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INEQUALITIES INVOLVING CERTAIN INTEGRAL OPERATORS

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Abstract. Two integral operators I_p^α $(\alpha > 0;\ p \in N)$ and $Q_{\beta,p}^\alpha$ $(\alpha > 0;\ \beta > -1;\ p \in N)$, where $N = \{1,2,....\}$, are introduced for functions of the form $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. The object of the present paper is to give an applications of the above operators to the differential inequalities.

1. Introduction

Let $A(p)$ denote the class of functions of the form :

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N = \{1,2,....\}) \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. In 1993 Jung et al. [2] introduced the following one-parameter families of integral operators :

$$I^\alpha f(z) = \frac{2^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)dt \qquad (\alpha > 0;\ f \in A(1)) \tag{1.2}$$

and

$$Q_\beta^\alpha f(z) = \binom{\alpha+\beta}{\beta} \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)dt$$
$$(\alpha > 0;\ \beta > -1;\ f \in A(1)). \tag{1.3}$$

They [2] showed that

$$I^\alpha f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\alpha a_n z^n \ , \tag{1.4}$$

and

$$Q_\beta^\alpha f(z) = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(p+n)}{\Gamma(\alpha+\beta+n)} a_n z^n. \tag{1.5}$$

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Motivated essentially by Jung et al. [2], Liu and Owa [3] generalized the operator Q_{β}^{α} as $Q_{\beta,p}^{\alpha}: A(p) \to A(p)$ defined as follows:

$$Q_{\beta,p}^{\alpha} f(z) = \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^{\beta}} \int_{0}^{z} (1-\frac{t}{z})^{\alpha-1} t^{\beta-1} f(t) dt$$

$$(\alpha > 0;\ \beta > -1;\ p \in N;\ f \in A(p)). \quad (1.6)$$

For $f(z) \in A(p)$ given by (1.1), Liu and Owa [3] have shown that

$$Q_{\beta,p}^{\alpha} f(z) = z^{p} + \frac{\Gamma(\alpha+\beta+p)}{\Gamma(\beta+p)} \sum_{n=1}^{\infty} \frac{\Gamma(\beta+p+n)}{\Gamma(\alpha+\beta+p+n)} a_{p+n} z^{p+n}$$

$$(\alpha > 0;\ \beta > -1;\ p \in N;\ f \in A(p)). \quad (1.7)$$

It is easily verified from the definition (1.7) that (see [3])

$$z(Q_{\beta,p}^{\alpha} f(z))' = (\alpha+\beta+p-1) Q_{\beta,p}^{\alpha-1} f(z) - (\alpha+\beta-1) Q_{\beta,p}^{\alpha} f(z). \quad (1.8)$$

We note that $Q_{\beta,1}^{\alpha} = Q_{\beta}^{\alpha}$.

Also Shams et al. [5] generalized the operator I^{α} as $I_{p}^{\alpha}: A(p) \to A(p)$, defined as follows :

$$I_{p}^{\alpha} f(z) = \frac{(p+1)^{\alpha}}{z\Gamma(\alpha)} \int_{0}^{z} \left(\log \frac{z}{t}\right)^{\alpha-1} f(t) dt \qquad (\alpha > 0;\ p \in N;\ f \in A(p)). \quad (1.9)$$

For $f(z) \in A(p)$ given by (1.1), Shams et al. [5] have shown that

$$I_{p}^{\alpha} f(z) = z^{p} + \sum_{n=1}^{\infty} \left(\frac{p+1}{p+n+1}\right)^{\alpha} a_{p+n} z^{p+n} \qquad (\alpha > 0;\ p \in N). \quad (1.10)$$

It is easily verified from the definition (1.10) that (see [5])

$$z(I_{p}^{\alpha} f(z))' = (p+1) I_{p}^{\alpha-1} f(z) - I_{p}^{\alpha} f(z). \quad (1.11)$$

Also we note that $I_{1}^{\alpha} = I^{\alpha}$.

