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AN INEQUALITY FOR POLYNOMIALS WITH POSITIVE COEFFICIENTS AND APPLICATIONS IN RATIONAL APPROXIMATION

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Abstract. We extend an inequality of Leviatan and Lubinsky ([3: Theorem 3.1]) to polynomials with positive coefficients. Two applications in approximation by rational functions with prescribed numerators are given.

1. Introduction

Let $C_{[a,b]}$ be the set of all continuous functions on $[a,b]$, $L^p_{[a,b]}$ the set of p power integrable functions on $[a,b]$ such that $\|f\|_{L^p_{[a,b]}} < \infty$, where

$$\|f\|_{L^p_{[a,b]}} = \left(\int_a^b |f(x)|^p dx\right)^{1/p}, \ 1 \leqslant p < +\infty.$$

In this paper, $L^\infty_{[a,b]}$ is interpreted as $C_{[a,b]}$, and equipped with the norm

$$\|f\|_{L^\infty_{[a,b]}} = \|f\|_{C_{[a,b]}} = \max_{a \leqslant x \leqslant b} |f(x)|$$

for $f(x) \in C_{[a,b]}$. Let $\omega(f,\delta)_{L^p_{[a,b]}}$ be the modulus of continuity in L^p norm of $f \in L^p_{[a,b]}$, that is,

$$\omega(f,\delta)_{L^p_{[a,b]}} = \sup_{0<h\leqslant\delta} \left\{\int_a^{b-h} |f(x+h)-f(x)|^p dx\right\}^{1/p}, \ 1 \leqslant p < +\infty,$$

and

$$\omega(f,\delta)_{L^\infty_{[a,b]}} = \sup\{|f(x)-f(y)| : |x-y| \leqslant \delta, x,y \in [a,b]\}.$$

For convenience, write

$$\omega(f,\delta)_{L^p_{[0,1]}} = \omega(f,\delta)_{L^p}, \ \|f\|_{L^p_{[0,1]}} = \|f\|_{L^p}.$$

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Denote by $\Pi_n(+)$ the set of all algebraic polynomials with positive coefficients of degree at most n on $[0,1]$, that is,

$$\Pi_n(+) = \left\{ p_n(x) : p_n(x) = \sum_{0 \leqslant k+l \leqslant n} a_{k,l} x^k (1-x)^l, a_{k,l} \geqslant 0 \right\}.$$

Approximation by reciprocals of polynomials is a special type of rational approximation. Because of the unique values in theories and applications, it has been investigated extensively. For last a dozen years, many important progresses in this direction have been achieved. Xu ([7]) established the following

THEOREM X. *Let $f \in C_{[0,1]}$, $f(x) \geqslant 0$, $x \in [0,1]$, and $f \not\equiv 0$. Then there is a sequence of polynomials $P_n \in \Pi_n(+)$ such that*

$$\|f - 1/P_n\| \leqslant C\omega_\varphi(f, n^{-1/2}),$$

where $\omega_\varphi(f,t)$ is the Ditzian-Totik modulus of smoothness with $\varphi(x) = \sqrt{x(1-x)}$.

Recently, Zhao and Zhou [8] generalized Theorem Xu to include the usual $L^p_{[0,1]}$ for $1 < p < +\infty$. Mei and Zhou [4] obtained an analogue in $L^1_{[0,1]}$ later by a different method. When f has finitely many sign changes in some finite interval I, it is impossible to approximate $f(x)$ by reciprocals of polynomials with real coefficients, and is also in the same situation for approximation by reciprocals of polynomials with positive coefficients. However, in this case, f can be approximated by rational functions with the numerators consisting of polynomials of degree l and denominators polynomials with positive (or real) coefficients (the class of this kind of rational functions can be denoted by $R^l_n(+)$). Zhou [9] investigated this type of problem by obtaining

THEOREM Z. *Let $f(x) \in C_{[0,1]}$ change sign exactly once, then there exist a $x_0 \in (0,1)$ and a $P_n(x) \in \Pi_n(+)$, such that*

$$\left\| f(x) - \frac{x - x_0}{P_n(x)} \right\|_C \leqslant C\omega(f, n^{-1/2}).$$

A very recent paper of Mei [6] generalized Theorem Zhou to $L^p_{[0,1]}$ spaces for $1 < p < +\infty$ as follows.

