Journal of
Mathematical
Inequalities
Volume 2, Number 4 (2008), 487–498

STEFFENSEN'S MEANS

J. JAKŠETIĆ AND J. PEČARIĆ

(*communicated by P. R. Mercer*)

Abstract. We define new means using Steffensen's inequality. Then we prove monotonicity of new means. Two new proofs of main result in [1] are given and conditions for Jensen-Steffensen inequality are relaxed.

1. Introduction

The well-known Steffensen inequality is given by ([1], p. 1):

THEOREM 1.1. *Suppose that f is decreasing and g is integrable on $[a,b]$ with $0 \leqslant g \leqslant 1$ and $\lambda = \int_a^b g(t)dt$. Then we have*

$$\int_{b-\lambda}^{b} f(t)dt \leqslant \int_a^b f(t)g(t)dt \leqslant \int_a^{a+\lambda} f(t)dt. \tag{1}$$

The inequalities are reversed for f increasing.

The following theorem and its corollary are proved in [1].

THEOREM 1.2. *Let f' be continuous and let g be integrable on $[a,b]$ with $0 \leqslant g \leqslant 1$ and $\lambda = \int_a^b g(t)dt$. Then there exists $\xi \in (a,b)$ such that*

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt = f'(\xi)\left[\int_a^b tg(t)dt - \lambda(a+\frac{\lambda}{2})\right].$$

COROLLARY 1.3. *For f', h' continuous and g integrable on $[a,b]$, with $0 \leqslant g \leqslant 1$ and $\lambda = \int_a^b g(t)dt$, there exist $\xi, \eta \in (a,b)$ such that*

$$\frac{\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt}{\int_a^b h(t)g(t)dt - \int_a^{a+\lambda} h(t)dt} = \frac{f'(\xi)}{h'(\xi)} \tag{2}$$

$$\frac{\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt}{\int_a^b h(t)g(t)dt - \int_{b-\lambda}^b h(t)dt} = \frac{f'(\eta)}{h'(\eta)} \tag{3}$$

Mathematics subject classification (2000): 26D15.
Keywords and phrases: Steffensen inequality, Jensen-Steffensen inequality, monotone function, means.

In the next section we generalize these results and then we define new means using some mean value theorems that we have already established.

2. Generalizations of Mercer's results

In [2], p. 184, it is shown that condition $0 \leqslant g \leqslant 1$ in Theorem 1.1 can be replaced with more general one:

THEOREM 2.1. *Assume that f and g are integrable functions on $[a,b]$. Then the inequalities in 1 hold for every decreasing function f iff*

$$0 \leqslant \int_x^b g(t)dt \leqslant b-x \text{ and } 0 \leqslant \int_a^x g(t)dt \leqslant x-a \quad \text{for every} \quad x \in [a,b]. \tag{4}$$

Now we can generalize Theorem 1.2

THEOREM 2.2. *Assume that f' is continuous and g is integrable function on $[a,b]$ such that*

$$0 \leqslant \int_x^b g(t)dt \leqslant b-x \text{ and } 0 \leqslant \int_a^x g(t)dt \leqslant x-a \quad \text{for every} \quad x \in [a,b]. \tag{5}$$

Then there exist $\xi,\ \eta \in (a,b)$ such that

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt = f'(\xi)\left[\int_a^b tg(t)dt - \lambda(a+\frac{\lambda}{2})\right]. \tag{6}$$

and

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^{b} f(t)dt = f'(\eta)\left[\int_a^b tg(t)dt - \lambda(b-\frac{\lambda}{2})\right]. \tag{7}$$

Proof. We will supply two proofs of this result.

First Proof. Since f' is continuous on $[a,b]$ there exists $m = \inf f'$ and $M = \sup f'$ both real numbers. We now first consider the following function $\tilde{f}(x) = Mx - f(x)$. Then $\tilde{f}'(x) = M - f'(x) \geqslant 0$, $x \in [a,b]$, so $\tilde{f}$ is an increasing function. From Theorem 1.1 we have

$$0 \leqslant \int_a^b \tilde{f}(t)g(t)dt - \int_a^{a+\lambda} \tilde{f}(t)dt = M\int_a^b tg(t)dt - \int_a^b f(t)g(t)dt -$$

$$-\frac{M}{2}[(a+\lambda)^2 - a^2] + \int_a^{a+\lambda} f(t)dt$$

i.e.