By using the operators I_{p}^{α} and $Q_{\beta,p}^{\alpha}$ we define the following classes of functions :

The Class Φ: Let Φ be the set of complex- valued functions $\phi(r,s,t)$;

$$\phi(r,s,t): C^{3} \to C \qquad (C \text{ is complex plane})$$

such that

(*i*) $\phi(r,s,t)$ is continuous in a domain $D \subset C^{3}$;
(*ii*) $(0,0,0) \in D$ and $|\phi(0,0,0)| < 1$;
(*iii*) $\left|\phi(e^{i\theta}, (\frac{\zeta+1}{p+1})e^{i\theta}, \frac{(1+3\zeta)e^{i\theta}+M}{(p+1)^{2}})\right| > 1$

whenever $\left(e^{i\theta},(\frac{\zeta+1}{p+1})e^{i\theta},\ \frac{(1+3\zeta)e^{i\theta}+M}{(p+1)^2}\right)\in D$ with $\operatorname{Re}\{e^{-i\theta}M\}\geqslant\zeta(\zeta-1)$, for all $\theta\in R$, and for all $\zeta\geqslant p\geqslant 1$.

The Class Ψ : Let Ψ be the set of complex-valued functions $\psi(r,s,t)$;

$$\psi(r,s,t):C^3\to C$$

such that

- (i) $\psi(r,s,t)$ is continuous in a domain $D\subset C^3$;
- (ii) $(0,0,0)\in D$ and $|\psi(0,0,0)|<1$;
- (iii)

$$\left|\psi\left(e^{i\theta},\frac{\zeta-1+\alpha+\beta}{\alpha+\beta+p-1}e^{i\theta},\ \frac{(\alpha+\beta-1)[2(\zeta-1)+\alpha+\beta]e^{i\theta}+L}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)}\right)\right|>1$$

whenever

$$\left(e^{i\theta},\frac{\zeta-1+\alpha+\beta}{\alpha+\beta+p-1}e^{i\theta},\ \frac{(\alpha+\beta-1)[2(\zeta-1)+\alpha+\beta]e^{i\theta}+L}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)}\right)\in D$$

with $\operatorname{Re}\{e^{-i\theta}L\}\geqslant\zeta(\zeta-1)$ for all $\theta\in R$, and for real $\zeta\geqslant p\geqslant 1$.

The Class H : Let H be the set of complex - valued functions $h(r,s,t)$;

$$h(r,s,t):C^3\to C$$

such that

- (i) $h(r,s,t)$ is continuous in a domain $D\subset C^3$;
- (ii) $(1,1,1)\in D$ and $|h(1,1,1)|<J\ (J>1)$;
- (iii)

$$\left|h\left(Je^{i\theta},\frac{\zeta+(p+1)Je^{i\theta}}{p+1},\frac{1}{p+1}\left\{\zeta+(p+1)Je^{i\theta}+\frac{\zeta-\zeta^2+(p+1)\zeta Je^{i\theta}+L}{\zeta+(p+1)Je^{i\theta}}\right\}\right)\right|\geqslant J,$$

whenever

$$\left(Je^{i\theta},\frac{\zeta+(p+1)Je^{i\theta}}{p+1},\frac{1}{p+1}\left\{\zeta+(p+1)Je^{i\theta}+\frac{\zeta-\zeta^2+(p+1)\zeta Je^{i\theta}+L}{\zeta+(p+1)Je^{i\theta}}\right\}\right)\in D$$

with $Re\{L\}\geqslant\zeta(\zeta-1)$ for all $\theta\in R$ and for all $\zeta\geqslant\dfrac{J-1}{J+1}$.

