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SOME SUFFICIENT CONDITIONS FOR CERTAIN INTEGRAL OPERATORS

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Abstract. In this paper, we consider some sufficient conditions for two integral operators to be starlike, close-to-convex and uniformly convex functions defined in the open unit disk.

1. Introduction and definitions

Let $\mathscr{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathscr{U} = \{z : |z| < 1\}$. Also let $\mathscr{S}^*(\alpha)$, $\mathscr{K}(\alpha)$, $\mathscr{C}(\alpha)$, and $\mathscr{UCV}$ denote the subclasses of $\mathscr{A}$ consisting of functions which are, respectively, starlike, convex, close -to-convex of order $\alpha (0 \leqslant \alpha < 1)$ in $\mathscr{U}$ and uniformly convex function . Thus, we have (see, for details, [4, 6]; see also [10, 11])

$$\mathscr{S}^*(\alpha) := \left\{ f : f \in \mathscr{A} \text{ and } \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathscr{U}; 0 \leqslant \alpha < 1) \right\} \tag{1.2}$$

$$\mathscr{K}(\alpha) := \left\{ f : f \in \mathscr{A} \text{ and } \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathscr{U}; 0 \leqslant \alpha < 1) \right\} \tag{1.3}$$

$$\mathscr{C}(\alpha) := \left\{ f : f \in \mathscr{A} \text{ and } \operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > \alpha \quad (z \in \mathscr{U}; 0 \leqslant \alpha < 1; g \in \mathscr{K}) \right\} \tag{1.4}$$

and

$$\mathscr{UCV} := \left\{ f : f \in \mathscr{A} \text{ and } \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in \mathscr{U}). \right\} \tag{1.5}$$

Recently, Breaz and Breaz in [1] introduced and studied the integral operator

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$$F_n(z)=\int_0^z\left(\frac{f_1(t)}{t}\right)^{\alpha_1}\cdots\left(\frac{f_n(t)}{t}\right)^{\alpha_n}dt \tag{1.6}$$

and the integral operator

$$F_{\alpha_1,\ldots,\alpha_n}(z)=\int_0^z\left(f_1'(t)\right)^{\alpha_1}\cdots\left(f_n'(t)\right)^{\alpha_n}dt \tag{1.7}$$

for $\alpha_i>0,$ was introduced by Breaz et al. [3].

In [2], Breaz and Güney considered the above integral operators and they obtained their properties on the classes $\mathscr{S}_{\alpha}^{*}(b)$, $\mathscr{C}_{\alpha}(b)$ of starlike and convex functions of complex order b and type α introduced and studied by the author [5].

In the present paper, we obtain some sufficient conditions for the above integral operators $F_n(z)$ and $F_{\alpha_1,\ldots,\alpha_n}(z)$ to be in the classes , $\mathscr{S}^*(0)=\mathscr{S}^*$, $\mathscr{C}(\alpha)$ and $\mathscr{UCV}$.

In order to derive our main results, we have to recall here the following results:

LEMMA 1.1. ([12]) *If $f\in\mathscr{A}$ satisfies*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}<\frac{3}{2}\qquad(z\in\mathscr{U})$$

then $f\in\mathscr{S}^*$.

LEMMA 1.2. ([7]) *If $f\in\mathscr{A}$ satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}\right|<2\qquad(z\in\mathscr{U})$$

then $f\in\mathscr{S}^*$.

LEMMA 1.3. ([9]) *If $f\in\mathscr{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right|<\frac{1}{2}\qquad(z\in\mathscr{U})$$

then $f\in\mathscr{UCV}$.

LEMMA 1.4. ([8]) *If $f\in\mathscr{A}$ satisfies*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\frac{1+3\alpha}{2(1+\alpha)},\qquad(z\in\mathscr{U};\,0\leqslant\alpha<1).$$

Then

$$\operatorname{Re}\{f'(z)\}>\frac{1+\alpha}{2},\qquad(z\in\mathscr{U};\,0\leqslant\alpha<1),$$

or equivalently,

$$f\in\mathscr{C}\left(\frac{1+\alpha}{2}\right),\qquad(0\leqslant\alpha<1).$$

LEMMA 1.5. ([8]) *If* $f \in \mathscr{A}$ *satisfies*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{3+2\alpha}{2+\alpha}, \qquad (z \in \mathscr{U};\, 0 \leqslant \alpha < 1).$$

Then

$$\operatorname{Re}\{f'(z)\} > -\alpha, \qquad (z \in \mathscr{U};\, 0 \leqslant \alpha < 1).$$

2. Sufficient conditions for the integral operator F_n

Applying Lemma 1.1, we derive

THEOREM 2.1. *Let* $\alpha_i,\ i \in \{1,\dots,n\}$ *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1,\dots,n\}$. *If* $f_i \in \mathscr{A}$ *for* $i \in \{1,\dots,n\}$ *satisfies*

$$\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) < 1 + \frac{1}{2\sum\limits_{i=1}^{n}\alpha_i} \tag{2.1}$$

then $F_n \in \mathscr{S}^*$.

