Journal of
Mathematical
Inequalities
Volume 2, Number 4 (2008), 549–553

A NOTE ON THE PAPER OF BREAZ AND GÜNEY

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(*communicated by J. Pečarić*)

Abstract. In this note, we generalize the Theorem 1 and Theorem 3 in [2]. Furthermore, we consider the strongly starlikeness and strongly convexity classes of the analytic functions and two integral operators.

1. Introduction

Let $\mathscr{A}$ be the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.

DEFINITION 1. [4] A function $f \in \mathscr{A}$ is said to be *starlike of complex order* b $(b \in \mathbb{C} - \{0\})$ *and type* α $(0 \leqslant \alpha < 1)$ if it satisfies the inequality:

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \alpha$$

for all $z \in \mathbb{U}$. We say that f is in the class $\mathscr{S}^*_\alpha(b)$ for such functions.

DEFINITION 2. [4] A function $f \in \mathscr{A}$ is said to be *convex of complex order* b $(b \in \mathbb{C} - \{0\})$ *and type* α $(0 \leqslant \alpha < 1)$ if it satisfies the inequality:

$$\operatorname{Re}\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > \alpha$$

for all $z \in \mathbb{U}$. We say that f is in the class $\mathscr{C}_\alpha(b)$ for such functions.

Mathematics subject classification (2000): 30C45.
Keywords and phrases: Integral operator, Analytic functions, Starlike functions, Convex functions, Complex order.

Paper JMI-02-49

We note that $f \in \mathscr{C}_\alpha(b)$ if and only if $zf' \in \mathscr{S}^*_\alpha(b)$.

In particular, the classes

$$\mathscr{S}^*_\alpha(1) := \mathscr{S}^*(\alpha),$$

and

$$\mathscr{C}_\alpha(1) := \mathscr{C}(\alpha)$$

are well known classes of functions which are starlike of order α and convex of order α, respectively.

Let we consider the integral operators

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt$$

and

$$F_{\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \left[f_1'(t)\right]^{\alpha_1} \cdots \left[f_n'(t)\right]^{\alpha_n} dt,$$

where $f_i \in \mathscr{A}$ and $\alpha_i > 0$ for all $i \in \{1,\ldots,n\}$.

These operators are introduced and studied in [1] and [3], respectively.

The purpose of this paper is to generalize the main results of [2].

2. Main Results

THEOREM 1. *Let $f_i \in \mathscr{S}^*_{\beta_i}(b)$ for $1 \leqslant i \leqslant n$ with $0 \leqslant \beta_i < 1$, $b \in \mathbb{C} - \{0\}$. Also let $\alpha_i > 0$, $1 \leqslant i \leqslant n$. If*

$$0 \leqslant 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1) < 1,$$

then $F_n \in \mathscr{C}_\lambda(b)$ with $\lambda = 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1)$.

Proof. After some calculus, we obtain

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1\right).$$

Then by multiplying the above relation with $1/b$, we have

$$\begin{aligned}\frac{1}{b}\frac{zF_n''(z)}{F_n'(z)} &= \sum_{i=1}^{n} \alpha_i \frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\\ &= \sum_{i=1}^{n} \alpha_i \left[1 + \frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right] - \sum_{i=1}^{n} \alpha_i\end{aligned}$$

or equivalently

$$1 + \frac{1}{b}\frac{zF_n''(z)}{F_n'(z)} = 1 + \sum_{i=1}^{n} \alpha_i \left[1 + \frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right] - \sum_{i=1}^{n} \alpha_i.$$

Since $f_i \in \mathscr{S}^*_{\beta_i}(b)$ for $1 \leqslant i \leqslant n$, we get

$$\begin{aligned}\operatorname{Re}\left\{1+\frac{1}{b}\frac{zF_n''(z)}{F_n'(z)}\right\} &= 1+\sum_{i=1}^{n}\alpha_i\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right\}-\sum_{i=1}^{n}\alpha_i\\ &> 1+\sum_{i=1}^{n}\alpha_i\beta_i-\sum_{i=1}^{n}\alpha_i\\ &= 1+\sum_{i=1}^{n}\alpha_i(\beta_i-1).\end{aligned}$$

So, $F_n \in \mathscr{C}_\lambda(b)$ with $\lambda = 1+\sum_{i=1}^{n}\alpha_i(\beta_i-1)$.

COROLLARY 2. *If we set* $\beta_1=\beta_2=\cdots=\beta_n=\alpha$ *in Theorem 1, we have Theorem 1 in* [2].

THEOREM 3. *Let* $f_i \in \mathscr{C}_{\beta_i}(b)$ *for* $1 \leqslant i \leqslant n$ *with* $0 \leqslant \beta_i < 1$, $b \in \mathbb{C}-\{0\}$. *Also let* $\alpha_i > 0$, $1 \leqslant i \leqslant n$. *If*

$$0 \leqslant 1+\sum_{i=1}^{n}\alpha_i(\beta_i-1) < 1$$

then the function $F_{\alpha_1,\dots,\alpha_n} \in \mathscr{C}_\mu(b)$ *with* $\mu = 1+\sum_{i=1}^{n}\alpha_i(\beta_i-1)$.