The Class G : Let G be the set of complex-valued functions $g(r,s,t)$;

$$g(r,s,t):C^3\to C$$

such that

- (i) $g(r,s,t)$ is continuous in a domain $D\subset C^3$;
- (ii) $(1,1,1)\in D$ and $|g(1,1,1)|<J\ (J>1)$;

(*iii*)

$$\left| g\left(Je^{i\theta}, \frac{-1+\zeta+(\alpha+\beta+p-1)Je^{i\theta}}{\alpha+\beta+p-2}, \frac{1}{\alpha+\beta+p-3}\left\{ -2+\zeta+(\alpha+\beta+p-1)Je^{i\theta}+ \frac{\zeta-\zeta^2+(\alpha+\beta+p-1)\zeta Je^{i\theta}+M}{-1+\zeta+(\alpha+\beta+p-1)Je^{i\theta}} \right\}\right)\right| \geqslant J,$$

whenever

$$\left(Je^{i\theta}, \frac{-1+\zeta+(\alpha+\beta+p-1)Je^{i\theta}}{\alpha+\beta+p-2}, \frac{1}{\alpha+\beta+p-3}\left\{ -2+\zeta+(\alpha+\beta+p-1)Je^{i\theta}+ \frac{\zeta-\zeta^2+(\alpha+\beta+p-1)\zeta Je^{i\theta}+M}{-1+\zeta+(\alpha+\beta+p-1)Je^{i\theta}} \right\}\right) \in D$$

with $\mathrm{Re}\{M\} \geqslant \zeta(\zeta-1)$ for all $\theta \in R$ and for all $\zeta \geqslant \dfrac{J-1}{J+1}$.

2. Main Results

We recall the following lemmas due to Miller and Mocanu [4].

LEMMA 1. *Let* $w(z) = b_p z^p + b_{p+1} z^{p+1} + \dots$ $(p \in N)$ *be regular in the unit disc* U *with* $w(z) \neq 0$ $(z \in U)$. *If* $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ *and* $|w(z_0)| = \max_{|z| \leqslant |z_0|} |w(z)|$, *then*

(*i*)
$$z_0 w'(z_0) = \zeta w(z_0) \tag{2.1}$$

and

(*ii*)
$$Re\left\{1 + \frac{z_0 w'(z_0)}{w'(z_0)}\right\} \geqslant \zeta \tag{2.2}$$

where ζ *is real and* $\zeta \geqslant p \geqslant 1$.

LEMMA 2. *Let* $w(z) = a + w_k z^k + \dots$ *be regular in* U *with* $w(z) \neq a$ *and* $k \geqslant 1$. *If* $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ *and* $|w(z_0)| = \max_{|z| \leqslant |z_0|} |w(z)|$, *then*

(*i*)
$$z_0 w'(z_0) = \zeta w(z_0)$$

(*ii*)
$$Re\left\{1 + \frac{z_0 w'(z_0)}{w'(z_0)}\right\} \geqslant \zeta$$

where ζ *is a real number and*

$$\zeta \geqslant k\frac{|w(z_0)-a|^2}{|w(z_0)|^2-|a|^2} \geqslant k\frac{|w(z_0)|-|a|}{|w(z_0)|+|a|}. \tag{2.3}$$

THEOREM 1. *Let $\phi(r,s,t) \in \Phi$ and let $f(z)$ belonging to the class $A(p)$ satisfy*

(*i*) $(I_p^{\alpha} f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z)) \in D \subset C^3$

and

(*ii*) $\left|\phi(I_p^{\alpha} f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z))\right| < 1$, *for* $\alpha > 2$, $p \in N$ *and* $z \in U$.

