Journal of
Mathematical
Inequalities
Volume 2, Number 4 (2008), 511–526

GENERAL DUAL EULER–SIMPSON FORMULAE

J. PEČARIĆ AND A. VUKELIĆ

(*communicated by N. Ujević*)

Abstract. We consider a general dual Simpson formulae, using some Euler-type identities. A number of inequalities, for functions whose derivatives are either functions of bounded variation or Lipschitzian functions or R-integrable functions, are proved.

1. Introduction

In the recent paper [3] the following two identities, named the extended Euler formulae, have been proved. For $n \geqslant 1$ and every $x \in [0,1]$

$$f(x) = \int_0^1 f(t)\mathrm{d}t + T_n(x) + R_n^1(x) \tag{1.1}$$

and

$$f(x) = \int_0^1 f(t)\mathrm{d}t + T_{n-1}(x) + R_n^2(x), \tag{1.2}$$

where $T_0(x) = 0$ and

$$T_m(x) = \sum_{k=1}^{m} \frac{B_k(x)}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right], \tag{1.3}$$

for $1 \leqslant m \leqslant n$, while

$$R_n^1(x) = -\frac{1}{n!}\int_0^1 B_n^*(x-t)\,\mathrm{d}f^{(n-1)}(t),$$

$$R_n^2(x) = -\frac{1}{n!}\int_0^1 [B_n^*(x-t) - B_n(x)]\,\mathrm{d}f^{(n-1)}(t).$$

Here, as in the rest of the paper, we write $\int_0^1 g(t)\,d\,\varphi(t)$ to denote the Riemann-Stieltjes integral with respect to a function $\varphi : [0,1] \to \mathbf{R}$ of bounded variation, and $\int_0^1 g(t)\,d\,t$ for the Riemann integral. The identities (1.1) and (1.2) extend the well known formula for the expansion of a function in Bernoulli polynomials [10, p. 17]. They hold for

Mathematics subject classification (2000): 26D15, 26D20, 26D99.
Keywords and phrases: General dual Euler-Simpson formulae, quadrature formulae, functions of bounded variation, Lipschitzian functions.

Paper JMI-02-46

every function $f:[0,1]\to\mathbf{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$. The functions $B_k(t)$ are the Bernoulli polynomials, $B_k=B_k(0)$ are the Bernoulli numbers, and $B_k^*(t),\ k\geqslant 0,$ are periodic functions of period 1, related to the Bernoulli polynomials as

$$B_k^*(t)=B_k(t),\ 0\leqslant t<1 \quad \text{and} \quad B_k^*(t+1)=B_k^*(t),\ t\in\mathbf{R}.$$

The Bernoulli polynomials $B_k(t),\ k\geqslant 0$ are uniquely determined by the following identities

$$B_k'(t)=kB_{k-1}(t),\ k\geqslant 1;\ B_0(t)=1,\ B_k(t+1)-B_k(t)=kt^{k-1},\ k\geqslant 0.$$

For some further details on the Bernoulli polynomials and the Bernoulli numbers see for example [1] or [2]. We have that $B_0^*(t)=1$ and $B_1^*(t)$ is a discontinuous function with a jump of -1 at each integer. It follows that $B_k(1)=B_k(0)=B_k$ for $k\geqslant 2$, so that $B_k^*(t)$ are continuous functions for $k\geqslant 2$. We get

$$B_k^{*\prime}(t)=kB_{k-1}^*(t),\ k\geqslant 1 \tag{1.4}$$

for every $t\in\mathbf{R}$ when $k\geqslant 3$, and for every $t\in\mathbf{R}\setminus\mathbf{Z}$ when $k=1,2$.

In the recent, many mathematicians are studying in the area which are related to Euler-Simpson's type formula, Euler summation formula etc. (see for example [6], [7], [8] and [9]).

In this paper we study, the general dual Simpson quadrature formula

$$\int_0^1 f(t)dt=\frac{1}{2u-v}\left[uf\left(\frac{1}{4}\right)-vf\left(\frac{1}{2}\right)+uf\left(\frac{3}{4}\right)\right]+E(f;u,v) \tag{1.5}$$

with $E(f;u,v)$ being the remainder, $u,v\in\mathbf{Z}^+,\ v<2u$ and the greatest common divisor of u and v is 1. The aim of this paper is to establish general dual Simpson formula (1.5) using identities (1.1) and (1.2) and give various error estimates for the quadrature rules based on such generalizations. In Section 2 we use the extended Euler formulae to obtain two new integral identities. We call them the general dual Euler-Simpson formulae. In Section 3, we prove a number of inequalities which give error estimates for the general dual Euler-Simpson formulae for functions whose derivatives are from the L_p-spaces.

2. General dual Euler-Simpson formulae

For $k\geqslant 1$ define the functions $G_k(t)$ and $F_k(t)$ as

$$G_k(t)=uB_k^*\left(\frac{1}{4}-t\right)-vB_k^*\left(\frac{1}{2}-t\right)+uB_k^*\left(\frac{3}{4}-t\right),\ t\in\mathbf{R}$$

and

$$F_k(t)=G_k(t)-\tilde{B}_k,\ t\in\mathbf{R},\ k\geqslant 1,$$

where

$$\tilde{B}_k = uB_k\left(\frac{1}{4}\right) - vB_k\left(\frac{1}{2}\right) + uB_k\left(\frac{3}{4}\right),\ k \geqslant 1.$$

Especially, using $B_1(t) = t - 1/2$ we get $\tilde{B}_1 = 0$. Also, for $k \geqslant 2$ we have $\tilde{B}_k = G_k(0)$, that is

$$F_k(t) = G_k(t) - G_k(0),\ k \geqslant 2, \text{ and } F_1(t) = G_1(t),\ t \in \mathbf{R}.$$

Obviously, $G_k(t)$ and $F_k(t)$ are periodic functions of period 1 and continuous for $k \geqslant 2$.

Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ exists on $[0,1]$ for some $n \geqslant 1$. We introduce the following notation

$$D(u,v) = \frac{1}{2u - v}\left[uf\left(\frac{1}{4}\right) - vf\left(\frac{1}{2}\right) + uf\left(\frac{3}{4}\right)\right].$$

Further, we define $\tilde{T}_0(u,v) = 0$ and, for $1 \leqslant m \leqslant n$,

$$\tilde{T}_m(u,v) = \frac{1}{2u - v}\left[uT_m\left(\frac{1}{4}\right) - vT_m\left(\frac{1}{2}\right) + uT_m\left(\frac{3}{4}\right)\right],$$

where $T_m(x)$ is given by (1.3). For $m \geqslant 1$

$$\tilde{T}_m(u,v) = \frac{1}{2u - v}\sum_{k=1}^{m}\frac{\tilde{B}_k}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right]. \tag{2.1}$$

In the next theorem we establish two formulae which play the key role in this paper. We call them the general dual Euler-Simpson formulae.

THEOREM 1. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \geqslant 1$. Then*

$$\int_0^1 f(t)\mathrm{d}t = D(u,v) - \tilde{T}_n(u,v) + \tilde{R}_n^1(f), \tag{2.2}$$

and

$$\int_0^1 f(t)\mathrm{d}t = D(u,v) - \tilde{T}_{n-1}(u,v) + \tilde{R}_n^2(f), \tag{2.3}$$

where

$$\tilde{R}_n^1(f) = \frac{1}{(2u - v)(n!)}\int_0^1 G_n(t)\,\mathrm{d}f^{(n-1)}(t),$$

and

$$\tilde{R}_n^2(f) = \frac{1}{(2u - v)(n!)}\int_0^1 F_n(t)\,\mathrm{d}f^{(n-1)}(t).$$

Proof. Put $x = 1/4,\ 1/2,\ 3/4$ in formula (1.1) to get three new formulae. Then multiply these new formulae by $u,\ -v,\ u$ respectively, and add. The result is formula (2.2). Formula (2.3) is obtained from (1.2) by the same procedure.

REMARK 1. If in Theorem 1 we chose $u = 2$ and $v = 1$ we get dual Euler Simpson formulae [4] and for $u = 8$ and $v = 1$ we get corrected dual Euler Simpson formulae [5].

By direct calculations we get

$$F_1(t) = G_1(t) = \begin{cases} (v-2u)t, & 0 \leqslant t \leqslant 1/4 \\ (v-2u)t+u, & 1/4 < t \leqslant 1/2 \\ (v-2u)t+u-v, & 1/2 < t \leqslant 3/4 \\ (v-2u)t+2u-v, & 3/4 < t \leqslant 1 \end{cases}, \tag{2.4}$$

$$G_2(t) = \begin{cases} (2u-v)t^2+(2v-u)/24, & 0 \leqslant t \leqslant 1/4 \\ (2u-v)t^2-2ut+(11u+2v)/24, & 1/4 < t \leqslant 1/2 \\ (2u-v)t^2+(2v-2u)t+(11u-22v)/24, & 1/2 < t \leqslant 3/4 \\ (2u-v)t^2+(2v-4u)t+(47u-22v)/24, & 3/4 < t \leqslant 1 \end{cases}, \tag{2.5}$$

$$F_2(t) = \begin{cases} (2u-v)t^2, & 0 \leqslant t \leqslant 1/4 \\ (2u-v)t^2-2ut+u/2, & 1/4 < t \leqslant 1/2 \\ (2u-v)t^2+(2v-2u)t+(u-2v)/2, & 1/2 < t \leqslant 3/4 \\ (2u-v)t^2+(2v-4u)t+2u-v, & 3/4 < t \leqslant 1 \end{cases}, \tag{2.6}$$

$$F_3(t) = G_3(t) = \begin{cases} (v-2u)t^3+(u-2v)t/8, & 0 \leqslant t \leqslant 1/4 \\ (v-2u)t^3+3ut^2-(11u+2v)t/8+3u/16, & 1/4 < t \leqslant 1/2 \\ (v-2u)t^3+(3u-3v)t^2 & \\ +(22v-11u)t/8+(3u-12v)/16, & 1/2 < t \leqslant 3/4 \\ (v-2u)t^3+(6u-3v)t^2 & \\ +(22v-47u)t/8+(15u-6v)/8, & 3/4 < t \leqslant 1 \end{cases}. \tag{2.7}$$

Now, we will prove some properties of the functions $G_k(t)$ and $F_k(t)$ defined above. The Bernoulli polynomials are symmetric with respect to $1/2$, that is $[1, 23.1.8]$

$$B_k(1-t) = (-1)^k B_k(t),\ \forall t \in \mathbf{R},\ k \geqslant 1. \tag{2.8}$$

Also, we have

$$B_k(1) = B_k(0) = B_k,\ k \geqslant 2,\ \ B_1(1) = -B_1(0) = \frac{1}{2}$$

and

$$B_{2j-1} = 0,\ j \geqslant 2.$$

Therefore, using $[1, 23.1.21, 23.1.22]$

$$B_{2j}\left(\frac{1}{2}\right) = -\left(1-2^{1-2j}\right)B_{2j},\ \ B_{2j}\left(\frac{1}{4}\right) = -2^{-2j}\left(1-2^{1-2j}\right)B_{2j} \quad j \geqslant 1,$$

we get

$$\tilde{B}_{2j-1} = 0,\ j \geqslant 1 \tag{2.9}$$

and for $j \geqslant 1$

$$\tilde{B}_{2j} = uB_{2j}\left(\frac{1}{4}\right) - vB_{2j}\left(\frac{1}{2}\right) + uB_{2j}\left(\frac{3}{4}\right) = (v - u \cdot 2^{1-2j})(1 - 2^{1-2j})B_{2j}. \tag{2.10}$$