Proof. From (1.6), we have

$$F_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdots \left(\frac{f_n(z)}{z}\right)^{\alpha_n} \tag{2.2}$$

and

$$F_n''(z) = \alpha_1\left(\frac{f_1'(z)}{f_1(z)} - \frac{1}{z}\right) + \dots + \alpha_n\left(\frac{f_n'(z)}{f_n(z)} - \frac{1}{z}\right)F_n'(z) \tag{2.3}$$

Then from (2.2) and (2.3) , we obtain

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)$$

or, equivalently,

$$1 + \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}\right) + 1 - \sum_{i=1}^{n}\alpha_i. \tag{2.4}$$

Taking the real part of both terms of (2.4), we have

$$\operatorname{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) = \sum_{i=1}^{n}\alpha_i\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) + 1 - \sum_{i=1}^{n}\alpha_i. \tag{2.5}$$

Using (2.5) and (2.1), we obtain

$$\operatorname{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) < \frac{3}{2}.$$

Hence by Lemma 1.1, we get $F_n \in \mathscr{S}^*$. This completes the proof.
Letting $n = 1$ in Theorem 2.1, we have

COROLLARY 2.2. *Let* $\alpha_1 > 0$. *If* $f_1 \in \mathscr{A}$ *and*

$$\operatorname{Re}\left(\frac{zf_1'(z)}{f_1(z)}\right) < 1 + \frac{1}{2\alpha_1} \tag{2.6}$$

then $F_1 \in \mathscr{S}^*$.

THEOREM 2.3. *Let* $\alpha_i,\ i \in \{1,\ldots,n\}$ *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1,\ldots,n\}$. *If* $f_i \in \mathscr{A}$ *for* $i \in \{1,\ldots,n\}$ *satisfies*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| < \frac{1}{\sum\limits_{i=1}^{n} \alpha_i} \tag{2.7}$$

then $F_n \in \mathscr{S}^*$.

Proof. From (2.4) we have

$$1 + \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + 1 \tag{2.8}$$

and hence

$$\left|1 + \frac{zF_n''(z)}{F_n'(z)}\right| \leqslant \sum_{i=1}^{n} |\alpha_i| \left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + 1 \tag{2.9}$$

Using (2.9), (2.7) and applying Lemma 1.2, we get $F_n \in \mathscr{S}^*$. Letting $n = 1$ in Theorem 2.3, we have

COROLLARY 2.4. *Let* $\alpha_1 > 0$. *If* $f_1 \in \mathscr{A}$ *and*

$$\left|\frac{zf_1'(z)}{f_1(z)} - 1\right| < \frac{1}{\alpha_1} \tag{2.10}$$

then $F_1 \in \mathscr{S}^*$.

Applying Lemma 1.3 and using (2.8), we easily get

THEOREM 2.5. *Let* $\alpha_i,\ i \in \{1,\ldots,n\}$ *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1,\ldots,n\}$. *If* $f_i \in \mathscr{A}$ *for* $i \in \{1,\ldots,n\}$ *satisfies*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| < \frac{1}{2\sum\limits_{i=1}^{n} \alpha_i} \tag{2.11}$$

then $F_n \in \mathscr{UCV}$.

Letting $n=1$ in Theorem 2.5, we have

COROLLARY 2.6. *Let* $\alpha_1>0$. *If* $f_1\in\mathscr{A}$ *and*

$$\left|\frac{zf_1'(z)}{f_1(z)}-1\right|<\frac{1}{2\alpha_1} \tag{2.12}$$

then $F_1\in\mathscr{UCV}$.

THEOREM 2.7. *Let* $\alpha_i,\ i\in\{1,\dots,n\}$ *be real numbers with the properties* $\alpha_i>0$ *for* $i\in\{1,\dots,n\}$. *If* $f_i\in\mathscr{A}$ *for* $i\in\{1,\dots,n\}$ *satisfies*

$$Re\left(\frac{zf_i'(z)}{f_i(z)}\right)>\frac{-1+\alpha+2(1+\alpha)\sum_{i=1}^{n}\alpha_i}{2(1+\alpha)\sum_{i=1}^{n}\alpha_i} \tag{2.13}$$

then $F_n\in\mathscr{C}\left(\frac{1+\alpha}{2}\right)$, *where* $0\leqslant\alpha<1$.