Then we have

$$\left|I_p^{\alpha} f(z)\right| < 1 \quad (z \in U). \tag{2.4}$$

Proof. We define the function $w(z)$ by

$$I_p^{\alpha} f(z) = w(z) \quad (\alpha > 2; p \in N) \tag{2.5}$$

for $f(z)$ belonging to the class $A(p)$. Then, it follows that $w(z) \in A(p)$ and $w(z) \neq 0$ $(z \in U)$. With the aid of the identily (1.11), we have

$$I_p^{\alpha-1} f(z) = \frac{1}{p+1}[w(z) + zw'(z)] \tag{2.6}$$

and

$$I_p^{\alpha-2} f(z) = \frac{1}{(p+1)^2}[w(z) + 3zw'(z) + z^2 w'(z)]. \tag{2.7}$$

Suppose that $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1; \theta \in R)$ and

$$|w(z_0)| = \max_{|z| \leqslant |z_0|} |w(z)| = 1. \tag{2.8}$$

Then, letting $w(z_0) = e^{i\theta}$ and using (2.1) of Lemma 1, we obtain

$$I_p^{\alpha} f(z_0) = w(z_0) = e^{i\theta}, \tag{2.9}$$

$$I_p^{\alpha-1} f(z_0) = \left(\frac{\zeta+1}{p+1}\right) w(z_0) = \left(\frac{\zeta+1}{p+1}\right) e^{i\theta}, \tag{2.10}$$

and

$$I_p^{\alpha-2} f(z_0) = \frac{1}{(p+1)^2}\left[(1+3\zeta)e^{i\theta} + z_0^2 w''(z_0)\right] = \frac{(1+3\zeta)e^{i\theta} + M}{(p+1)^2}, \tag{2.11}$$

where $M = z_0^2 w'(z_0)$ and $\zeta \geqslant p \geqslant 1$.

Further, an application of (2.2) in Lemma 1 gives

$$\operatorname{Re}\left\{\frac{z_0 w''(z_0)}{w'(z_0)}\right\} = \operatorname{Re}\left\{\frac{z_0^2 w''(z_0)}{\zeta e^{i\theta}}\right\} \geqslant \zeta - 1, \tag{2.12}$$

or

$$\operatorname{Re}\left\{e^{-i\theta} M\right\} \geqslant \zeta(\zeta - 1) \qquad (\theta \in R; \zeta \geqslant 1). \tag{2.13}$$

Since $\phi(r,s,t) \in \Phi$, we also have

$$\begin{aligned}&\left|\phi(I_p^{\alpha} f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z))\right| \\ &\quad = \left|\phi(e^{i\theta}, (\frac{\zeta+1}{p+1})e^{i\theta}, \frac{(1+3\zeta)e^{i\theta} + M}{(p+1)^2})\right| > 1\end{aligned} \tag{2.14}$$

which contradicts the condition (*ii*) of Theorem 1. Therefore, we conclude that

$$|w(z)| = \left|I_p^{\alpha} f(z)\right| < 1 \qquad (z \in U; \alpha > 2). \tag{2.15}$$

This completes the proof of Theorem 1.

COROLLARY 1. *Let $\phi_1(r,s,t) = s$ and let $f(z) \in A(p)$ satisfy the conditions in Theorem 1 for $\alpha > 2,\ p \in N$ and $z \in U$. Then*

$$\left|I_p^{\alpha+i} f(z)\right| < 1 \qquad (i = 0,1,2,....;\ \alpha > 2; p \in N; z \in U). \tag{2.16}$$

Proof. Note that $\phi_1(r,s,t) = s$ is in Φ, with the aid of Theorem 1, we have

$$\left|I_p^{\alpha-1} f(z)\right| < 1 \Longrightarrow \left|I_p^{\alpha} f(z)\right| < 1 \quad (\alpha > 2; p \in N)$$

$$\Longrightarrow \left|I_p^{\alpha+i} f(z)\right| < 1 \qquad (i = 0,1,2,....;\ \alpha > 2; p \in N; z \in U).$$

THEOREM 2. *Let $\psi(r,s,t) \in \Psi$ and let $f(z)$ belonging to the class $A(p)$ satisfying*