Now, by (2.9) we have

$$F_{2j-1}(t) = G_{2j-1}(t), \; j \geqslant 1, \tag{2.11}$$

and, by (2.10),

$$F_{2j}(t) = G_{2j}(t) - \tilde{B}_{2j} = G_{2j}(t) - (v - u \cdot 2^{1-2j})(1 - 2^{1-2j})B_{2j}, \; j \geqslant 1. \tag{2.12}$$

Further, the points 0 and 1 are the zeros of $F_k(t) = G_k(t) - G_k(0), \; k \geqslant 2,$ that is

$$F_k(0) = F_k(1) = 0, \; k \geqslant 2.$$

As we shall see below, for $j \geqslant 1$, 0 and 1 are the only zeros of $F_{2j}(t)$ for $u/2 \leqslant v < 2u$. Next, setting $t = 1/2$ in (2.8) we get

$$B_k\left(\frac{1}{2}\right) = (-1)^k B_k\left(\frac{1}{2}\right), \; k \geqslant 1.$$

which implies that

$$B_{2j-1}\left(\frac{1}{2}\right) = 0, \; j \geqslant 1.$$

Using the above formulae, we get

$$F_{2j-1}\left(\frac{1}{2}\right) = G_{2j-1}\left(\frac{1}{2}\right) = 0, \; j \geqslant 1.$$

We shall see that for $j \geqslant 2, \; 0, \; 1/2$ and 1 are the only zeros of $F_{2j-1}(t) = G_{2j-1}(t)$ for $u/2 \leqslant v < 2u$. Also, note that

$$G_{2j}\left(\frac{1}{2}\right) = uB_{2j}\left(\frac{3}{4}\right) - vB_{2j} + uB_{2j}\left(\frac{1}{4}\right) = \left[-v - u \cdot 2^{1-2j}(1 - 2^{1-2j})\right] B_{2j}, \; j \geqslant 1,$$

$$F_{2j}\left(\frac{1}{2}\right) = G_{2j}\left(\frac{1}{2}\right) - \tilde{B}_{2j} = -2v(1 - 2^{-2j})B_{2j}, \; j \geqslant 1. \tag{2.13}$$

LEMMA 1. For $k \geqslant 2$ we have

$$G_k(1-t) = (-1)^k G_k(t), \; 0 \leqslant t \leqslant 1,$$

and

$$F_k(1-t) = (-1)^k F_k(t), \; 0 \leqslant t \leqslant 1.$$

Proof. As we noted in introduction, the functions $B_k^*(t)$ are periodic with period 1 and continuous for $k \geqslant 2$. Therefore, for $k \geqslant 2$ and $0 \leqslant t \leqslant 1$ we have

$$\begin{aligned}
G_k(1-t) &= uB_k^*\left(-\frac{3}{4}+t\right) - vB_k^*\left(-\frac{1}{2}+t\right) + uB_k^*\left(-\frac{1}{4}+t\right) \\
&= \begin{cases} uB_k^*\left(\frac{1}{4}+t\right) - vB_k^*\left(\frac{1}{2}+t\right) + uB_k^*\left(\frac{3}{4}+t\right), & 0 \leqslant t \leqslant 1/4, \\ uB_k^*\left(\frac{1}{4}+t\right) - vB_k^*\left(\frac{1}{2}+t\right) + uB_k^*\left(-\frac{1}{4}+t\right), & 1/4 < t \leqslant 1/2, \\ uB_k^*\left(\frac{1}{4}+t\right) - vB_k^*\left(-\frac{1}{2}+t\right) + uB_k^*\left(-\frac{1}{4}+t\right), & 1/2 < t \leqslant 3/4, \\ uB_k^*\left(-\frac{3}{4}+t\right) - vB_k^*\left(-\frac{1}{2}+t\right) + uB_k^*\left(-\frac{1}{4}+t\right), & 3/4 < t \leqslant 1, \end{cases} \\
&= (-1)^k \times \begin{cases} uB_k^*\left(\frac{3}{4}-t\right) - vB_k^*\left(\frac{1}{2}-t\right) + uB_k^*\left(\frac{1}{4}-t\right), & 0 \leqslant t \leqslant 1/4, \\ uB_k^*\left(\frac{3}{4}-t\right) - vB_k^*\left(\frac{1}{2}-t\right) + uB_k^*\left(\frac{5}{4}-t\right), & 1/4 < t \leqslant 1/2, \\ uB_k^*\left(\frac{3}{4}-t\right) - vB_k^*\left(\frac{3}{2}-t\right) + uB_k^*\left(\frac{5}{4}-t\right), & 1/2 < t \leqslant 3/4, \\ uB_k^*\left(\frac{7}{4}-t\right) - vB_k^*\left(\frac{3}{2}-t\right) + uB_k^*\left(\frac{5}{4}-t\right), & 3/4 < t \leqslant 1, \end{cases} \\
&= (-1)^k G_k(t),
\end{aligned}$$

which proves the first identity. Further, we have $F_k(t) = G_k(t) - G_k(0)$ and $(-1)^k G_k(0) = G_k(0)$, since $G_{2j+1}(0) = 0$, so that we have

$$F_k(1-t) = G_k(1-t) - G_k(0) = (-1)^k \left[G_k(t) - G_k(0)\right] = (-1)^k F_k(t),$$

which proves the second identity.

Note that the identities established in Lemma 1 are valid for $k = 1$, too, except at the points $1/4$, $1/2$ and $3/4$ of discontinuity of $F_1(t) = G_1(t)$.

