Journal of
Mathematical
Inequalities
Volume 2, Number 4 (2008), 479–486

ERROR TERMS FOR STEFFENSEN'S, YOUNG'S, AND CHEBYCHEV'S INEQUALITIES

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(*communicated by J. Pečarić*)

Abstract. We recast the well-known Steffensen's inequality, and a number of its relations, as an equality involving an error term. The same general idea leads to error terms for Young's and Chebychev's inequalities.

1. Error Terms for Steffensen's Inequalities

Owing to its utility as well as its intrinsic appeal, the following result has enjoyed considerable attention in the literature.

STEFFENSEN'S INEQUALITIES (1918). *Suppose that f is decreasing and g is integrable on $[a,b]$, with $0 \leqslant g \leqslant 1$ and $\lambda = \int_a^b g(t)dt$. Then we have*

$$\int_{b-\lambda}^{b} f(t)dt \leqslant \int_a^b f(t)g(t)dt \leqslant \int_a^{a+\lambda} f(t)dt.$$

The inequalities are reversed for f increasing. They are nearly obvious: For example, thinking of f (decreasing) as fixed, the largest that $\int_a^b f(t)g(t)dt$ can be, if $0 \leqslant g \leqslant 1$ and $\lambda = \int_a^b g(t)dt$, is for

$$g(t) = \begin{cases} 1 & t \in [a, a+\lambda] \\ 0 & t \in (a+\lambda, b]. \end{cases}$$

Mitrinović's 1969 proof ([4], p. 318), which uses an idea due to Apéry (1951), can be refined to yield an error term for the right-hand inequality, as follows.

Mathematics subject classification (2000): 26D15.

THEOREM 1. *Let f' be continuous and let g be integrable on $[a,b]$, with $0 \leqslant g \leqslant 1$ and $\lambda = \int_a^b g(t)dt$. Then there exists $\xi \in (a,b)$ such that*

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt = f'(\xi)\left[\int_a^b tg(t)dt - \lambda(a+\frac{\lambda}{2})\right].$$

Proof. We have

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt$$
$$= \int_a^{a+\lambda} [f(a+\lambda)-f(t)][1-g(t)]dt + \int_{a+\lambda}^b [f(t)-f(a+\lambda)]g(t)dt.$$

By the Mean Value Theorem there are $p \in (a, a+\lambda)$ and $q \in (a+\lambda, b)$ such that the right-hand side

$$= \int_a^{a+\lambda} f'(p)[a+\lambda-t][1-g(t)]dt + \int_{a+\lambda}^b f'(q)[t-(a+\lambda)]g(t)dt.$$

Each of the three expressions $[\cdot]$ above is nonnegative, so by the Mean Value Theorem for Integrals there are $r,s \in (a,b)$ such that this

$$= f'(r)\int_a^{a+\lambda} [a+\lambda-t][1-g(t)]dt + f'(s)\int_{a+\lambda}^b [t-(a+\lambda)]g(t)dt.$$

Each integral is nonnegative and so by the Intermediate Value Theorem there is $\xi \in (a,b)$ such that this

$$= f'(\xi)\left[\int_a^{a+\lambda} [a+\lambda-t][1-g(t)]dt + \int_{a+\lambda}^b [t-(a+\lambda)]g(t)dt\right]$$
$$= f'(\xi)\left[\int_a^b tg(t)dt - \lambda(a+\frac{\lambda}{2})\right].$$

REMARK 1.1. The term $[\cdot]$ on the right-hand side above is $\leqslant 0$. Similarly (or replace g with $1-g$), we can obtain an error term for the left-hand inequality:

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt = f'(\xi)\left[\int_a^b tg(t)dt - \lambda(b-\frac{\lambda}{2})\right].$$

Applying essentially Theorem 5 of [7] we obtain the following.