(*i*)
$$(Q_{\beta,p}^{\alpha} f(z),\ Q_{\beta,p}^{\alpha-1} f(z), Q_{\beta,p}^{\alpha-2} f(z)) \in D \subset C^3$$

and

(*ii*)
$$\left|\psi(Q_{\beta,p}^{\alpha} f(z),\ Q_{\beta,p}^{\alpha-1} f(z), Q_{\beta,p}^{\alpha-2} f(z))\right| < 1$$

for $\alpha > 2,\ \beta > -1,\ p \in N$ and $z \in U$. Then we have

$$\left|Q_{\beta,p}^{\alpha} f(z)\right| < 1, \qquad (z \in U). \tag{2.17}$$

Proof. Defining $w(z)$ by

$$Q_{\beta,p}^{\alpha} f(z) = w(z), \qquad (\alpha > 2; \beta > -1; p \in N), \tag{2.18}$$

we have $w(z) \in A(p)$ and $w(z) \neq 0\ \ (z \in U)$. With the aid of the identily (1.8), we have

$$Q_{\beta,p}^{\alpha-1} f(z) = \frac{1}{(\alpha+\beta+p-1)} \left\{(\alpha+\beta-1)w(z) + zw'(z)\right\} \tag{2.19}$$

and

$$Q_{\beta,p}^{\alpha-2} f(z) = \frac{1}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)} \left\{(\alpha+\beta-1)(\alpha+\beta-2)w(z) \right.$$
$$\left. +2(\alpha+\beta-1)zw'(z) + z^2 w''(z)\right\}. \tag{2.20}$$

Suppose that $z_0 = r_0 e^{i\theta}(0 < r_0 < 1; \theta \in R)$ and

$$|w(z_0)| = \max_{|z| \leqslant |z_0|} |w(z)| = 1. \tag{2.21}$$

Then letting $w(z_0) = e^{i\theta}$ and using (2.1), we obtain

$$Q^{\alpha}_{\beta,p} f(z_0) = w(z_0) = e^{i\theta}, \tag{2.22}$$

$$Q^{\alpha-1}_{\beta,p} f(z_0) = \frac{1}{(\alpha+\beta+p-1)} \{\zeta - 1 + \alpha + \beta\} e^{i\theta}, \tag{2.23}$$

and

$$Q^{\alpha-2}_{\beta,p} f(z_0) = \frac{1}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)} \left\{(\alpha+\beta-1)[2(\zeta-1)+\alpha+\beta]e^{i\theta} + L\right\}, \tag{2.24}$$

where $L = z_0^2 w''(z_0)$ and $\zeta \geqslant 1$. Moreover, we find from (2.2) that

$$\mathrm{Re}\left\{\frac{z_0^2 w''(z_0)}{w'(z_0)}\right\} = \mathrm{Re}\left\{\frac{z_0^2 w''(z_0)}{\zeta e^{i\theta}}\right\} \geqslant \zeta - 1, \tag{2.25}$$

or

$$\mathrm{Re}\{e^{-i\theta} L\} \geqslant \zeta(\zeta - 1) \quad (\theta \in R; \zeta \geqslant 1). \tag{2.26}$$

Since $\psi(r,s,t) \in \Psi$, we also have

$$\begin{aligned} \left|\psi(Q^{\alpha}_{\beta,p} f(z), Q^{\alpha-1}_{\beta,p} f(z), Q^{\alpha-2}_{\beta,p} f(z))\right| = \Bigg|\psi(e^{i\theta}, \frac{\zeta-1+\alpha+\beta}{\alpha+\beta+p-1} e^{i\theta},& \\ \frac{(\alpha+\beta-1)[2(\zeta-1)+\alpha+\beta]e^{i\theta}+L}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)})\Bigg| > 1& \end{aligned} \tag{2.27}$$

which contradicts the hypothesis (ii) of Theorem 2. Therefore, we conclude that

$$|w(z)| = \left|Q^{\alpha}_{\beta,p} f(z)\right| < 1 \qquad (z \in U; \alpha > 2; \beta > -1; p \in N), \tag{2.28}$$

which completes the proof of Theorem 2.