LEMMA 2. For $k \geqslant 2$ and $u/2 \leqslant v < 2u$ the function $G_{2k-1}(t)$ has no zeros in the interval $(0, 1/2)$. The sign of this function is determined by

$$(-1)^{k-1} G_{2k-1}(t) > 0,\ 0 < t < \frac{1}{2}.$$

Proof. For $k = 2$, $G_3(t)$ is given by (2.7) and it is easy to see that for $u/2 \leqslant v < 2u$

$$-G_3(t) > 0,\ 0 < t < \frac{1}{2},$$

Thus, our assertion is true for $k = 2$. Now, assume that $k \geqslant 3$. Then $2k-1 \geqslant 5$ and $G_{2k-1}(t)$ is continuous and at least twice differentiable function. Using (1.4) we get

$$G'_{2k-1}(t) = -(2k-1)G_{2k-2}(t)$$

and

$$G''_{2k-1}(t) = (2k-1)(2k-2)G_{2k-3}(t).$$

Let us suppose that G_{2k-3} has no zeros in the interval $(0, 1/2)$. We know that 0 and $1/2$ are the zeros of $G_{2k-1}(t)$. Let us suppose that some α, $0 < \alpha < 1/2$, is also a zero of $G_{2k-1}(t)$. Then inside each of the intervals $(0, \alpha)$ and $(\alpha, 1/2)$ the derivative

$G'_{2k-1}(t)$ must have at least one zero, say $\beta_1,\ 0<\beta_1<\alpha$ and $\beta_2,\ \alpha<\beta_2<1/2$. Therefore, the second derivative $G''_{2k-1}(t)$ must have at least one zero inside the interval (β_1,β_2). Thus, from the assumption that $G_{2k-1}(t)$ has a zero inside the interval $(0,1/2)$, it follows that $(2k-1)(2k-2)G_{2k-3}(t)$ also has a zero inside this interval. Thus, $G_{2k-1}(t)$ can not have a zero inside the interval $(0,1/2)$. To determine the sign of $G_{2k-1}(t)$, note that

$$G_{2k-1}\left(\frac{1}{4}\right)=-vB_{2k-1}\left(\frac{1}{4}\right).$$

We have [1, 23.1.14]

$$(-1)^kB_{2k-1}(t)>0,\ 0<t<\frac{1}{2},$$

which implies

$$(-1)^{k-1}G_{2k-1}\left(\frac{1}{4}\right)=(-1)^kvB_{2k-1}\left(\frac{1}{4}\right)>0.$$

So, we proved our assertions.

COROLLARY 1. For $k\geqslant 2$ and $u/2\leqslant v<2u$ the functions $(-1)^kF_{2k}(t)$ and $(-1)^kG_{2k}(t)$ are strictly increasing on the interval $(0,1/2)$, and strictly decreasing on the interval $(1/2,1)$. Further, for $k\geqslant 2$ and $u/2\leqslant v<2u$ we have

$$\max_{t\in[0,1]}|F_{2k}(t)|=2v\left(1-2^{-2k}\right)|B_{2k}|,$$

and

$$\max_{t\in[0,1]}|G_{2k}(t)|=\left[v+u\cdot 2^{1-2k}(1-2^{1-2k})\right]|B_{2k}|.$$

Proof. Using (1.4) we get

$$\left[(-1)^kF_{2k}(t)\right]'=\left[(-1)^kG_{2k}(t)\right]'=2k(-1)^{k-1}G_{2k-1}(t)$$

and $(-1)^{k-1}G_{2k-1}(t)>0$ for $0<t<1/2$, by Lemma 2. Thus, $(-1)^kF_{2k}(t)$ and $(-1)^kG_{2k}(t)$ are strictly increasing on the interval $(0,1/2)$. Also, by Lemma 1, we have $F_{2k}(1-t)=F_{2k}(t),\ 0\leqslant t\leqslant 1$ and $G_{2k}(1-t)=G_{2k}(t),\ 0\leqslant t\leqslant 1$, which implies that $(-1)^kF_{2k}(t)$ and $(-1)^kG_{2k}(t)$ are strictly decreasing on the interval $(1/2,1)$. Further, $F_{2k}(0)=F_{2k}(1)=0$, which implies that $|F_{2k}(t)|$ achieves its maximum at $t=1/2$, that is

$$\max_{t\in[0,1]}|F_{2k}(t)|=\left|F_{2k}\left(\frac{1}{2}\right)\right|=2v\left(1-2^{-2k}\right)|B_{2k}|.$$

Also

$$\max_{t\in[0,1]}|G_{2k}(t)|=\max\left\{|G_{2k}(0)|,\left|G_{2k}\left(\frac{1}{2}\right)\right|\right\}=\left[v+u\cdot 2^{1-2k}(1-2^{1-2k})\right]|B_{2k}|,$$

which completes the proof.

COROLLARY 2. For $k \geqslant 2$ and $u/2 \leqslant v < 2u$ we have

$$\int_0^1 |F_{2k-1}(t)|\,\mathrm{d}t = \int_0^1 |G_{2k-1}(t)|\,\mathrm{d}t = \frac{2v}{k}(1-2^{-2k})\,|B_{2k}|\,.$$

Also, we have

$$\int_0^1 |F_{2k}(t)|\,\mathrm{d}t = \left|\tilde{B}_{2k}\right| = (v-u\cdot 2^{1-2j})(1-2^{1-2j})\,|B_{2k}|$$

and

$$\int_0^1 |G_{2k}(t)|\,\mathrm{d}t \leqslant 2\left|\tilde{B}_{2k}\right| = 2(v-u\cdot 2^{1-2j})(1-2^{1-2j})\,|B_{2k}|\,.$$

