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ON A QUADRATIC ESTIMATE OF SHAFER

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Abstract. In this note, a upper bound for $\arctan x$ is obtained. We would point out that the numbers $80/3$ and $256/\pi^2$ are the best constants in Shafer-type inequality.

1. Introduction

Shafer [1-3] established the following results for $\arctan x$ and $\tanh^{-1} x$:

THEOREM 1. *Let* $x > 0$. *Then*

$$\arctan x > \frac{8x}{3+\sqrt{25+\frac{80}{3}x^2}}. \tag{1}$$

THEOREM 2. *Let* $0 < x < \sqrt{15}/4$. *Then*

$$\tanh^{-1} x < \frac{8x}{3+\sqrt{25-\frac{80}{3}x^2}}. \tag{2}$$

In fact, we can show a upper bound for $\arctan x$, and obtain the following theorem:

THEOREM 3 (SHAFER-TYPE INEQUALITY). *Let* $x > 0$. *Then*

$$\frac{8x}{3+\sqrt{25+\frac{80}{3}x^2}} < \arctan x < \frac{8x}{3+\sqrt{25+\frac{256}{\pi^2}x^2}}. \tag{3}$$

Furthermore, $80/3$ *and* $256/\pi^2$ *are the best constants in (3).*

In this note, we shall prove Theorem 3 by elementary method, and show a new simple proof of Theorem 2 in concise way.

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2. Two Lemmas

LEMMA 1. *Let* $|x| < \pi$. *Then*

$$\cot^2 t = \frac{1}{t^2} - \frac{2}{3} + \sum_{n=2}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|(2n-1)t^{2n-2}. \tag{4}$$

Proof. The following power series expansion can be found in [4, 1.3.1.4 (2); 5, 1.3.10]:

$$\cot t = \frac{1}{t} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n-1}, |t| < \pi \tag{5}$$

or

$$t\cot t = 1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n}, |t| < \pi. \tag{6}$$

(For further comprehension of the even-indexed Bernoulli numbers B_{2n}, refer to pp.231-232 in [6].) Then

$$\csc^2 t = -(\cot t)' = \frac{1}{t^2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|(2n-1)t^{2n-2}, |t| < \pi, \tag{7}$$

and

$$\cot^2 t = \csc^2 t - 1 = \frac{1}{t^2} - \frac{2}{3} + \sum_{n=2}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|(2n-1)t^{2n-2}, |t| < \pi.$$

LEMMA 2 ([7-9]). *Let* a_n *and* b_n $(n = 0,1,2,\cdots)$ *be real numbers, and let the power series* $A(t) = \sum_{n=0}^{\infty} a_n t^n$ *and* $B(t) = \sum_{n=0}^{\infty} b_n t^n$ *be convergent for* $|t| < R$. *If* $b_n > 0$ *for* $n = 0,1,2,\cdots$, *and if* a_n/b_n *is strictly increasing (or decreasing) for* $n = 0,1,2,\cdots$, *then the function* $A(t)/B(t)$ *is strictly increasing (or decreasing) on* $(0,R)$.

3. Proof of Theorem 3

Let

$$F(x) = \frac{(\frac{8x}{\arctan x} - 3)^2 - 25}{x^2},$$

and $\arctan x = t$. Then $x = \tan t$ for $t \in (0,\pi/2)$, and we have $F(x) = 16G(t)$, where

$$\begin{aligned} G(t) =& \frac{4\tan^2 t - 3t\tan t - t^2}{t^2\tan^2 t} = \frac{4 - 3t\cot t - t^2\cot^2 t}{t^2} \\ =& \frac{1}{t^2}[4 - 3(1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n}) - (1 - \frac{2}{3}t^2 + \sum_{n=2}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|(2n-1)t^{2n})] \\ =& \frac{5}{3} + \sum_{n=2}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|(4-2n)t^{2n-2} \end{aligned} \tag{8}$$

by (6) and (4).

Since $G'(t)=\sum_{n=2}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|(4-2n)(2n-2)t^{2n-3}<0$, We conclude that $G(t)$ is decreasing on $(0,\pi/2)$. This leads to $F(x)$ is decreasing on $(0,+\infty)$.