COROLLARY. *For f', h' continuous and g integrable on $[a,b]$, with $0 \leqslant g \leqslant 1$ and $\lambda = \int_a^b g(t)dt$, there exist $\xi, \eta \in (a,b)$ such that*

$$\frac{\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt}{\int_a^b h(t)g(t)dt - \int_a^{a+\lambda} h(t)dt} = \frac{f'(\xi)}{h'(\xi)} \quad \text{and} \quad \frac{\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt}{\int_a^b h(t)g(t)dt - \int_{b-\lambda}^b h(t)dt} = \frac{f'(\eta)}{h'(\eta)}.$$

Proof. Fix g, define the linear functional L via $L(f) = \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt$, and set $\phi(t) = f(t)L(h) - h(t)L(f)$. By Theorem 1 we have

$$L(\phi) = \phi'(\xi)\left[\int_a^b tg(t)dt - \lambda(a + \frac{\lambda}{2})\right].$$

But $L(\phi) = 0$ and so $f'(\xi)L(h) - h'(\xi)L(f) = 0$, thus proving the first statement. The proof of the second statement is entirely similar.

Clearly, such an argument can be applied to obtain many results akin to Cauchy's Mean Value Theorem.

2. Error Terms for Inequalities Resulting From Steffensen's

In principle Theorem 1 can be used whenever Steffensen's Inequality is used, if f has sufficient regularity. Our focus is on convexity so we apply Theorem 1, which reads

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt = f'(\xi)\left[\int_a^b tg(t)dt - \int_a^{a+\lambda} tdt\right],$$

to f' and various functions g, in order to recast inequalities as equalities involving an error term:

$$\text{Error}(f) = \frac{f''(\xi)}{2}\,\text{Error}(t^2).$$

We omit any calculations which follow familiar lines. (See [3-5], for example.)

HERMITE-HADAMARD INEQUALITIES. *Suppose that f is convex on $[a,b]$. Then*

$$f\Big(\frac{a+b}{2}\Big)(b-a) \leqslant \int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2}(b-a).$$

Suppose instead that f'' is continuous on $[a,b]$, and let $c = \frac{a+b}{2}$.

Applying Theorem 1 to f' and $g(t) = \begin{cases} \frac{t-a+\frac{b-a}{2}}{b-a} & t \in [a,c] \\ \frac{t-b+\frac{b-a}{2}}{b-a} & t \in (c,b], \end{cases}$ (here $\lambda = \int_a^b g = \frac{b-a}{2}$)

we obtain the

TRAPEZOID RULE:
$$\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}(b-a) = -\frac{f''(\xi_1)}{2}\frac{(b-a)^3}{6}.$$

Applying Remark 1.1 to f' and $g(t) = \frac{t-a}{b-a}$, (again $\lambda = \int_a^b g = \frac{b-a}{2}$) we obtain the

MIDPOINT RULE:
$$\int_a^b f(t)dt - f\Big(\frac{a+b}{2}\Big)(b-a) = \frac{f''(\xi_2)}{2}\frac{(b-a)^3}{12}.$$

That the Hermite-Hadamard Inequalities follow from Steffensen's seems to have been overlooked in the literature.

The rest of the inequalities we consider contain weights w, which we normalize for convenience, but with no loss of generality. That is, we assume $\sum_{j=1}^m w_j = 1$ or $\int_a^b w(x)dx = 1$ as the case may be.

JENSEN-STEFFENSEN INEQUALITY. *Let $\{a_j\}_1^m \subset [0,\infty)$ be nonincreasing, and let $\{w_j\}_1^m$ satisfy*

$$0 \leqslant \sum_{j=1}^k w_j \leqslant 1 \qquad \text{for } k = 1,2,\cdots,m.$$

Then for f convex on $[0,a_1]$ we have

$$f\left(\sum_{j=1}^m a_j w_j\right) \leqslant \sum_{j=1}^m f(a_j)w_j.$$

Again we suppose that f'' is continuous and here we define g on $[0,a_1]$ via

$$g = \sum_{j=1}^k w_j \quad \text{on } (a_{k+1},a_k], \text{ for } k = 1,2,\cdots,m \quad (a_{m+1} := 0).$$

Then $0 \leqslant g \leqslant 1$ and

$$\lambda = \int_0^{a_1} g(t)dt = \sum_{j=1}^m a_j w_j.$$

So we apply Theorem 1 to f' and g to get

$$\sum_{j=1}^m f(a_j)w_j - f\left(\sum_{j=1}^m a_j w_j\right) = \frac{f''(\xi)}{2}\left[\sum_{j=1}^m a_j^2 w_j - \left(\sum_{j=1}^m a_j w_j\right)^2\right].$$

Requiring also that $w_j \geqslant 0$ gives the less general JENSEN'S INEQUALITY. In this context the same error term was obtained in [2]. From here one can use standard limiting arguments to obtain the following [7].