Proof. Using (1.4) it is easy to see that

$$G_m'(t) = -mG_{m-1}(t),\ m \geqslant 3. \tag{2.14}$$

Now, using Lemma 1, Lemma 2 and (2.14) we get

$$\begin{aligned}\int_0^1 |G_{2k-1}(t)|\,\mathrm{d}t &= 2\left|\int_0^{1/2} G_{2k-1}(t)\mathrm{d}t\right| = 2\left|-\frac{1}{2k}\,G_{2k}(t)|_0^{1/2}\right| \\ &= \frac{1}{k}\left|G_{2k}\left(\frac{1}{2}\right) - G_{2k}(0)\right| = \frac{2v}{k}(1-2^{-2k})\,|B_{2k}|\,,\end{aligned}$$

which proves the first assertion. By Corollary 1 and because $F_{2k}(0) = F_{2k}(1) = 0$, $F_{2k}(t)$ does not change its sign on the interval $(0,1)$. Therefore, using (2.12) and (2.14), we get

$$\begin{aligned}\int_0^1 |F_{2k}(t)|\,\mathrm{d}t = \left|\int_0^1 F_{2k}(t)\mathrm{d}t\right| &= \left|\int_0^1 \left[G_{2k}(t) - \tilde{B}_{2k}\right]\mathrm{d}t\right| \\ &= \left|-\frac{1}{2k+1}\,G_{2k+1}(t)|_0^1 - \tilde{B}_{2k}\right| = \left|\tilde{B}_{2k}\right|,\end{aligned}$$

which proves the second assertion. Finally, we use (2.12) again and the triangle inequality to obtain

$$\int_0^1 |G_{2k}(t)|\,\mathrm{d}t = \int_0^1 \left|F_{2k}(t) + \tilde{B}_{2k}\right|\mathrm{d}t \leqslant \int_0^1 |F_{2k}(t)|\,\mathrm{d}t + \left|\tilde{B}_{2k}\right| = 2\left|\tilde{B}_{2k}\right|,$$

which proves the third assertion.

3. Inequalities related to the general dual Euler-Simpson formulae

In this section we use formulae established in Theorem 1 to prove a number of inequalities using L_p norms for $1 \leqslant p \leqslant \infty$. These inequalities are generally sharp (in case $p = 1$ the best possible).

THEOREM 2. *Assume* (p,q) *is a pair of conjugate exponents,* $1 \leqslant p,q \leqslant \infty$. *Let* $\left|f^{(n)}\right|^p : [0,1] \to \mathbf{R}$ *is R-integrable function for some* $n \geqslant 1$. *Then, we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v) + \tilde{T}_{n-1}(u,v)\right| \leqslant K(n,p;u,v) \cdot \|f^{(n)}\|_p, \tag{3.1}$$

and

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v) + \tilde{T}_n(u,v)\right| \leqslant K^*(n,p;u,v) \cdot \|f^{(n)}\|_p, \tag{3.2}$$

where

$$K(n,p;u,v) = \frac{1}{(2u-v)(n!)}\left[\int_0^1 |F_n(t)|^q \,\mathrm{d}t\right]^{1/q} \quad \text{and}$$

$$K^*(n,p;u,v) = \frac{1}{(2u-v)(n!)}\left[\int_0^1 |G_n(t)|^q \,\mathrm{d}t\right]^{1/q}.$$

The constants $K(n,p;u,v)$ *and* $K^*(n,p;u,v)$ *are sharp for* $1 < p \leqslant \infty$ *and the best possible for* $p = 1$.

Proof. Applying the Hölder inequality we have

$$\left|\frac{1}{(2u-v)(n!)}\int_0^1 F_n(t) f^{(n)}(t)\mathrm{d}t\right| \leqslant \frac{1}{(2u-v)(n!)}\left[\int_0^1 |F_n(t)|^q \,\mathrm{d}t\right]^{1/q} \cdot \left\|f^{(n)}\right\|_p$$
$$= K(n,p;u,v) \cdot \|f^{(n)}\|_p.$$

Using the above inequality from (2.3) we get estimate (3.1). In the same manner, from (2.2) we get estimate (3.2). Now, we consider the optimality of $K(n,p;u,v)$. We shall find a function f such that

$$\left|\int_0^1 F_n(t) f^{(n)} dt\right| = \left(\int_0^1 |F_n(t)|^q dt\right)^{1/q} \left(\int_0^1 |f^{(n)}(t)|^p dt\right)^{1/p}.$$

For $1 < p < \infty$ take f to be such that

$$f^{(n)}(t) = \operatorname{sgn} F_n(t) \cdot |F_n(t)|^{\frac{1}{p-1}} \tag{3.3}$$

where for $p = \infty$ we put $f^{(n)}(t) = \operatorname{sgn} F_n(t)$. For constant $K^*(n,p;u,v)$ the proof of sharpness is analogous. For $p = 1$ we shall prove that

$$\left|\int_0^1 F_n(t) f^{(n)}(t) dt\right| \leqslant \max_{t\in[0,1]} |F_n(t)| \int_0^1 |f^{(n)}(t)| dt \tag{3.4}$$

is the best possible inequality. Suppose that $|F_n(t)|$ attains its maximum at $t_0 \in (0,1)$. First, we assume that $F_n(t_0) > 0$. For ε small enough define $f_\varepsilon^{(n-1)}(t)$ by

$$f_\varepsilon^{(n-1)}(t) = \begin{cases} 0, & t \leqslant t_0 \\ \frac{1}{\varepsilon}(t - t_0), & t \in [t_0, t_0+\varepsilon] \\ 1, & t \geqslant t_0 + \varepsilon \end{cases}.$$

Then, for ε small enough

$$\left|\int_0^1 F_n(t) f_\varepsilon^{(n)}(t)dt\right| = \left|\int_{t_0}^{t_0+\varepsilon} F_n(t)\frac{1}{\varepsilon}dt\right| = \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} F_n(t)dt.$$

Now, from inequality (3.4) we have

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} F_n(t)dt \leqslant F_n(t_0)\int_{t_0}^{t_0+\varepsilon}\frac{1}{\varepsilon}dt = F_n(t_0).$$

Since,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} F_n(t)dt = F_n(t_0)$$

the statement follows. If $F_n(t_0) < 0$, then we take

$$f_\varepsilon^{(n-1)}(t) = \begin{cases} 1, & t \leqslant t_0 \\ -\frac{1}{\varepsilon}(t-t_0-\varepsilon), & t \in [t_0, t_0+\varepsilon] \\ 0, & t \geqslant t_0+\varepsilon \end{cases}$$

and the rest of proof is the same as above. Proof of the best possibility of the second inequality is similar.