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \leqslant M[\int_a^b tg(t)dt - \lambda(a+\frac{\lambda}{2})].$$

Similarly, if we consider decreasing function $\hat{f}(x) = f(x) - mx$ we will get

$$m[\int_a^b tg(t)dt - \lambda(a+\frac{\lambda}{2})] \leqslant \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt.$$

We can now conclude that there exists $\xi \in (a,b)$ that we looking in (6). With the same technique we can proof existence of η in (7).
Second proof.

$$\begin{aligned}
&\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt = \\
&= \int_a^{a+\lambda} [f(a+\lambda) - f(t)][1-g(t)]dt + \int_{a+\lambda}^b [f(t) - f(a+\lambda)]g(t)dt = \\
&= \int_a^{a+\lambda} \left(\int_t^{a+\lambda} f'(x)dx\right)[1-g(t)]dt + \int_{a+\lambda}^b \left(\int_{a+\lambda}^t f'(x)dx\right)g(t)dt = \\
&= \int_a^{a+\lambda} f'(x)\left(\int_a^x [1-g(t)]dt\right)dx + \int_{a+\lambda}^b f'(x)\left(\int_x^b g(t)dt\right)dx \\
&= \int_a^b G(x)f'(x)dx,
\end{aligned} \tag{8}$$

where

$$G(x) = \begin{cases} \int_a^x (1-g(t))dt, & a \leqslant x \leqslant a+\lambda; \\ \int_x^b g(t)dt, & a+\lambda \leqslant x \leqslant b. \end{cases} \tag{9}$$

Since $G(x) \geqslant 0$, $x \in [a,b]$ we conclude that there exists $\xi \in (a,b)$ such that

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt = f'(\xi)\int_a^b G(x)dx = f'(\xi)\left[\int_a^b tg(t)dt - \lambda(a+\frac{\lambda}{2})\right]. \quad (10)$$

So we again have proved (6).

We can prove (7) in a similar fashion:

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt = -\int_a^b F(x)f'(x)dx \quad (11)$$

where

$$F(x) = \begin{cases} \int_a^x g(t)dt, & a \leqslant x \leqslant b-\lambda; \\ \int_x^b (1-g(t))dt, & b-\lambda \leqslant x \leqslant b. \end{cases} \quad (12)$$

Since $F(x) \geqslant 0$, $x \in [a,b]$ we conclude that there exists $\eta \in (a,b)$ such that

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt = f'(\eta)\int_a^b (-F(x))dx = f'(\eta)\left[\int_a^b tg(t)dt - \lambda(b-\frac{\lambda}{2})\right]. \quad (13)$$

REMARK 2.3. From the proof of previous theorem and especially from (9) we see that if g differs from function $x \mapsto 1_{[a,a+\lambda]}(x)$ on a set of positive measure then the left hand side of (6) is different from 0. Similarly, if g differs from function $x \mapsto 1_{[b-\lambda,b]}(x)$ on a set of positive measure then the left hand side of (7) is different from 0. These remarks are connected with equations (2) and (3) and similar equations that will follow below.

We are now able to make estimation of Steffensen's inequality using Höelder inequality and integral representations (8) i (11):

COROLLARY 2.4. *Assume that f' is continuous and g is integrable function on $[a,b]$ such that*

$$0 \leqslant \int_x^b g(t)dt \leqslant b-x \text{ and } 0 \leqslant \int_a^x g(t)dt \leqslant x-a \quad \text{for every} \quad x \in [a,b]. \quad (14)$$

Then

$$\left| \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \right| \leqslant \| f' \|_p \| G \|_q;$$

$$\left| \int_a^b f(t)g(t)dt - \int_{b-\lambda}^{b} f(t)dt \right| \leqslant \| f' \|_p \| F \|_q,$$

where G *and* F *are given by* (9) *and* (12) *and* $1 < p, g < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

We are also able to restate Corollary 1.3 with more general conditions:

COROLLARY 2.5. *For* f', h' *continuous and* g *integrable on* $[a,b]$, *with* $\lambda = \int_a^b g(t)dt$ *and*

$$0 \leqslant \int_x^b g(t)dt \leqslant b - x \text{ and } 0 \leqslant \int_a^x g(t)dt \leqslant x - a \quad \text{for every} \quad x \in [a,b]. \tag{15}$$

there exist $\xi, \eta \in (a,b)$ *such that*

$$\frac{\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt}{\int_a^b h(t)g(t)dt - \int_a^{a+\lambda} h(t)dt} = \frac{f'(\xi)}{h'(\xi)} \tag{16}$$

$$\frac{\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt}{\int_a^b h(t)g(t)dt - \int_{b-\lambda}^b h(t)dt} = \frac{f'(\eta)}{h'(\eta)} \tag{17}$$

