)\prod_{i=1}^n x_i^{\alpha_i} \leqslant \sum_{i=1}^n \alpha_i x_i. \tag{1}$$

The hypotheses $n \geqslant 2$, $x_i \geqslant 0$ for $i=1,\dots,n$ and $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1$ will be maintained throughout this note without further mention.

Proof. The following refinement of a standard consequence of Hölder's inequality on a probability space appears in [Al, Theorem 3.1]: Let $0 < r < s/2 < \infty$, and let $0 \leqslant f \in L^s$ satisfy $\|f\|_s > 0$. Then

$$\|f\|_r \leqslant \|f\|_s \left[1-\frac{2r}{s}\left(1-\frac{\|f^{s/2}\|_1}{\|f^{s/2}\|_2}\right)\right]^{1/r}. \tag{2}$$

From this, a well known argument yields the refinement of the AM-GM indicated in the theorem. Let $f:[0,\infty)\to[0,\infty)$ be the identity $f(x)=x$, and let $\mu := \sum_{i=1}^n \alpha_i \delta_{x_i}$, where δ_{x_i} denotes the Dirac measure supported on $\{x_i\}$. Writing

$$c := 1-\frac{\sum_{i=1}^n \alpha_i x_i^{1/2}}{\left(\sum_{i=1}^n \alpha_i x_i\right)^{1/2}}$$

and setting $s=1$ in (2), we get

$$\left(\sum_{i=1}^n \alpha_i x_i^r\right)^{1/r} \leqslant \left(\sum_{i=1}^n \alpha_i x_i\right)(1-2rc)^{1/r}. \tag{3}$$

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Taking the limit as $r \downarrow 0$, it follows (for instance, by L'Hôpital's rule) that

$$\prod_{i=1}^{n} x_i^{\alpha_i} \leqslant \left(\sum_{i=1}^{n} \alpha_i x_i\right) e^{-2c}. \quad \square$$

Regarding the meaning of (2), it essentially says that, for fixed r and s, if the variance of $f^{s/2}/\|f^{s/2}\|_2$ is large, then $\|f\|_r \ll \|f\|_s$. For simplicity, set $s = 1$, which is the case we used. To see that $1 - \int f^{1/2}/\left(\int f\right)^{1/2}$ is a measure of the dispersion of $f^{1/2}$ about its mean value, and in fact, comparable to the variance $\operatorname{Var}\left(f^{1/2}/\|f^{1/2}\|_2\right)$ of its normalization in L^2, observe first that $\int f^{1/2}/\left(\int f\right)^{1/2} \leqslant 1$, since $\operatorname{Var}\left(f^{1/2}\right) \geqslant 0$. Now, for all $t \in [0,1]$

$$2^{-1}(1-t^2) = 2^{-1}(1+t)(1-t) \leqslant 1-t \leqslant 1-t^2, \tag{4}$$

so, setting $t = \|f^{1/2}\|_1/\|f^{1/2}\|_2$, we obtain

$$\frac{1}{2}\operatorname{Var}\left(\frac{f^{1/2}}{\|f^{1/2}\|_2}\right) \leqslant 1 - \frac{\|f^{1/2}\|_1}{\|f^{1/2}\|_2} \leqslant \operatorname{Var}\left(\frac{f^{1/2}}{\|f^{1/2}\|_2}\right). \tag{5}$$

Using (5), we see that (1) entails the following inequality:

$$\exp\left(\operatorname{Var}\left(\frac{x^{1/2}}{\|x^{1/2}\|_2}\right)\right)\prod_{i=1}^{n} x_i^{\alpha_i} \leqslant \sum_{i=1}^{n} \alpha_i x_i. \tag{6}$$

Thus, (6) gives a quantitative bound of the deviation from equality, in terms of the variance of $x^{1/2}/\|x^{1/2}\|_2$; if the variance is large, so is the difference between the AM and the GM.

Next we ask ourselves how "efficient" the refinement in (1) is. We study its average performance in the classical equal weights case, modified by the change of variables $x_i = y_i^2$. The AM-GM inequality bounds the GM-AM ratio by 1, always, while for $n \gg 1$ and after the said change of variables, inequality (1) gives a "typical" upper bound smaller than 0.82 (with probability at least $1 - 1/n$). We prove this next.