COROLLARY 2. *Let* $\psi_0(r,s,t) = s$ *and let* $f(z) \in A(p)$ *satisfy the conditions in Theorem 2 for* $\alpha > 2$, $\beta > -1$, $p \in N$ *and* $z \in U$. *Then*

$$\left|Q^{\alpha+i}_{\beta,p} f(z)\right| < 1, \qquad (i = 0,1,2,\ldots; \alpha > 2; \beta > -1; p \in N; z \in U). \tag{2.29}$$

Proof. Note that $\psi_0(r,s,t) = s$ is in Ψ, so with the aid of Theorem 2, we have

$$\begin{aligned} \left|Q^{\alpha-1}_{\beta,p} f(z)\right| < 1 \Longrightarrow \left|Q^{\alpha}_{\beta,p} f(z)\right| < 1 \qquad &(\alpha > 2; \beta > -1; p \in N; z \in U) \\ \Longrightarrow \left|Q^{\alpha+i}_{\beta,p} f(z)\right| < 1 \qquad &(i = 0,1,2,\ldots; \alpha > 2; \beta > -1; p \in N; z \in U). \end{aligned}$$

REMARK 1. Putting $p = 1$ in the above results we obtain the results obtained by Aouf et al. [1].

THEOREM 3. *Let $h(r,s,t)\in H$, and let $f(z)$ belonging to $A(p)$ satisfying*

(*i*)
$$\left(\frac{I_p^{\alpha-1}f(z)}{I_p^{\alpha}f(z)},\frac{I_p^{\alpha-2}f(z)}{I_p^{\alpha-1}f(z)},\frac{I_p^{\alpha-3}f(z)}{I_p^{\alpha-2}f(z)}\right)\in D\subset C^3$$

and

(*ii*)
$$\left|h\left(\frac{I_p^{\alpha-1}f(z)}{I_p^{\alpha}f(z)},\frac{I_p^{\alpha-2}f(z)}{I_p^{\alpha-1}f(z)},\frac{I_p^{\alpha-3}f(z)}{I_p^{\alpha-2}f(z)}\right)\right|<J$$

for some α, p, J $(\alpha>3; p\in N; J>1)$ and for all $z\in U$. Then we have

$$\left|\frac{I_p^{\alpha-1}f(z)}{I_p^{\alpha}f(z)}\right|<J\quad(z\in U). \tag{2.30}$$

Proof. We define the function $w(z)$ by

$$\frac{I_p^{\alpha-1}f(z)}{I_p^{\alpha}f(z)}=w(z)\qquad(\alpha>3;p\in N) \tag{2.31}$$

for $f(z)$ belonging to the class $A(p)$. Then, it follows that $w(z)$ is either analytic or meromorphic in U, $w(0)=1$, and $w(z)\neq 1$. With the aid of the idenitily (1.11), we have

$$\frac{I_p^{\alpha-2}f(z)}{I_p^{\alpha-1}f(z)}=\frac{1}{p+1}\left[(p+1)w(z)+\frac{zw'(z)}{w(z)}\right] \tag{2.32}$$

and

$$\frac{I_p^{\alpha-3}f(z)}{I_p^{\alpha-2}f(z)}=\frac{1}{p+1}\left\{(p+1)w(z)+\frac{zw'(z)}{w(z)}+\frac{(p+1)zw'(z)+\frac{zw'(z)}{w(z)}+\frac{zw''(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2}{(p+1)w(z)+\frac{zw'(z)}{w(z)}}\right\}. \tag{2.33}$$