Proof. After some calculus, we obtain

$$\frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)} = \sum_{i=1}^{n}\alpha_i\frac{zf_i''(z)}{f_i'(z)}.$$

Then by multiplying the above relation with $1/b$, we have

$$\frac{1}{b}\frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)} = \sum_{i=1}^{n}\alpha_i\frac{1}{b}\frac{zf_i''(z)}{f_i'(z)} = \sum_{i=1}^{n}\alpha_i\left(1+\frac{1}{b}\frac{zf_i''(z)}{f_i'(z)}\right)-\sum_{i=1}^{n}\alpha_i$$

or equivalently

$$1+\frac{1}{b}\frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)} = 1+\sum_{i=1}^{n}\alpha_i\left(1+\frac{1}{b}\frac{zf_i''(z)}{f_i'(z)}\right)-\sum_{i=1}^{n}\alpha_i.$$

Since $f_i \in \mathscr{C}_{\beta_i}(b)$ for $1 \leqslant i \leqslant n$, we get

$$\operatorname{Re}\left\{1+\frac{1}{b}\frac{zf_i''(z)}{f_i'(z)}\right\} > \beta_i.$$

So we obtain

$$\begin{aligned}\operatorname{Re}\left\{1+\frac{1}{b}\frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)}\right\} &= 1+\sum_{i=1}^{n}\alpha_i\operatorname{Re}\left\{1+\frac{1}{b}\frac{zf_i''(z)}{f_i'(z)}\right\}-\sum_{i=1}^{n}\alpha_i\\ &> 1+\sum_{i=1}^{n}\alpha_i(\beta_i-1).\end{aligned}$$

So, $F_{\alpha_1,\dots,\alpha_n} \in \mathscr{C}_\mu(b)$ with $\mu = 1+\sum_{i=1}^{n}\alpha_i(\beta_i-1)$.

COROLLARY 4. *If we set* $\beta_1 = \beta_2 = \cdots = \beta_n = \alpha$ *in Theorem 3, we have Theorem 3 in* [2].

3. Strongly Starlikeness and Strongly Convexity

Firstly, we introduce two new classes:

DEFINITION 3. If $f \in \mathscr{A}$ satisfies

$$\left|\arg\left[1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) - \alpha\right]\right| < \frac{\pi}{2}\rho \quad (z \in \mathbb{U})$$

for some α $(0 \leqslant \alpha < 1)$, ρ $(0 < \rho \leqslant 1)$ and $b \in \mathbb{C} - \{0\}$, then f is said to be *strongly starlike of complex order* b, *real order* ρ *and type* α in $\mathbb{U}$, and denoted by $f \in \mathscr{S}^*_\alpha(b, \rho)$.

DEFINITION 4. If $f \in \mathscr{A}$ satisfies

$$\left|\arg\left(1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi}{2}\rho \quad (z \in \mathbb{U})$$

for some α $(0 \leqslant \alpha < 1)$, ρ $(0 < \rho \leqslant 1)$ and $b \in \mathbb{C} - \{0\}$, then f is said to be *strongly convex of complex order* b, *real order* ρ *and type* α in $\mathbb{U}$, and denoted by $f \in \mathscr{C}_\alpha(b, \rho)$.

It is clear that $f \in \mathscr{C}_\alpha(b, \rho)$ if and only if $zf' \in \mathscr{S}^*_\alpha(b, \rho)$.
Also, we note that $\mathscr{S}^*_\alpha(1, 1) \equiv \mathscr{S}^*(\alpha)$ and $\mathscr{C}_\alpha(1, 1) \equiv \mathscr{C}(\alpha)$.
In this section, we investigate strongly starlikeness and strongly convexity properties for the integral operator of Alexander

$$F(z) = \int_0^z \frac{f(t)}{t} dt$$

and the integral operator

$$G(z) = \int_0^z f'(t) dt,$$

where $f \in \mathscr{A}$.

THEOREM 5. *Let* $f \in \mathscr{S}^*_\alpha(b, \rho)$. *Then* $F \in \mathscr{C}_\alpha(b, \rho)$.

Proof. After some calculus, we obtain

$$1 + \frac{1}{b}\frac{zF''(z)}{F'(z)} = 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right).$$

Since $f \in \mathscr{S}^*_\alpha(b, \rho)$, we get

$$\left|\arg\left(1 + \frac{1}{b}\frac{zF''(z)}{F'(z)} - \alpha\right)\right| = \left|\arg\left[1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) - \alpha\right]\right| < \frac{\pi}{2}\rho.$$

So, $F \in \mathscr{C}_\alpha(b, \rho)$.

THEOREM 6. *Let* $f \in \mathscr{C}_\alpha(b,\rho)$. *Then* $G \in \mathscr{C}_\alpha(b,\rho)$.

Proof. After some calculus, we obtain

$$1+\frac{1}{b}\frac{zG''(z)}{G'(z)} = 1+\frac{1}{b}\frac{zf''(z)}{f'(z)}.$$

Since $f \in \mathscr{C}_\alpha(b,\rho)$, we have

$$\left|\arg\left(1+\frac{1}{b}\frac{zG''(z)}{G'(z)}-\alpha\right)\right| = \left|\arg\left(1+\frac{1}{b}\frac{zf''(z)}{f'(z)}-\alpha\right)\right| < \frac{\pi}{2}\rho.$$

Thus, $G \in \mathscr{C}_\alpha(b,\rho)$.

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(Received June 2, 2008)

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