Furthermore, $\lim\limits_{x\longrightarrow+\infty}F(x)=16\lim\limits_{t\longrightarrow\frac{\pi}{2}-}G(t)=16\lim\limits_{t\longrightarrow\frac{\pi}{2}-}\frac{4\tan^2 t-3t\tan t-t^2}{t^2\tan^2 t}=\frac{256}{\pi^2}$, and $\lim\limits_{x\longrightarrow 0^+}F(x)=16\cdot\frac{5}{3}=\frac{80}{3}$. The proof of Theorem 3 is complete.

4. A Concise Proof of Theorem 2

Let

$$H(x)=\frac{25-(\frac{8x}{\tanh^{-1}x}-3)^2}{x^2},$$

and $\tanh^{-1}x=t$. Then $x=\tanh t$ for $t\in(0,\tanh^{-1}\sqrt{15}/4)$, and we have $H(x)=16I(t)$, where

$$I(t)=\frac{-4\sinh^2 t+3t\sinh t\cosh t+t^2\cosh^2 t}{t^4\cosh^2 t}=\frac{A(t)}{B(t)},$$

and

$$\begin{aligned}
A(t)=&-4\sinh^2 t+3t\sinh t\cosh t+t^2\cosh^2 t\\
=&-2(\cosh 2t-1)+\frac{3}{2}t\sinh 2t+\frac{1}{2}t^2(\cosh 2t+1)\\
=&-2\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n}+\frac{3}{2}\sum_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}t^{2n+2}+\frac{1}{2}t^2+\frac{1}{2}\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n+2}\\
=&-2\sum_{n=2}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n}+\frac{3}{2}\sum_{n=1}^{\infty}\frac{2^{2n+1}}{(2n+1)!}t^{2n+2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n+2}\\
=&-2\sum_{n=1}^{\infty}\frac{2^{2n+2}}{(2n+2)!}t^{2n+2}+\frac{3}{2}\sum_{n=1}^{\infty}\frac{2^{2n+1}}{(2n+1)!}t^{2n+2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n+2}\\
=&\sum_{n=1}^{\infty}[-2\frac{2^{2n+2}}{(2n+2)!}++\frac{3}{2}\frac{2^{2n+1}}{(2n+1)!}+\frac{1}{2}\frac{2^{2n}}{(2n)!}]t^{2n+2}\\
=&\sum_{n=1}^{\infty}a_n t^{2n+2},
\end{aligned}$$

$$\begin{aligned}
B(t)=&t^4\cosh^2 t=\frac{1}{2}t^4(1+\cosh 2t)=\frac{1}{2}t^4+\frac{1}{2}\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n+4}\\
=&t^4+\frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n+4}=t^4+\frac{1}{2}\sum_{n=2}^{\infty}\frac{2^{2n-2}}{(2n-2)!}t^{2n+2}\\
=&\sum_{n=1}^{\infty}b_n t^{2n+2},
\end{aligned}$$

$a_1 = \frac{5}{3}, a_n = \frac{-4\cdot 2^{2n+2}+3(2n+2)2^{2n+1}+(2n+2)(2n+1)2^{2n}}{2(2n+2)!}$, $b_1 = 1, b_n = \frac{2^{2n-2}}{2(2n-2)!} > 0$, $n \geqslant 2$, and $n \in \mathbf{N}^+$.
So $\frac{a_1}{b_1} > \frac{a_2}{b_2}$, and for $n \geqslant 2$ we have

$$c_n = \frac{a_n}{b_n} = \frac{4(2n+2)(2n+1)+24(2n+2)-64}{(2n+2)(2n+1)2n(2n-1)}.$$

We conclude that c_n is decreasing for $n = 1, 2, \cdots$, and $I(t) = \frac{A(t)}{B(t)}$ is decreasing on $(0, \tanh^{-1}\sqrt{15}/4)$ by Lemma 2. This leads to $H(x)$ is decreasing on $(0, \sqrt{15}/4)$. At the same time, $\lim\limits_{x\to 0^+} H(x) = \frac{80}{3}$. So the proof of Theorem 2 is complete.

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