Proof. We are showing (16) first. Let us define the linear functional $L(f) = \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt$. Next, we define $\phi(t) = f(t)L(h) - h(t)L(f)$. By Theorem 2.2 there exists $\xi \in (a,b)$ such that

$$L(\phi) = \phi'(\xi) \left[\int_a^b t g(t)dt - \lambda (a + \frac{\lambda}{2}) \right].$$

From $L(\phi) = 0$ it follows $f'(\xi)L(h) - h'(\xi)L(f) = 0$ and (16) is proved.

The proof of (17) is quite similar.

3. Error Terms for Jensen-Steffensen inequality

In this section we will generalize conditions for Jensen-Steffensen inequality given in [1]. We first consider discrete case. For convenience, we will take that weights w satisfy $\sum_{j=1}^m w_j = 1$.

THEOREM 3.1. *Let* $\{a_j\}_1^m \subset [0,\infty)$ *be nonincreasing, and let* $\{w_j\}_1^m$ *satisfy*

$$0 \leqslant \sum_{j=1}^{m} w_j(a_j - x) \leqslant a_1 - x, \text{ for } x \in (a_{k+1}, a_k] \; k = 1,2,\ldots,m, \tag{18}$$

and

$$0 \leqslant \sum_{j=k+1}^{m} w_j(x - a_j) \leqslant x, \text{ for } x \in (a_{k+1}, a_k] \; k = 1,2,\ldots,m. \tag{19}$$

Then for f *convex on* $[0,a_1]$, $f \in C^2([0,a_1])$, *we have*

$$\sum_{j=1}^{m} f(a_j)w_j - f(\sum_{j=1}^{m} a_j w_j) = \frac{f''(\xi)}{2}\left[\sum_{j=1}^{m} a_j^2 w_j - (\sum_{j=1}^{m} a_j w_j)^2\right], \tag{20}$$

for some $\xi \in (0,a_1)$.

Proof. The condition (18) is equivalent to

$$0 \leqslant \int_x^{a_1} g(t)dt \leqslant a_1 - x, \;\; x \in [0,a_1]$$

and (19) is equivalent to

$$0 \leqslant \int_0^{x} g(t)dt \leqslant x, \; x \in [0,a_1],$$

where $g : [0,a_1] \to \mathbb{R}$ is simple function defined by

$$g(x) = \sum_{j=1}^{k} w_j, \text{ for } x \in (a_{k+1}, a_k], \; k = 1,2,\ldots,m \quad (a_{m+1} := 0).$$

Further

$$\lambda = \int_0^{a_1} g(t)dt = \sum_{j=1}^{m} a_j w_j.$$

Now we can apply Theorem 2.2, relation (6), to increasing function f' :

$$\int_0^{a_1} f'(t)g(t)dt - \int_0^{\lambda} f'(t)dt = f''(\xi)\left[\int_0^{a_1} tg(t)dt - \int_0^{\lambda} tdt\right].$$

i.e.

$$\sum_{j=1}^{m} f(a_j)w_j - f(\sum_{j=1}^{m} a_j w_j) = \frac{f''(\xi)}{2}\left[\sum_{j=1}^{m} a_j^2 w_j - (\sum_{j=1}^{m} a_j w_j)^2\right].$$

Next we consider error term for continuous Jensen-Steffensen inequality. Again, we suppose $\int_0^1 w(t)dt = 1$.