THEOREM M. *Let $l \geqslant 1$. If $f(x) \in L^p_{[0,1]}$, $1 < p < +\infty$, changes sign l times in $(0,1)$, then there exist $0 < b_1 < b_2 < \cdots < b_l < 1$, a polynomial $P_n(x) \in \Pi_n(+)$ and a positive integer $N(b)$ only depending on b such that*

$$\left\| f(x) - \frac{\prod_{j=1}^{l}(x - b_j)}{P_n(x)} \right\|_{L^p} \leqslant C_{p,b,l}\,\omega(f, n^{-1/2})_{L^p}$$

holds for $n > N(b)$, where $b = \min\{|b_{j+1} - b_j| : j = 1, 2, \cdots, l-1\}$, $C_{p,b,l}$ is a positive constant only depending on p, b and l (independent of n and the function if b keeps unchanged).

The following definition of sign change of a function f in L^p spaces is adopted ([5]).

DEFINITION. Let $f(x) \in L^p_{[0,1]}$, $1 \leqslant p < \infty$. If there are l points $0 < a_1 < a_2 < \cdots < a_l < 1$ such that

$$\sigma(\Pi_{j=1}^l (x-a_j))f(x) \geqslant 0, x \in [0,1], \sigma = \pm 1,$$

and for every $j = 1,2,\cdots,l$ and any $0 < \eta < a_{j+1} - a_j (a_{l+1} = 1)$,

$$\mathrm{mes}(\{x \in (a_j, a_{j+1}) : f(x) \neq 0\} \cap (a_j, a_j + \eta)) > 0,$$

then we say $f(x)$ changes sign exactly l times at $a_1, a_2, \cdots, a_l$.

In fact, it is Leviatan and Lubinsky ([3]) who first established such kind of results for polynomials with real coefficients for $f(x)$ changing sign exactly l times. Their main tool used in the proof is the following important inequality:

THEOREM LL. *There is an absolute constant $C > 0$ with the following property: Let $-1 < b_1 < b_2 < \cdots < b_l < 1$, and set*

$$\rho(x) := \Pi_{j=1}^l (x - b_j).$$

Then there exists, for $n \geqslant 3l$, a polynomial $S(x)$ of degree $\leqslant n$ such that for $x \in [-1,1]$,

$$0 \leqslant 1 - \frac{|\rho(x)|}{S(x)} \leqslant \min\left\{1, \frac{Cl}{n} \sum_{j=1}^{l} \frac{\sqrt{1-b_j^2}}{|x-b_j|}\right\}.$$

We will establish an important inequality for polynomials with positive coefficients analogue to Theorem LL, and improve Theorem Z and Theorem M as applications.

In the present paper, C always stands for an absolute positive constant, and $C_{p,b}$ a positive constant only depending on p and b, their values may be different even in the same line.

2. An Inequality analogue to Theorem LL

THEOREM 2.1. *For any $0 < b_1 < b_2 < \cdots < b_l < 1$, let*

$$\rho(x) = \prod_{j=1}^{l} (x - b_j).$$

Then there exists a polynomial $S_n(x) \in \Pi_n(+)$ such that for any $x \in [0,1]$, $n \geqslant l$,

$$0 \leqslant 1 - \frac{|\rho(x)|}{S_n(x)} \leqslant \min\left\{1, \frac{Cl}{\sqrt{n}} \sum_{j=1}^{l} \frac{\varphi(x)}{|x-b_j|}\right\}. \tag{1}$$

REMARK 1. Obviously, inequality (1) has better estimate than that of Theorem LL in the sense that we use $\varphi(x)$ instead of $\varphi(b_j)$ in the right hand, which we believe will play important roles in establishing pointwise estimates.