Let $\alpha_i = 1/n$ for $i = 1, \ldots, n$, write $x_i = y_i^2$, and set $y = (y_1, \ldots, y_n)$, where y_i is now allowed to take negative values. In terms of the GM-AM ratio (1) becomes

$$\frac{\prod_{i=1}^{n} |y_i|^{1/n}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} y_i^2}} \leqslant \exp\left(\frac{\frac{1}{n}\sum_{i=1}^{n}|y_i|}{\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right)^{1/2}} - 1\right). \tag{7}$$

The ℓ_1^n and ℓ_2^n norms of the vector $y \in \mathbb{R}^n$ are given by $\|y\|_1 := \sum_{i=1}^{n} |y_i|$ and $\|y\|_2 := \sum_{i=1}^{n} y_i^2$ respectively. Since both sides of (7) are positive homogeneous functions of degree zero (so replacing y by $y/\|y\|_2$ does not change anything) we may assume that $\|y\|_2 = 1$. Probability statements then mean that y is chosen at random (i.e., uniformly) from the euclidean unit sphere. Setting $\|y\|_2 = 1$, inequality (7) becomes

$$\sqrt{n}\prod_{i=1}^{n} |y_i|^{1/n} \leqslant \exp\left(n^{-1/2}\|y\|_1 - 1\right). \tag{8}$$

Denote by P^{n-1} the normalized area, or Haar measure, on the euclidean unit sphere $\mathbb{S}_2^{n-1} = \{\|y\|_2 = 1\} \subset \mathbb{R}^n$.

THEOREM 2. *For all n sufficiently high and with probability at least $1 - 1/n$ on $\mathbb{S}_2^{n-1}$, we have*

$$\exp\left(n^{-1/2}\|y\|_1 - 1\right) < 0.82. \tag{9}$$

The proof consists in computing the expectation of $n^{-1/2}\|y\|_1$ over $\mathbb{S}_2^{n-1}$ (a standard calculation) and then using the known fact that typically $n^{-1/2}\|y\|_1$ is very close to its mean, provided n is large enough. Details follow.

Recall that the area of $\mathbb{S}_2^{n-1}$ is $|\mathbb{S}_2^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$, and that $\|y\|_2 \leqslant \|y\|_1 \leqslant \sqrt{n}\|y\|_2$. It turns out that the average of $\|\cdot\|_1$ over $\mathbb{S}_2^{n-1}$ is closer to $\sqrt{n}$ than to 1.

LEMMA 3. *The expectation of $\|\cdot\|_1$ over $\mathbb{S}_2^{n-1}$ is given by*

$$E\left(\|y\|_1\right) := \int_{\mathbb{S}_2^{n-1}} \|y\|_1 dP^{n-1}(y) = \frac{n\Gamma\left(\frac{n}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{n+1}{2}\right)}. \tag{10}$$

Thus,

$$\sqrt{\frac{2}{\pi}} \leqslant E\left(\frac{\|y\|_1}{\sqrt{n}}\right) \leqslant \left(\frac{n}{n-1}\right)^{1/2}\sqrt{\frac{2}{\pi}}. \tag{11}$$

Proof. We integrate the left hand side of (12) below in two ways, first in polar coordinates and then as a product, via Fubini's Theorem. Using $|\mathbb{S}_2^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ and the fact that $\|\cdot\|_1$ is a positive homogeneous function of degree one, we get

$$\int_{\mathbb{R}^n} \|y\|_1 e^{-\|y\|_2^2} dy = \int_0^\infty t^n e^{-t^2} dt \left(\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\right) \int_{\{\|y\|_2=1\}} \|y\|_1 dP^{n-1}(y). \tag{12}$$

Given $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we denote by $\hat{y}_i \in \mathbb{R}^{n-1}$ the vector obtained from y by deleting the i-th coordinate: $\hat{y}_i = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$. Now the first two integrals in (12) can either be computed or expressed in terms of the Gamma function:

$$\int_0^\infty t^n e^{-t^2} dt = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right), \tag{13}$$

and

$$\int_{\mathbb{R}^n} \left(\sum_{i=1}^n |y_i|\right) \exp\left(-\sum_{i=1}^n y_i^2\right) dy$$
$$= \sum_{i=1}^n 2\int_0^\infty y_i e^{-y_i^2} dy_i \int_{\mathbb{R}^{n-1}} \exp\left(-\|\hat{y}_i\|_2^2\right) d\hat{y}_i = n\pi^{(n-1)/2}. \tag{14}$$