THEOREM 3.2. *Let* $h:[0,1]\to\mathbb{R}$ *be monotonic function with* $h(0)=\alpha\neq\beta=h(1)$. *Let* $w:[0,1]\to\mathbb{R}$ *satisfy*

$$0 \leqslant \int_{h^{-1}((\beta-\alpha)x+\alpha)}^{1} w(t)\left[\frac{h(t)-\alpha}{\beta-\alpha}-x\right]dt \leqslant 1-x,\ 0\leqslant x\leqslant 1 \tag{21}$$

and

$$0 \leqslant \int_0^{h^{-1}((\beta-\alpha)x+\alpha)} w(t)\frac{h(t)-\alpha}{\beta-\alpha}dt + x\int_{h^{-1}((\beta-\alpha)x+\alpha)}^{1} w(t)dt \leqslant x,\ 0\leqslant x\leqslant 1. \tag{22}$$

Then for f *convex on* $[0,1]$, $f\in C^2([0,1])$, *we have*

$$\int_0^1 f\Big(\frac{h(t)-\alpha}{\beta-\alpha}\Big)w(t)dt - f\left(\int_0^1 \frac{h(t)-\alpha}{\beta-\alpha}w(t)dt\right)$$
$$= \frac{f''(\xi)}{2}\left[\int_0^1\left(\tfrac{h(t)-\alpha}{\beta-\alpha}\right)^2 w(t)dt - \left(\int_0^1\frac{h(t)-\alpha}{\beta-\alpha}w(t)dt\right)^2\right] \tag{23}$$

for some $\xi\in(0,1)$.

Proof. The condition (21) is equivalent to

$$0\leqslant\int_x^1 g(t)dt\leqslant 1-x,\ \ x\in[0,1]$$

and (22) is equivalent to

$$0\leqslant\int_0^x g(t)dt\leqslant x,\ x\in[0,1],$$

where $g:[0,1]\to\mathbb{R}$ is a function defined by

$$g(x)=\int_{h^{-1}((\beta-\alpha)x+\alpha)}^{1} w(t).$$

Further

$$\lambda = \int_0^1 g(t)dt = \int_0^1 \frac{h(t)-\alpha}{\beta-\alpha} w(t)dt.$$

After we apply Theorem 2.2, relation (6), to increasing function f' we get (23).

REMARK 3.3. Theorems 3.1 and 3.2 are extensions of relation (6) from [1].

4. Steffensen's means

Corollary 2.5 enables us to define various types of means, because if f'/h' has inverse, from (16) we have ($\lambda = \int_x^y g(t)dt$)

$$\xi = \left(\frac{f'}{h'}\right)^{-1} \left(\frac{\int_x^y f(t)g(t)dt - \int_x^{x+\lambda} f(t)dt}{\int_x^y h(t)g(t)dt - \int_x^{x+\lambda} h(t)dt} \right) \tag{24}$$

which means that ξ is a mean of numbers x and y. Specially, if we take substitutions $f(t) = t^{p-1}$, $h(t) = t^{q-1}$ in (16) and using continuous extension we consider the following expressions

$$M_1(g;x,y;p,q) = \left\{ \frac{q-1}{p-1} \frac{\int_x^y t^{p-1} g(t)dt - \frac{(x+\lambda)^p - x^p}{p}}{\int_x^y t^{q-1} g(t)dt - \frac{(x+\lambda)^q - x^q}{q}} \right\}^{\frac{1}{p-q}}, \tag{25}$$

where $p \neq q$, $y > x > 0$.

In order to cover all continuous extensions of (25) we consider the following function

$$\phi(u) = \begin{cases} \frac{1}{u-1}\left[\int_x^y t^{u-1} g(t)dt - \frac{(x+\lambda)^u - x^u}{u}\right] & u \neq 0; \\ \ln(\frac{x+\lambda}{x}) - \int_x^y \frac{g(t)}{t} dt, & u = 0; \\ \int_x^y g(t)\ln t dt - (x+\lambda)\ln(x+\lambda) + x\ln x + \lambda, & u = 1. \end{cases} \tag{26}$$

It is easy to see that ϕ is a continuous function. Now

$$M_1(g;x,y;p,q) = \left(\frac{\phi(p)}{\phi(q)}\right)^{\frac{1}{p-q}}$$

and continuous extensions of (25) are now obvious but the case $p = q$:

$$M_1(g;x,y;q,q) = Exp\left(\lim_{p\to q}\left(\frac{\phi(p)}{\phi(q)} - 1\right)\frac{1}{p-q}\right) = Exp\left(\frac{\phi'(q)}{\phi(q)}\right);$$

$$M_1(g;x,y;q,q) =$$

$$\operatorname{Exp}\left\{\frac{\int_x^y t^{q-1}g(t)\ln t dt - \frac{1}{q-1}\int_x^y t^{q-1}g(t)dt - \frac{q(x+\lambda)^q\ln(x+\lambda)-(x+\lambda)^q-qx^q\ln x+x^q}{q^2}+\frac{(x+\lambda)^q-x^q}{q(q-1)}}{\int_x^y t^{q-1}g(t)dt-\frac{(x+\lambda)^q-x^q}{q}}\right\}$$