LEMMA 2.1. ([2: Corollary 4.2]) *If* $f(x)$ *is convex on* $[0,1]$, *then*

$$B_n(f,x) \geqslant B_{n+1}(f,x) \geqslant f(x), 0 < x < 1,$$

where $B_n(f,x)$ *is the Bernstein polynomial of degree* n *of* $f(x)$ *defined as*

$$B_n(f,x) = \sum_{k=0}^{n} f(k/n) p_{n,k}(x),$$

and

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Proof of Theorem 2.1. For any $x_0 \in (0,1)$, Let $g(x) = |x - x_0|$. By noting that $g(x)$ is convex on $[0,1]$, thus, by Lemma 2.1, we have

$$B_n(g,x) \geqslant g(x),\ 0 < x < 1.$$

Therefore,

$$0 \leqslant 1 - \frac{g(x)}{B_n(g,x)} \leqslant 1,\ 0 < x < 1. \tag{2}$$

For any $0 < \alpha \leqslant 2$ (see DeVore [1]),

$$|B_n(f,x) - f(x)| \leqslant C \left(\frac{x(1-x)}{n} \right)^{\alpha/2}$$

if only if $\omega_2(f,h) = O(h^\alpha)$. It implies that

$$|B_n(g,x) - g(x)| \leqslant C \frac{\varphi(x)}{\sqrt{n}}. \tag{3}$$

Combining Lemma 2.1 with (3), we get

$$\begin{aligned} 0 \leqslant 1 - \frac{g(x)}{B_n(g,x)} &= \frac{|B_n(g,x) - g(x)|}{B_n(g,x)} \\ &\leqslant C \frac{\varphi(x)}{\sqrt{n} B_n(g,x)} \leqslant C \frac{\varphi(x)}{\sqrt{n} g(x)}. \end{aligned} \tag{4}$$

Set $B_{n,x_0}(x) := B_n(g,x) \in \Pi_n(+)$. By (2) and (4), we deduce that for any $x_0 \in (0,1)$,

$$0 \leqslant 1 - \frac{|x - x_0|}{B_{n,x_0}(x)} \leqslant \min\left\{ 1, \frac{C\varphi(x)}{\sqrt{n}|x - x_0|} \right\},\ 0 < x < 1. \tag{5}$$

Since $B_n(g,0) = g(0)$ and $B_n(g,1) = g(1)$, inequality (5) also holds for all $x \in [0,1]$.

From the proof of (5), for every $b_j,\ j = 1,2,\cdots,l,$ we actually find a polynomial $B_{n,b_j}(x)$ such that

$$0 \leqslant 1 - \frac{|x - b_j|}{B_{n,b_j}(x)} \leqslant \min\left\{ 1, \frac{C\varphi(x)}{\sqrt{n}|x - b_j|} \right\},\ 0 \leqslant x \leqslant 1.$$

Define

$$S_n(x) := \prod_{j=1}^{l} B_{[n/l],b_j}(x),$$

where $[x]$ denotes the greatest integer not exceeding x. We will proceed to prove Theorem 2.1 by the same manner as that of [3].

Obviously, $S_n(x)$ is polynomial with positive coefficients and has degree at most $l([n/l]) \leqslant n$. Moreover, by (5), for all $1 \leqslant j \leqslant l$ and $x \in [0,1]$,

$$B_{[n/l],b_j}(x) \geqslant |x - b_j|,$$

hence

$$S_n(x) \geqslant \prod_{j=1}^{l} |x - b_j| = |\rho(x)|.$$

Next, (5) also gives

$$\begin{aligned} 0 \leqslant 1 - \frac{|\rho(x)|}{S_n(x)} &= 1 - \prod_{j=1}^{l} \left(1 - \left(1 - \frac{|x-b_j|}{B_{[n/l],b_j}}\right)\right) \\ &\leqslant 1 - \prod_{j=1}^{l} \left(1 - \min\left\{1, \frac{Cl\varphi(x)}{\sqrt{n}|x-b_j|}\right\}\right) \\ &\leqslant \sum_{j=1}^{l} \min\left\{1, \frac{Cl\varphi(x)}{\sqrt{n}|x-b_j|}\right\} \end{aligned}$$

where we have used the inequality (see [3])

$$1 - \prod_{j=1}^{l}(1-y_j) \leqslant \sum_{j=1}^{l} y_j,\ y_j \in [0,1],\ 1 \leqslant j \leqslant l.$$

Together with the earlier estimates, we finish the proof of Theorem 2.1. $\square$

3. Applications

3.1. Generalization of Theorem Z

THEOREM 3.1. *Let $l \geqslant 1$. There exists an absolute constant $C > 0$ with the following property: If $f \in C_{[0,1]}$ changes sign exactly l times in $(0,1)$, say at $b_1, b_2, \cdots, b_l$, then for each $n \geqslant 1$, there is a polynomial $P_n \in \Pi_n(+)$, having the same sign as f in $(b_l, 1)$, and such that for $x \in [-1,1]$,*

$$\left| f(x) - \frac{(x-b_1)(x-b_2)\cdots(x-b_l)}{P_n(x)} \right| \leqslant C(l+1)^2 \omega_\varphi(f, n^{-1/2}).$$

REMARK 2. The rational function constructed in Theorem 3.1 is copositive with $f(x)$, while that of Theorem Z may not have this property since x_0 may not be the same sign changing point of $f(x)$.