Putting together (12), (13) and (14), and solving for the expectation, we get

$$E(\|y\|_1) = \frac{n\Gamma\left(\frac{n}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{n+1}{2}\right)}. \tag{15}$$

Now (11) follows from (15) by using the following known and elementary estimate (cf. Exercise 5, pg. 216 of [Web]; the result is an immediate consequence of the log-convexity of the Γ function):

$$\sqrt{\frac{2}{n}} \leqslant \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \leqslant \sqrt{\frac{2}{n-1}}. \quad \square \tag{16}$$

An expression similar to (11) can also be obtained from (15) by using the very well known asymptotic expansion

$$\Gamma(z) = e^{-z} z^{z-1/2} \sqrt{2\pi} \left(1 + \frac{1}{12z} + O(z^{-2})\right).$$

Given a real valued random variable f on a probability space (X, μ), a median M_f of f is a constant such that $\mu\{f \geqslant M_f\} \geqslant 1/2$ and $\mu\{f \leqslant M_f\} \geqslant 1/2$. It is a well known fact that "reasonable functions" on $\mathbb{S}_2^{n-1}$, when observed at the right scale, exhibit the concentration of measure phenomenon. That is, they are almost constant over large portions of the sphere, taking values very close to their medians. In the particular case of $\|\cdot\|_1$, the right scale means dividing by $\sqrt{n}$, which is precisely what we have in the right hand side of (8). We use the following facts (they can be found in [MiSch], within the proof of the Lemma in pg. 19):

1) $\left|E(\|y\|_1) - M_{\|y\|_1}\right| \leqslant \pi/2$.

2) $P^{n-1}\{\left|M_{\|y\|_1} - \|y\|_1\right| > t\} \leqslant \sqrt{\frac{\pi}{2}} e^{-t^2/2}$.

Proof of Theorem 2. Let $t = \sqrt{\log((\pi/2)n^2)}$. By 1),

$$\{|E(\|y\|_1) - \|y\|_1| > t + \pi/2\} \subset \{\left|\|y\|_1 - M_{\|y\|_1}\right| > t\}.$$

By the preceding inclusion and 2),

$$P^{n-1}\{\|y\|_1 > t + \pi/2 + E(\|y\|_1)\} \leqslant P^{n-1}\{|E(\|y\|_1) - \|y\|_1| > t + \pi/2\}$$
$$\leqslant P^{n-1}\{\left|\|y\|_1 - M_{\|y\|_1}\right| > t\} \leqslant 1/n.$$

It follows from (11) and the previous bound that for all n sufficiently high,

$$\|y\|_1 \leqslant t + \frac{\pi}{2} + \left(\frac{n}{n-1}\right)^{1/2} \sqrt{\frac{2n}{\pi}},$$

with probability at least $1 - 1/n$. Since $n^{-1/2}(t + \pi/2) = O(n^{-1/4})$ and $(n/(n-1))^{1/2} = 1 + O(n^{-1})$, again for all n sufficiently high and with probability at least $1 - 1/n$ we have

$$\exp\left(n^{-1/2}\|y\|_1 - 1\right) \leqslant \exp\left(\sqrt{\frac{2}{\pi}}\left(1 + O\left(\frac{1}{n^{1/4}}\right)\right) - 1\right) < 0.82. \quad \square \tag{17}$$

In some unlikely cases (that is, with low P^{n-1}-probability) the refined AM-GM inequality (1) performs badly. Suppose $n \gg 1$, and let $0 < y_1 = \ldots = y_n$. Then both sides of (7) equal 1. Letting $y_1 \downarrow 0$, the left hand side drops to zero, while the right hand side remains essentially unchanged.

Finally, given any correction factor in a refinement of the AM-GM inequality, how far down could it go on some large set? Not lower than $1/2$. Actually, it is possible to be more precise: In [GluMi] E. Gluskin and V. Milman show that $0.394 < n^{1/2}\prod_{i=1}^{n}|y_i|^{1/n}$ asymptotically in n, with probability approaching 1, and their method can be used to prove that $n^{1/2}\prod_{i=1}^{n}|y_i|^{1/n}$ concentrates around the value $\sqrt{2}\exp[\Gamma'(1/2)/(2\Gamma(1/2))] \approx 0.529$ (cf. [Al2, Theorem 2.8]).

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