JENSEN'S INEQUALITY. (Integral Form) *For* $h:[0,1]\to R,\ w:[0,1]\to R^{+}$ *with* $\int_a^b w(x)dx = 1$, *and for* f'' *continuous we have*

$$\int_0^1 f(h(t))w(t)dt - f\left(\int_0^1 h(t)w(t)dt\right) = \frac{f''(\xi)}{2}\left[\int_0^1 h^2(t)w(t)dt - \left(\int_0^1 h(t)w(t)dt\right)^2\right].$$

We mention also the

JENSEN-STEFFENSEN INTEGRAL INEQUALITY. *Let* $h:[0,1]\to R$ *be monotonic with* $h(0)=\alpha\neq\beta=h(1)$. *Let* $w:[0,1]\to R$ *satisfy*

$$0\leqslant\int_x^1 w(t)dt\leqslant 1 \qquad \text{for } x\in[0,1].$$

Then for f *convex on* $[0,1]$ *we have*

$$f\left(\int_0^1 \frac{h(t)-\alpha}{\beta-\alpha}w(t)dt\right)\leqslant\int_0^1 f\Big(\frac{h(t)-\alpha}{\beta-\alpha}\Big)w(t)dt.$$

Let $g(x)=\int_{h^{-1}((\beta-\alpha)x+\alpha)}^{1} w(t)dt$. Then $0\leqslant g\leqslant 1$ and

$$\lambda=\int_0^1 g(x)dx=\int_0^1 w(x)\frac{h(x)-\beta}{\beta-\alpha}dx.$$

Supposing again that f'' is continuous, we apply Theorem 1 to f' and g to get

$$\int_0^1 f\Big(\frac{h(t)-\alpha}{\beta-\alpha}\Big)w(t)dt - f\left(\int_0^1 \tfrac{h(t)-\alpha}{\beta-\alpha}w(t)dt\right)$$
$$=\frac{f''(\xi)}{2}\left[\int_0^1 \big(\tfrac{h(t)-\alpha}{\beta-\alpha}\big)^2 w(t)dt-\left(\int_0^1 \tfrac{h(t)-\alpha}{\beta-\alpha}w(t)dt\right)^2\right].$$

Error terms for many other inequalities can be similarly obtained — for example, Olkin's Inequality and its relations, due to Bellman, Szegö etc. [3-5]. Of course the Corollary may be applied in any of these contexts as well. For example, in the case of the Jensen-Steffensen Inequality, the conclusion reads

$$\frac{\sum_{j=1}^m f(a_j)w_j - f\left(\sum_{j=1}^m a_jw_j\right)}{\sum_{j=1}^m h(a_j)w_j - h\left(\sum_{j=1}^m a_jw_j\right)}=\frac{f''(\xi)}{h''(\xi)}.$$

3. Error Terms for Young's and Chebyshev's Inequalities

YOUNG'S INEQUALITY (1912). *Let f be continuous and strictly increasing on $[0,A]$, with $f(0)=0$. Let $a\in[0,A]$ and $b\in[0,f(A)]$. Then*

$$\int_0^a f(x)dx+\int_0^b f^{-1}(x)dx \geqslant ab.$$

We have equality if and only if $b=f(a)$. The inequality is reversed for f strictly decreasing.

Here we modify an argument by Hoorfar and Qi [1] to obtain an error term, as follows.

THEOREM 2. *Let f be continuous and strictly monotonic on $[0,A]$, with f' continuous and $f(0)=0$. Let $a\in[0,A]$ and $b\in[0,f(A)]$. Then there is ξ between $f^{-1}(b)$ and a such that*

$$\int_0^a f(x)dx+\int_0^b f^{-1}(x)dx-ab=\frac{f'(\xi)}{2}[a-f^{-1}(b)]^2.$$

Proof. As in [1], it is easily verified that

$$\int_0^a f(x)dx+\int_0^b f^{-1}(x)dx-ab=\int_0^a f(x)dx+\int_0^{f^{-1}(b)} xf'(x)dx-ab=\int_{f^{-1}(b)}^a [f(x)-b]dx.$$

We may assume that $f(a)\neq b$. Then we have

$$\int_{f^{-1}(b)}^a [f(x)-b]dx=\int_{f^{-1}(b)}^a [f(x)-f(f^{-1}(b))]dx=\int_{f^{-1}(b)}^a f'(\eta_x)[x-f^{-1}(b)]dx$$

(where η_x is between x and $f^{-1}(b)$, by the Mean Value Theorem).