$$M_1(g;x,y;0,0) = \operatorname{Exp}\left\{\frac{\ln(\frac{x+\lambda}{x})[1+\frac{\ln(x(x+\lambda))}{2}]-\int_x^y \frac{g(t)\ln t}{t}dt-\int_x^y \frac{g(t)}{t}dt}{\ln(\frac{x+\lambda}{x})-\int_x^y \frac{g(t)}{t}dt}\right\}$$

$$M_1(g;x,y;1,1) = \operatorname{Exp}\left\{\frac{\frac{1}{2}\int_x^y g(t)\ln^2 t dt+(x+\lambda)\ln(x+\lambda)-x\ln x-\frac{1}{2}[(x+\lambda)\ln^2(x+\lambda)-x\ln^2 x]-\lambda}{\int_x^y g(t)\ln t dt-(x+\lambda)\ln(x+\lambda)+x\ln x+\lambda}\right\}$$

From (17) we have also

$$\eta = \left(\frac{f'}{g'}\right)^{-1}\left(\frac{\int_x^y f(t)g(t)dt-\int_{y-\lambda}^y f(t)dt}{\int_x^y h(t)g(t)dt-\int_{y-\lambda}^y h(t)dt}\right) \tag{27}$$

which enables us, if we take $f(t)=t^{p-1}$, $h(t)=t^{q-1}$, to define new means

$$M_2(g;x,y;p,q) = \left\{\frac{q-1}{p-1}\frac{\int_x^y t^{p-1}g(t)dt-\frac{y^p-(y-\lambda)^p}{p}}{\int_x^y t^{q-1}g(t)dt-\frac{y^q-(y-\lambda)^q}{q}}\right\}^{\frac{1}{p-q}},$$

where $p \neq q$, $y > x > 0$.
In this case a continuous function that enables us continuous extensions is

$$\psi(u) = \begin{cases} \frac{1}{u-1}\left[\int_x^y t^{u-1}g(t)dt-\frac{y^u-(y-\lambda)^u}{u}\right] & u\neq 0;\\ \ln(\frac{y}{y-\lambda})-\int_x^y \frac{g(t)}{t}dt, & u=0;\\ \int_x^y g(t)\ln t dt - y\ln y+(y-\lambda)\ln(y-\lambda)+\lambda, & u=1.\end{cases} \tag{28}$$

Now

$$M_2(g;x,y;p,q) = \left(\frac{\psi(p)}{\psi(q)}\right)^{\frac{1}{p-q}}$$

and all extensions can be now easily deduced but the case $p=q$ ::

$$M_2(g;x,y;q,q) =$$

$$\operatorname{Exp}\left\{\frac{\int_x^y t^{q-1}g(t)\ln t dt\,\frac{1}{q-1}\int_x^y t^{q-1}g(t)dt-\frac{qy^q\ln y-y^q-q(y-\lambda)^q\ln(y-\lambda)+(y-\lambda)^q}{q^2}+\frac{y^q-(y-\lambda)^q}{q(q-1)}}{\int_x^y t^{q-1}g(t)dt-\frac{y^q-(y-\lambda)^q}{q}}\right\}$$

$$M_2(g;x,y;0,0)=\operatorname{Exp}\left\{\frac{\ln(\frac{y}{(y-\lambda)})[1+\frac{\ln((y-\lambda)(y))}{2}]-\int_x^y\frac{g(t)\ln t}{t}dt-\int_x^y\frac{g(t)}{t}dt}{\ln(\frac{y}{(y-\lambda)})-\int_x^y\frac{g(t)}{t}dt}\right\}$$

$$M_2(g;x,y;1,1)=\operatorname{Exp}\left\{\frac{\frac{1}{2}\int_x^y g(t)\ln^2 t\,dt+y\ln y-x\ln x-\frac{1}{2}[y\ln^2 y-(y-\lambda)\ln^2(y-\lambda)]-\lambda}{\int_x^y g(t)\ln t\,dt-y\ln y+(y-\lambda)\ln(y-\lambda)+\lambda}\right\}$$

THEOREM 4.1. *Let $r\leqslant u$, $t\leqslant v$. Then*

$$M_1(g;x,y;r,t)\leqslant M_1(g;x,y;u,v) \tag{29}$$

and

$$M_2(g;x,y;r,t)\leqslant M_2(g;x,y;u,v). \tag{30}$$

For the proof, we need the following two lemmas.