COROLLARY 3. Let $f:[0,1]\to\mathbf{R}$ be given function.
If f is L-Lipschitzian on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v)\right| \leqslant \frac{2u+v}{8(2u-v)}\cdot L.$$

If f' is L-Lipschitzian on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v)\right| \leqslant \frac{2u^2(3v+\sqrt{2uv})+uv(5v-\sqrt{2uv})+2v^2(v+3\sqrt{2uv})}{48(2u-v)(v+\sqrt{2uv})(2u+v+2\sqrt{2uv})}\cdot L.$$

Proof. Using (2.4) and (2.5) we get

$$\int_0^1 |F_1(t)|\,\mathrm{d}t = \frac{2u+v}{8} \quad \text{and}$$

$$\int_0^1 |F_2(t)|\,\mathrm{d}t = \frac{2u^2(3v+\sqrt{2uv})+uv(5v-\sqrt{2uv})+2v^2(v+3\sqrt{2uv})}{24(v+\sqrt{2uv})(2u+v+2\sqrt{2uv})}.$$

Therefore, applying (3.1) with $n=1,2$ and $p=\infty$ we get the above inequalities.

REMARK 2. The first inequality in Corollary 3 achieves an infimum of $1/24$ and the second inequality an infimum of 0 for $u\to\infty$ and $v=1$.

REMARK 3. Let $f:[0,1]\to\mathbf{R}$ is such that $f^{(n-1)}$ is an L-Lipschitzian function on $[0,1]$ for some $n\geqslant 3$. Then from Corollary 2 for $u/2\leqslant v<2u$ we get

$$K(2k-1,\infty;u,v)=\frac{2v}{(2u-v)\,[(2k)!]}(1-2^{-2k})\,|B_{2k}|,$$

$$K^*(2k,\infty;u,v)=\frac{1}{(2u-v)\,[(2k)!]}(v-u\cdot 2^{1-2k})(1-2^{1-2k})\,|B_{2k}|$$

and

$$K(2k,\infty;u,v)=\frac{2}{(2u-v)\,[(2k)!]}(v-u\cdot 2^{1-2k})(1-2^{1-2k})\,|B_{2k}|.$$

COROLLARY 4. Let $f:[0,1]\to\mathbf{R}$ be given function.
If f is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t-D(u,v)\right|\leqslant\frac{2u+v}{4(2u-v)}\cdot V_0^1(f).$$

If f' is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t-D(u,v)\right|\leqslant\frac{1}{64(2u-v)}[2u+3v+|2u-5v|]\cdot V_0^1(f').$$

Proof. ¿From explicit expressions (2.4) and (2.5), we get

$$\max_{t\in[0,1]}|F_1(t)|=\max\left\{\frac{2u-v}{4},\frac{2u+v}{4}\right\}=\frac{2u+v}{4}\text{ and}$$
$$\max_{t\in[0,1]}|F_2(t)|=\max\left\{\frac{2u-v}{16},\frac{v}{4}\right\}=\frac{1}{32}[2u+3v+|2u-5v|].$$

Therefore, applying (3.1) with $n=1,2$ and $p=1$ we get the above inequalities.

REMARK 4. The first inequality in Corollary 4 achieves an infimum of $1/4$ and the second inequality an infimum of 0 for $u=\to\infty$ and $v=1$.

REMARK 5. Let $f:[0,1]\to\mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ for some $n\geqslant 3$. Then from Corollary 1 for $u/2\leqslant v<2u$ we get

$$K(2k-1,1;u,v)=\frac{1}{(2u-v)\,[(2k-1)!]}\max_{t\in[0,1]}|F_{2k-1}(t)|,$$

$$K^*(2k,1;u,v)=\frac{2v}{(2u-v)\,[(2k)!]}(1-2^{-2k})\,|B_{2k}|$$

and

$$K(2k,1;u,v)=\frac{1}{(2u-v)\,[(2k)!]}\left[v+u\cdot 2^{1-2k}(1-2^{1-2k})\right]|B_{2k}|$$

Now, we calculate the optimal constant for $p=2$.

COROLLARY 5. Let $\left|f^{(n)}\right|^2:[0,1]\to\mathbf{R}$ be a R-integrable function for some $n\geqslant 1$. Then, we have

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v) + \tilde{T}_{n-1}(u,v)\right| \leqslant \frac{1}{(2u-v)}\left[\frac{(-1)^{n-1}}{(2n)!}\left[2u^2+v^2-(2u^2-uv\cdot 2^{2-2n})(1-2^{1-2n})\right]B_{2n} + \frac{\tilde{B}_n^2}{(n!)^2}\right]^{1/2}\|f^{(n)}\|_2,$$

and

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v) + \tilde{T}_n(u,v)\right| \leqslant \frac{1}{(2u-v)}\left[\frac{(-1)^{n-1}}{(2n)!}\left[2u^2+v^2-(2u^2-uv\cdot 2^{2-2n})(1-2^{1-2n})\right]B_{2n}\right]^{1/2}\|f^{(n)}\|_2.$$

Proof. Using integration by part and also using Lemma 1 from [3] we have

$$\begin{aligned}\int_0^1 G_n^2(t)dt &= (-1)^{n-1}\frac{n(n-1)\dots 2}{(n+1)(n+2)\dots(2n-1)}\times\\ &\qquad\times\left[-\frac{1}{2n}G_{2n}(t)G_1(t)|_0^1+\frac{1}{2n}\int_0^1 G_{2n}(t)dG_1(t)\right]\\ &= (-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[(v-2u)\int_0^1 G_{2n}(t)dt+2uG_{2n}\left(\frac{1}{4}\right)-vG_{2n}\left(\frac{1}{2}\right)\right]\\ &= (-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[-4uvB_{2n}\left(\frac{1}{4}\right)+2u^2B_{2n}\left(\frac{1}{2}\right)+(2u^2+v^2)B_{2n}\right]\\ &= (-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[2u^2+v^2-(2u^2-uv\cdot 2^{2-2n})(1-2^{1-2n})\right]B_{2n}.\end{aligned}$$