Suppose that $z_0=r_0e^{i\theta}$ $(0<r_0<1;\theta\in R)$ and $|w(z_0)|=\max_{|z|\leqslant|z_0|}|w(z)|=J$. Letting $w(z_0)=Je^{i\theta}$ and using Lemma 2 with $a=k=1$, we see that

$$\frac{I_p^{\alpha-2}f(z_0)}{I_p^{\alpha-1}f(z_0)}=\frac{1}{p+1}[\zeta+(p+1)Je^{i\theta}] \tag{2.34}$$

and

$$\frac{I_p^{\alpha-3}f(z_0)}{I_p^{\alpha-2}f(z_0)}=\frac{1}{p+1}\left\{\zeta+(p+1)Je^{i\theta}+\frac{\zeta-\zeta^2+(p+1)\zeta Je^{i\theta}+L}{\zeta+(p+1)Je^{i\theta}}\right\}, \tag{2.35}$$

where $L = \dfrac{z_0 w''(z_0)}{w(z_0)}$ and $\zeta \geqslant \dfrac{J-1}{J+1}$.

Further, an appliciation of (ii) in Lemma 2 gives

$$\mathrm{Re}\{L\} \geqslant \zeta(\zeta-1).$$

Since $h(r,s,t) \in H$, we have

$$\left| h\left(\frac{I_p^{\alpha-1} f(z_0)}{I_p^{\alpha} f(z_0)}, \frac{I_p^{\alpha-2} f(z_0)}{I_p^{\alpha-1} f(z_0)}, \frac{I_p^{\alpha-3} f(z_0)}{I_p^{\alpha-2} f(z_0)} \right) \right|$$
$$= \left| h\left(Je^{i\theta}, \frac{\zeta+(p+1)Je^{i\theta}}{p+1}, \frac{1}{p+1}\left\{ \zeta+(p+1)Je^{i\theta} + \frac{\zeta-\zeta^2+(p+1)\zeta Je^{i\theta}+L}{\zeta+(p+1)Je^{i\theta}} \right\} \right) \right| \geqslant J, \quad (2.36)$$

which contradicts condition (ii) of Theorem 3. Therefore, we conclude that

$$|w(z)| = \left| \frac{I_p^{\alpha-1} f(z)}{I_p^{\alpha} f(z)} \right| < J \qquad (2.37)$$

for all $\alpha > 3, p \in N$ and $z \in U$. This compltes the proof of Theorem 3.

THEOREM 4. *Let $g(r,s,t) \in G$, and let $f(z)$ belonging to $A(p)$ satisfying*

(*i*)
$$\left(\frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)}, \frac{Q_{\beta,p}^{\alpha-2} f(z)}{Q_{\beta,p}^{\alpha-1} f(z)}, \frac{Q_{\beta,p}^{\alpha-3} f(z)}{Q_{\beta,p}^{\alpha-2} f(z)} \right) \in D \subset C^3$$

and

(*ii*)
$$\left| g\left(\frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)}, \frac{Q_{\beta,p}^{\alpha-2} f(z)}{Q_{\beta,p}^{\alpha-1} f(z)}, \frac{Q_{\beta,p}^{\alpha-3} f(z)}{Q_{\beta,p}^{\alpha-2} f(z)} \right) \right| < J$$

for some α, β, p, J $(\alpha > 3;\ \beta > -1;\ p \in N;\ J > 1)$ and for all $z \in U$. Then we have

$$\left| \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)} \right| < J \qquad (z \in U). \qquad (2.38)$$

Proof. We define the function $w(z)$ by

$$\frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha} f(z)} = w(z) \qquad (\alpha > 3;\ \beta > -1;\ p \in N) \qquad (2.39)$$

for $f(z)$ belonging to the class $A(p)$. Then, it follows that $w(z)$ is either analytic or neromorphic in U, $w(0) = 0$, and $w(z) \neq 1$. With the aid of the identily (1.8), we have

$$\frac{Q_{\beta,p}^{\alpha-2} f(z)}{Q_{\beta,p}^{\alpha-1} f(z)} = \frac{1}{(\alpha+\beta+p-2)} \left[-1 + (\alpha+\beta+p-1)w(z) + \frac{zw'(z)}{w(z)} \right] \qquad (2.40)$$