LEMMA 4.2. *Let f be log-convex function and if, $x_1\leqslant y_1$, $x_2\leqslant y_2$, $x_1\neq x_2$, $y_1\neq y_2$, then the following inequality is valid:*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{1/(x_2-x_1)}\leqslant\left(\frac{f(y_2)}{f(y_1)}\right)^{1/(y_2-y_1)}. \tag{31}$$

Proof. This follows from [2], Remark 1.2.

LEMMA 4.3. *The functions ϕ and ψ defined by* (26) *and* (28) *are log-convex functions.*

Proof. Let us consider the following function

$$f(x)=p^2\varphi_r(x)+2pq\varphi_z(x)+q^2\varphi_s(x)\qquad\text{where } z=\frac{r+s}{2}\text{ and } p,q\in\mathbb{R},$$

and

$$\varphi_u(x)=\begin{cases}\frac{x^u}{u}, & u\neq 0;\\ \ln x, & u=0.\end{cases}$$

Now

$$\begin{aligned}f'(x)&=p^2x^{r-1}+2pqx^{z-1}+q^2x^{s-1}\\&=\left(px^{(r-1)/2}+qx^{(s-1)/2}\right)^2\geqslant 0.\end{aligned}$$

This implies f is monotonically increasing.

We now define the function

$$\phi(u) = \int_x^y \varphi_{u-1}(t)g(t)dt - \int_x^{x+\lambda} \varphi_{u-1}(t)dt. \tag{32}$$

It is easy to see that

$$\phi(t) = \begin{cases} \frac{1}{u-1}\left[\int_x^y t^{u-1}g(t)dt - \frac{(x+\lambda)^u - x^u}{u}\right] & u \neq 0; \\ \ln(\frac{x+\lambda}{x}) - \int_x^y \frac{g(t)}{t}dt, & u = 0; \\ \int_x^y g(t)\ln t dt - (x+\lambda)\ln(x+\lambda) + x\ln x + \lambda, & u = 1, \end{cases} \tag{33}$$

and that ϕ is a continuous function.

Using function f in inequality (1) we have

$$p^2\phi(r) + 2pq\phi(z) + q^2\phi(s) \geqslant 0$$

i.e.

$$\phi^2(z) \leqslant \phi(r)\cdot\phi(s) \quad \text{where } z = \frac{r+s}{2}.$$

So ϕ is log-convex in Jensen sense and because we prove that ϕ is continuous, it is a log-convex function.

In the same fashion we can prove that ψ is a continuous function.

Proof. [Proof of Theorem 4.1] We now apply inequality Lemma 4.2 for $f = \phi$, $r \leqslant u$, $t \leqslant v$, $r \neq t$, $u \neq v$ $(r,t,u,v \neq 0,1,2)$ to deduce that

$$\left\{\frac{t-1}{r-1}\frac{\int_x^y s^{r-1}g(s)ds - \frac{(x+\lambda)^r - x^r}{r}}{\int_x^y s^{t-1}g(s)ds - \frac{(x+\lambda)^t - x^t}{t}}\right\}^{\frac{1}{r-t}} \leqslant \left\{\frac{v-1}{u-1}\frac{\int_x^y s^{u-1}g(s)ds - \frac{(x+\lambda)^u - x^u}{u}}{\int_x^y s^{v-1}g(s)ds - \frac{(x+\lambda)^v - x^v}{v}}\right\}^{\frac{1}{u-v}}$$

Since $M_1(g;x,y;r,t)$ is continuous we have, for $r \leqslant u$, $t \leqslant v$,

$$M_1(g;x,y;r,t) \leqslant M_1(g;x,y;u,v).$$

The same arguments stand for $M_2(g;x,y;r,t)$.

Acknowledgements. The research of the second author was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grant 117-1170889-0888.

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(Received September 18, 2008)

J. Jakšetić
Department of Mathematics
University of Zagreb
Croatia
e-mail: julije@math.hr

J. Pečarić
University Of Zagreb
Faculty Of Textile Technology Zagreb
Croatia
e-mail: pecaric@mahazu.hazu.hr