Proof. As in the proof of Theorem 2.1 . From (2.5) and (2.13), we obtain

$$\text{Re}\left(1+\frac{zF_n''(z)}{F_n'(z)}\right)>\frac{1+3\alpha}{2(1+\alpha)}$$

Hence by Lemma 1.4, we get $F_n\in\mathscr{C}\left(\frac{1+\alpha}{2}\right)$, $\quad(0\leqslant\alpha<1)$. Letting $n=1$ in Theorem 2.7, we have

COROLLARY 2.8. *Let* $\alpha_1>0$. *If* $f_1\in\mathscr{A}$ *and*

$$\text{Re}\left(\frac{zf_1'(z)}{f_1(z)}\right)>\frac{-1+\alpha+2(1+\alpha)\alpha_1}{2(1+\alpha)\alpha_1} \tag{2.14}$$

then $F_1\in\mathscr{C}\left(\frac{1+\alpha}{2}\right)$, *where* $0\leqslant\alpha<1$.

Using Lemma 1.5 and applying similar proof as in Theorem 2.7, we obtain

THEOREM 2.9. *Let* $\alpha_i,\ i\in\{1,\dots,n\}$ *be real numbers with the properties* $\alpha_i>0$ *for* $i\in\{1,\dots,n\}$. *If* $f_i\in\mathscr{A}$ *for* $i\in\{1,\dots,n\}$ *satisfies*

$$\text{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right)<\frac{1+\alpha+(2+\alpha)\sum_{i=1}^{n}\alpha_i}{(2+\alpha)\sum_{i=1}^{n}\alpha_i} \tag{2.15}$$

then

$$\text{Re}\{F_n'(z)\}>-\alpha,\qquad(z\in\mathscr{U};\,0\leqslant\alpha<1) \tag{2.16}$$

Letting $n = 1$ in Theorem 2.9, we have

COROLLARY 2.10. *Let* $\alpha_1 > 0$. *If* $f_1 \in \mathscr{A}$ *and*

$$\operatorname{Re}\left(\frac{zf_1'(z)}{f_1(z)}\right) < \frac{1+\alpha+(2+\alpha)\alpha_1}{(2+\alpha)\alpha_1} \tag{2.17}$$

$$\operatorname{Re}\{F_1'(z)\} > -\alpha, \qquad (z \in \mathscr{U};\ 0 \leqslant \alpha < 1) \tag{2.18}$$

3. Sufficient conditions for the integral operator $F_{\alpha_1,\ldots,\alpha_n}$

Applying Lemma 1.1 , we derive

THEOREM 3.1. *Let* α_i, $i \in \{1,\ldots,n\}$ *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1,\ldots,n\}$. *If* $f_i \in \mathscr{A}$ *for* $i \in \{1,\ldots,n\}$ *satisfies*

$$\operatorname{Re}\left(1+\frac{zf_i''(z)}{f_i'(z)}\right) < 1+\frac{1}{2\sum_{i=1}^{n}\alpha_i} \tag{3.1}$$

then $F_{\alpha_1,\ldots,\alpha_n} \in \mathscr{S}^*$.

Proof. As in the proof of Theorem 2.1, we have

$$\frac{F''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)} = \alpha_1\frac{f_1''(z)}{f_1'(z)}+\ldots+\alpha_n\frac{f_n''(z)}{f_n'(z)} \tag{3.2}$$

or, equivalently,

$$\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)}\right). \tag{3.3}$$

From (3.3) we have,

$$\operatorname{Re}\left(1+\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)}\right) = \sum_{i=1}^{n}\alpha_i\operatorname{Re}\left(1+\frac{zf_i''(z)}{f_i'(z)}\right)+1-\sum_{i=1}^{n}\alpha_i \tag{3.4}$$

Using (3.4) and (3.1), we obtain

$$\operatorname{Re}\left(1+\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)}\right) < \frac{3}{2}.$$

Hence by Lemma 1.1, we get $F_{\alpha_1,\ldots,\alpha_n} \in \mathscr{S}^*$.

Using (3.3) , (3.4) and Lemmas 1.2, 1.3, 1.4 and 1.5, we have, respectively

THEOREM 3.2. *Let* α_i, $i \in \{1,\ldots,n\}$ *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1,\ldots,n\}$. *If* $f_i \in \mathscr{A}$ *for* $i \in \{1,\ldots,n\}$ *satisfies*

$$\left|1+\frac{zf_i''(z)}{f_i'(z)}\right| < \frac{1}{\sum_{i=1}^{n}\alpha_i}+1 \tag{3.5}$$

then $F_{\alpha_1,\ldots,\alpha_n} \in \mathscr{S}^*$.