Now,

$$\begin{aligned}\int_0^1 F_n^2(t)dt &= \int_0^1\left[G_n(t)-\tilde{B}_n\right]^2dt\\ &= \int_0^1\left[G_n^2(t)-2G_n(t)\tilde{B}_n+\tilde{B}_n^2\right]dt = \int_0^1 G_n^2(t)dt+\tilde{B}_n^2\\ &= (-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[2u^2+v^2-(2u^2-uv\cdot 2^{2-2n})(1-2^{1-2n})\right]B_{2n}+\tilde{B}_n^2.\end{aligned}$$

Finally, we give the values of optimal constant for $n=1$ and arbitrary p from Theorem 2.

REMARK 6. Note that $K^*(1,p;u,v) = K(1,p;u,v)$, for $1 < p \leqslant \infty$, since $G_1(t) = F_1(t)$. Also, for $1 < p \leqslant \infty$ we can easily calculate $K(1,p;u,v)$. We get

$$K(1,p;u,v) = \frac{1}{(2u-v)}\left[\frac{(2u-v)^{q+1} + (2u+v)^{q+1} - 2^{q+1}v^{q+1}}{(2u-v)(q+1)2^{2q+1}}\right]^{\frac{1}{q}}, \ 1 < p \leqslant \infty.$$

Now we use the formula (2.2) and one technical result from [11] to obtain Grüss type inequality related to that general dual Euler-Simpson formula:

THEOREM 3. *Suppose that $f:[0,1] \to \mathbf{R}$ is such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n \geqslant 1$. Assume that*

$$m_n \leqslant f^{(n)}(t) \leqslant M_n, \ 0 \leqslant t \leqslant 1,$$

for some constants m_n and M_n. Then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v) + \tilde{T}_n(u,v)\right| \leqslant C_n(M_n - m_n), \tag{3.5}$$

where $C_n = \frac{1}{(2u-v)(n!)}\int_0^1 |G_n(t)|dt$.

Our final results are connected with the series expansion of a function in Bernoulli polynomials.

THEOREM 4. *If $f:[0,1] \to \mathbf{R}$ is such that $f^{(2k)}$ is a continuous function on $[0,1]$, for some $k \geqslant 2$, then for $u/2 \leqslant v < 2u$ there exists a point $\eta \in [0,1]$ such that*

$$\tilde{R}^2_{2k}(f) = -\frac{(v - u\cdot 2^{1-2k})(1-2^{1-2k})B_{2k}}{(2u-v)[(2k)!]} f^{(2k)}(\eta). \tag{3.6}$$

Proof. We can rewrite $\tilde{R}^2_{2k}(f)$ as $\tilde{R}^2_{2k}(f) = (-1)^k \frac{J_k}{2[(2k)!]}$, where $J_k = \int_0^1 (-1)^k F^x_{2k}(s) f^{(2k)}(s)\mathrm{d}s$. From Corollary 1 follows that $(-1)^k F^x_{2k}(s) \geqslant 0, \ 0 \leqslant s \leqslant 1$ and the claim follows from the mean value theorem for integrals and Corollary 2.

REMARK 7. For $k = 2$ formula (3.6) reduces to

$$\tilde{R}^2_4(f) = \frac{7(8v-u)}{46080(2u-v)} f^{(4)}(\eta).$$

4. General dual Euler-Simpson formulae with nonsymmetric coefficients

In this section we study, the general Simpson quadrature formula

$$\int_0^1 f(t)dt = \frac{1}{u+-v+w}\left[uf\left(\frac{1}{4}\right) - vf\left(\frac{1}{2}\right) + wf\left(\frac{3}{4}\right)\right] + E(f;u,v,w) \tag{4.1}$$

with $E(f;u,v,w)$ being the remainder, $u,v,w \in \mathbf{Z}^+$ and $u + w > v$. We are using identities (1.1) and (1.2) to get two new identities of Euler type.

THEOREM 5. *Let $f:[0,1]\to\mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n\geqslant 1$. Then*

$$\int_0^1 f(t)\mathrm{d}t = D(u,v,w) - \overline{T}_n(u,v,w) + \overline{R}_n^1(f), \tag{4.2}$$

and

$$\int_0^1 f(t)\mathrm{d}t = D(u,v,w) - \overline{T}_{n-1}(u,v,w) + \overline{R}_n^2(f), \tag{4.3}$$

where

$$D(u,v,w) = \frac{1}{u-v+w}\left[uf\left(\frac{1}{4}\right) - vf\left(\frac{1}{2}\right) + wf\left(\frac{3}{4}\right)\right],$$

$$\overline{R}_n^1(f) = \frac{1}{(u-v+w)(n!)}\int_0^1 \overline{G}_n(t)\,\mathrm{d}f^{(n-1)}(t),$$

$$\overline{R}_n^2(f) = \frac{1}{(u-v+w)(n!)}\int_0^1 \overline{F}_n(t)\,\mathrm{d}f^{(n-1)}(t),$$

$$\overline{G}_k(t) = uB_k^*\left(\frac{1}{4}-t\right) - vB_k^*\left(\frac{1}{2}-t\right) + wB_k^*\left(\frac{3}{4}-t\right),\ \ t\in\mathbf{R},$$

$$\overline{F}_k(t) = \overline{G}_k(t) - \overline{B}_k,\ \ t\in\mathbf{R},\ \ k\geqslant 1,$$

$$\overline{B}_k = uB_k\left(\frac{1}{4}\right) - vB_k\left(\frac{1}{2}\right) + wB_k\left(\frac{3}{4}\right),\ \ k\geqslant 1$$

and

$$\overline{T}_m(u,v,w) = \frac{1}{u-v+w}\left[uT_m\left(\frac{1}{4}\right) - vT_m\left(\frac{1}{2}\right) + wT_m\left(\frac{3}{4}\right)\right].$$

Proof. Put $x=1/4,\ 1/2,\ 3/4$ in formula (1.1) to get three new formulae. Then multiply these new formulae by $u,\ -v,\ w$ respectively, and add. The result is formula (4.2). Formula (4.3) is obtained from (1.2) by the same procedure.