and

$$\frac{Q_{\beta,p}^{\alpha-3}f(z)}{Q_{\beta,p}^{\alpha-2}f(z)} = \frac{1}{(\alpha+\beta+p-3)}\left\{-2+(\alpha+\beta+p-1)w(z)+\frac{zw'(z)}{w(z)}\right.$$

$$\left.+\frac{(\alpha+\beta+p-1)zw'(z)+\frac{zw'(z)}{w(z)}+\frac{z^2w''(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2}{-1+(\alpha+\beta+p-1)w(z)+\frac{zw'(z)}{w(z)}}\right\}. \quad (2.41)$$

Suppose that $z_0 = r_0e^{i\theta}$ $(0 < r_0 < 1;\ \theta \in R)$ and $|w(z_0)| = \max_{|z|\leqslant|z_0|}|w(z)| = J$. Letting $w(z_0) = Je^{i\theta}$ and using Lemma 2 with $a = k = 1$, we see that

$$\frac{Q_{\beta,p}^{\alpha-2}f(z_0)}{Q_{\beta,p}^{\alpha-1}f(z_0)} = \frac{1}{(\alpha+\beta+p-2)}\left[-1+\zeta+(\alpha+\beta+p-1)Je^{i\theta}\right] \quad (2.42)$$

and

$$\frac{Q_{\beta,p}^{\alpha-3}f(z_0)}{Q_{\beta,p}^{\alpha-2}f(z_0)} = \frac{1}{(\alpha+\beta+p-3)}\left\{-2+\zeta+(\alpha+\beta+p-1)Je^{i\theta}+\right.$$

$$\left.\frac{\zeta-\zeta^2+(\alpha+\beta+p-1)\zeta Je^{i\theta}+M}{-1+\zeta+(\alpha+\beta+p-1)Je^{i\theta}}\right\}, \quad (2.43)$$

where $M = \dfrac{z_0w''(z_0)}{w(z_0)}$ and $\zeta \geqslant \dfrac{J-1}{J+1}$.

Further, an appliciation of (*ii*) in Lemma 2 gives

$$\mathrm{Re}\{M\} \geqslant \zeta(\zeta-1).$$

Since $g(r,s,t) \in G$, we have

$$\left|g\left(\frac{Q_{\beta,p}^{\alpha-1}f(z_0)}{Q_{\beta,p}^{\alpha}f(z_0)},\frac{Q_{\beta,p}^{\alpha-2}f(z_0)}{Q_{\beta,p}^{\alpha-1}f(z_0)},\frac{Q_{\beta,p}^{\alpha-3}f(z_0)}{Q_{\beta,p}^{\alpha-2}f(z_0)}\right)\right|$$

$$=\left|g\left(Je^{i\theta},\frac{-1+\zeta+(\alpha+\beta+p-1)Je^{i\theta}}{(\alpha+\beta+p-2)},\right.\right.$$

$$\frac{1}{(\alpha+\beta+p+3)}\left\{-2+\zeta+(\alpha+\beta+p-1)Je^{i\theta}+\right.$$

$$\left.\left.\left.\frac{\zeta-\zeta^2+(\alpha+\beta+p-1)\zeta Je^{i\theta}+M}{-1+\zeta+(\alpha+\beta+p-1)Je^{i\theta}}\right\}\right)\right| \geqslant J, \quad (2.44)$$

which contradicts condition (ii) of Theorem 4. Therefore, we conclude that

$$|w(z)| = \left|\frac{Q_{\beta,p}^{\alpha-1}f(z)}{Q_{\beta,p}^{\alpha}f(z)}\right| < J \quad (2.45)$$

for all $\alpha > 3$, $\beta > -1$, $p \in N$ and $z \in U$. This completes the proof of Theorem 4.

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