THEOREM 3.3. *Let* α_i, $i \in \{1,\ldots,n\}$ *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1,\ldots,n\}$. *If* $f_i \in \mathscr{A}$ *for* $i \in \{1,\ldots,n\}$ *satisfies*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| < \frac{1}{2\sum_{i=1}^{n}\alpha_i} \tag{3.6}$$

then $F_{\alpha_1,\ldots,\alpha_n} \in \mathscr{UCV}$.

THEOREM 3.4. *Let* α_i, $i \in \{1,\ldots,n\}$ *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1,\ldots,n\}$. *If* $f_i \in \mathscr{A}$ *for* $i \in \{1,\ldots,n\}$ *satisfies*

$$\operatorname{Re}\left(1+\frac{zf_i''(z)}{f_i'(z)}\right) > \frac{-1+\alpha+2(1+\alpha)\sum_{i=1}^{n}\alpha_i}{2(1+\alpha)\sum_{i=1}^{n}\alpha_i} \tag{3.7}$$

then $F_{\alpha_1,\ldots,\alpha_n} \in \mathscr{C}\left(\frac{1+\alpha}{2}\right)$, *where* $0 \leqslant \alpha < 1$.

THEOREM 3.5. *Let* α_i, $i \in \{1,\ldots,n\}$ *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1,\ldots,n\}$. *If* $f_i \in \mathscr{A}$ *for* $i \in \{1,\ldots,n\}$ *satisfies*

$$\operatorname{Re}\left(1+\frac{zf_i''(z)}{f_i'(z)}\right) < \frac{1+\alpha+(2+\alpha)\sum_{i=1}^{n}\alpha_i}{(2+\alpha)\sum_{i=1}^{n}\alpha_i} \tag{3.8}$$

then

$$\operatorname{Re}\{F'_{\alpha_1,\ldots,\alpha_n}(z)\} > -\alpha, \qquad (z \in \mathscr{U};\ 0 \leqslant \alpha < 1) \tag{3.9}$$

Letting $n = 1$ in Theorems 3.1-3.5, we have, respectively:

COROLLARY 3.6. *Let* $\alpha_1 > 0$. *If* $f_1 \in \mathscr{A}$ *and*

$$\operatorname{Re}\left(1+\frac{zf_1''(z)}{f_1'(z)}\right) < 1+\frac{1}{2\alpha_1} \tag{3.10}$$

then $F_{\alpha_1} \in \mathscr{S}^*$.

COROLLARY 3.7. *Let* $\alpha_1 > 0$. *If* $f_1 \in \mathscr{A}$ *and*

$$\left|1+\frac{zf_1''(z)}{f_1'(z)}\right| < \frac{1}{\alpha_1}+1 \tag{3.11}$$

then $F_{\alpha_1} \in \mathscr{S}^*$.

COROLLARY 3.8. *Let* $\alpha_1 > 0$. *If* $f_1 \in \mathscr{A}$ *and*

$$\left|\frac{zf_1''(z)}{f_1'(z)}\right| < \frac{1}{2\alpha_1} \tag{3.12}$$

then $F_{\alpha_1} \in \mathscr{UCV}$.

COROLLARY 3.9. *Let* $\alpha_1 > 0$. *If* $f_1 \in \mathscr{A}$ *and*

$$\operatorname{Re}\left(1+\frac{zf_1''(z)}{f_1'(z)}\right) > \frac{-1+\alpha+2(1+\alpha)\alpha_1}{2(1+\alpha)\alpha_1} \tag{3.13}$$

then $F_{\alpha_1} \in \mathscr{C}\left(\frac{1+\alpha}{2}\right)$, *where* $0 \leqslant \alpha < 1$.

COROLLARY 3.10. *Let* $\alpha_1 > 0$. *If* $f_1 \in \mathscr{A}$ *and*

$$\operatorname{Re}\left(1+\frac{zf_1''(z)}{f_1'(z)}\right) < \frac{1+\alpha+(2+\alpha)\alpha_1}{(2+\alpha)\alpha_1} \tag{3.14}$$

then

$$\operatorname{Re}\{F_{\alpha_1}'(z)\} > -\alpha, \qquad (z \in \mathscr{U};\ 0 \leqslant \alpha < 1). \tag{3.15}$$

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