THEOREM 6. *Assume (p_1,q_1) and (p_2,q_2) are two pairs of conjugate exponents, $1\leqslant p_1,q_1,p_2,q_2\leqslant\infty$. Let $|f^{(n)}|^{p_1}:[0,x]\to\mathbf{R}$ and $|f^{(n)}|^{p_2}:[x,1]\to\mathbf{R}$ are R-integrable functions for some $n\geqslant 1$. Then, we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v,w) + \overline{T}_{n-1}(u,v,w)\right| \leqslant K(n,p_1;u,v,w,x)\cdot\|f^{(n)}\|_{L_{p_1}[0,x]} + K(n,p_2;u,v,w,x)\cdot\|f^{(n)}\|_{L_{p_2}[x,1]}, \tag{4.4}$$

and

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v,w) + \overline{T}_n(u,v,w)\right| \leqslant K^*(n,p_1;u,v,w,x)\cdot\|f^{(n)}\|_{L_{p_1}[0,x]} + K^*(n,p_2;u,v,w,x)\cdot\|f^{(n)}\|_{L_{p_2}[x,1]}, \tag{4.5}$$

where

$$K(n,p_1;u,v,w,x)=\frac{1}{(u-v+w)(n!)}\left[\int_0^x |\bar{F}_n(t)|^{q_1}\,\mathrm{d}t\right]^{1/q_1},$$

$$K(n,p_2;u,v,w,x)=\frac{1}{(u-v+w)(n!)}\left[\int_x^1 |\bar{F}_n(t)|^{q_2}\,\mathrm{d}t\right]^{1/q_2},$$

$$K^*(n,p_1;u,v,w,x)=\frac{1}{(u-v+w)(n!)}\left[\int_0^x \left|\bar{G}_n(t)\right|^{q_1}\mathrm{d}t\right]^{1/q_1} \quad \textit{and}$$

$$K^*(n,p_2;u,v,w,x)=\frac{1}{(u-v+w)(n!)}\left[\int_x^1 \left|\bar{G}_n(t)\right|^{q_2}\mathrm{d}t\right]^{1/q_2}.$$

The constants $K(n,p_1;u,v,w,x)$, $K(n,p_2;u,v,w,x)$, $K^*(n,p_1;u,v,w,x)$ *and* $K^*(n,p_2;u,v,w,x)$ *are sharp for* $1<p_1,p_2\leqslant\infty$ *and the best possible for* $p_1=1$ *or* $p_2=1$.

Proof. Applying the Hölder inequality we have

$$\begin{aligned}
&\left|\frac{1}{(u-v+w)(n!)}\int_0^1 \bar{F}_n(t)f^{(n)}(t)dt\right|\\
&=\left|\frac{1}{(u-v+w)(n!)}\int_0^x \bar{F}_n(t)f^{(n)}(t)dt+\frac{1}{(u-v+w)(n!)}\int_x^1 \bar{F}_n(t)f^{(n)}(t)dt\right|\\
&\leqslant \frac{1}{(u-v+w)(n!)}\left\{\left[\int_0^x |\bar{F}_n(t)|^{q_1}dt\right]^{1/q_1}\|f^{(n)}\|_{L_{p_1}[0,x]}\right.\\
&\quad+\left.\left[\int_x^1 |\bar{F}_n(t)|^{q_2}dt\right]^{1/q_2}\|f^{(n)}\|_{L_{p_2}[x,1]}\right\}\\
&=K(n,p_1;u,v,w,x)\|f^{(n)}\|_{L_{p_1}[0,x]}+K(n,p_2;u,v,w,x)\|f^{(n)}\|_{L_{p_2}[x,1]}.
\end{aligned}$$

Using the above inequality from (2.3) we get estimate (4.4). In the same manner, from (2.2) we get estimate (4.5). The proof of sharpness and best possibility is similar as in the proof of Theorem 2.

REMARK 8. For $n=1$, $\frac{1}{4}\leqslant u\leqslant w\leqslant\frac{1}{2}$ and $u-v+w=1$ in inequality (4.4) we get inequality

$$\begin{aligned}
&\left|\int_0^1 f(t)\mathrm{d}t-D(u,v,w)\right|\\
&\leqslant\left[\frac{(w-u)^{q_1+1}+(u-w+1)^{q_1+1}+(3u+w-1)^{q_1+1}+(2-3u-w)^{q_1+1}}{4^{q_1+1}(q_1+1)}\right]^{1/q_1}\\
&\times\|f'\|_{L_{p_1}[0,1/2]}\\
&+\left[\frac{-(w-u)^{q_2+1}+(w-u+1)^{q_2+1}+(3w+u-1)^{q_2+1}+(2-u-3w)^{q_2+1}}{4^{q_2+1}(q_2+1)}\right]^{1/q_2}\\
&\times\|f'\|_{L_{p_2}[1/2,1]}.
\end{aligned}$$

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(Received June 28, 2008)

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6
10000 Zagreb, Croatia
e-mail: pecaric@mahazu.hazu.hr

A. Vukelić
Faculty of Food Technology and Biotechnology
Mathematics department
University of Zagreb
Pierottijeva 6
10000 Zagreb, Croatia
e-mail: avukelic@pbf.hr