In either of the cases $f^{-1}(b)<a$ or $a<f^{-1}(b)$, $[x-f^{-1}(b)]$ does not change sign over the interval of integration, and so by the Mean Value Theorem for Integrals there is ξ between $f^{-1}(b)$ and a such that this

$$=f'(\xi)\int_{f^{-1}(b)}^a [x-f^{-1}(b)]dx=\frac{f'(\xi)}{2}[a-f^{-1}(b)]^2,$$

as desired. □

REMARK 2.1. Observing merely that $\int_{f^{-1}(b)}^a [f(x)-b]dx\geqslant 0$ proves Young's Inequality.

REMARK 2.2. It is quite standard to apply Young's inequality to $f(x) = x^{p-1}$ $(p > 1)$ to obtain

$$\frac{1}{p}a^p + \frac{1}{q}b^q \geqslant ab \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

From here Hölder's and Minkowski's inequalities are just a few steps away. Theorem 2 could provide error terms for these as well. We mention incidentally that for $p = 2$ Young's inequality reads

$$\frac{1}{2}a^2 + \frac{1}{2}b^2 - ab \geqslant 0, \quad \text{while Theorem 2 reads} \quad \frac{1}{2}a^2 + \frac{1}{2}b^2 - ab = \frac{1}{2}(a-b)^2.$$

CHEBYSHEV'S INEQUALITY (1882). *Let f and g be similarly monotonic on* $[a,b]$. Then

$$\int_a^b f(x)g(x)dx \geqslant \frac{1}{b-a}\int_a^b f(x)dx \int_a^b g(x)dx.$$

The inequality is reversed for f and g of opposite monotonicity.

Here we apply the same basic idea to a 1888 proof by Andreief to obtain an error term, as follows.

THEOREM 3. *Let f' and g' be continuous on $[a,b]$. Then there are $\sigma, \tau \in (a,b)$ such that*

$$\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \int_a^b g(x)dx = \tfrac{1}{12} f'(\sigma)g'(\tau)(b-a)^3.$$

Proof. We have

$$(b-a)\int_a^b f(x)g(x)dx - \int_a^b f(x)dx \int_a^b g(x)dx$$

$$= \tfrac{1}{2}\int_a^b\int_a^b (f(x) - f(y)(g(x) - g(y))dxdy = \tfrac{1}{2}\int_a^b\int_a^b f'(p)g'(q)(x-y)^2 dxdy$$

(for some $p, q \in (a,b)$, by the Mean Value Theorem)

$$= \tfrac{1}{2} f'(\sigma)g'(\tau)\int_a^b\int_a^b (x-y)^2 dxdy$$

(for some $\sigma, \tau \in (a,b)$, by the Mean Value Theorem for Integrals)

$$= \tfrac{1}{12} f'(\sigma)g'(\tau)(b-a)^3,$$

as desired. □

REMARK 3.1. This result was obtained by Ostrowski [6], in an entirely different way.

REMARK 3.2. Suppose that f'' is continuous. Applying Theorem 3 to f' and $g(t) = t - \frac{a+b}{2}$ $(\int_a^b g = 0)$ we obtain the Trapezoid Rule. Applying Theorem 3 to f' and

$$g(t) = \begin{cases} t-a & t \in [a,c] \\ t-b & t \in (c,b], \end{cases}$$

(here $\int_a^c g + \int_c^b g = 0$) we obtain the Midpoint Rule. That the Hermite-Hadamard Inequalities follow from Chebyshev's also seems to have been overlooked in the literature.

Acknowledgement. The author is grateful to Rex Mercer for a number of excellent suggestions.

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(Received September 2, 2008)

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