Let $s,t \in [0,1/2]$, set $a := \frac{s+t}{2}$, we claim that

$$\varphi(a) \geqslant \frac{1}{2}\varphi(s),\ \varphi(a) \geqslant \frac{1}{2}\varphi(t). \tag{6}$$

In fact, without loss of generality, we may assume that $a \geqslant 1/2$, then

$$\varphi^2(a) = a(1-a) \geqslant \frac{1}{2}(1-a) = \frac{1}{2}\left(\frac{1-s}{2} + \frac{1-t}{2}\right)$$

$$\geqslant \frac{1}{4}(1-s) \geqslant \frac{1}{4}\varphi^2(s).$$

By (6) and a similar discussion of [3, Lemma 3.5], we obtain

LEMMA 3.1. *There exists an absolute constant C such that for $s,t \in [0,1]$ and $f \in C_{[0,1]}$,*

$$|f(s) - f(t)| \min\left\{1, \frac{\varphi(s)}{\sqrt{n}|s-t|}\right\} \leqslant C\omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right). \tag{7}$$

LEMMA 3.2. *If $f \in C_{[0,1]}$ has a zero in $[0,1]$, then there exists an absolute constant C such that*

$$|f(x)| \leqslant \omega_\varphi(f,4).$$

Proof. Let $f(b) = 0$. For any $x \in [0,1]$, write

$$a := \frac{1}{2}(x+b); \quad h\varphi(a) := |x-b|.$$

Then

$$|f(x)| = |f(x) - f(b)| = \left|f\left(a + \frac{h}{2}\varphi(a)\right) - f\left(a - \frac{h}{2}\varphi(a)\right)\right| \leqslant \omega_\varphi(f,h).$$

Hence we only need to prove $h \leqslant 4$.

If both x and b are no less than $\frac{1}{2}$, then by noting that (see (6))

$$\varphi(a) \geqslant \frac{1}{2}\varphi(x), \quad \varphi(a) \geqslant \frac{1}{2}\varphi(b),$$

we have

$$h = \left|\frac{1-x-(1-b)}{\varphi(a)}\right| \leqslant \frac{\max\{1-x, 1-b\}}{\varphi(a)}$$

$$\leqslant \max\left\{2\frac{1-x}{\varphi(x)}, 2\frac{1-b}{\varphi(b)}\right\} \leqslant 4\max\left\{\frac{x(1-x)}{\varphi(x)}, \frac{b(1-b)}{\varphi(b)}\right\} \leqslant 4.$$

If both x and b are no larger than $\frac{1}{2}$, then a similar discussion also leads to $h \leqslant 4$.

If one of x and b is no large than $\frac{1}{2}$ and the other is no less than $\frac{1}{2}$, say $a \geqslant \frac{1}{2}$, then

$$\varphi^2(a) = a(1-a) \geqslant \frac{1}{2}(1-a) = \frac{1}{2}\left(\frac{1-x}{2} + \frac{1-b}{2}\right) \geqslant \frac{1}{8},$$

so

$$h = |x-b|/\varphi(a) \leqslant 2\sqrt{2}.$$

Proof of Theorem 3.1 Theorem 3.1 can be proved by Lemma 3.1, Lemma 3.2 with following the line of [3, Theorem 2.1], we omit the details here.

3.2. Improvement of Theorem M.

We improve Theorem M by establishing that

THEOREM 3.2. *Let l be a nonnegative integer. If $f(x) \in L^p_{[0,1]}$, $1 < p \leqslant \infty$, changes sign exactly l times on $(0,1)$, then there exist $0 < b_1 < b_2 < \cdots < b_l < 1$, a polynomial $P_n(x) \in \Pi_n(+)$ and a positive integer $N(b)$ only depending on b such that*

$$\left\| f(x) - \frac{\prod_{j=1}^{l}(x-b_j)}{P_n(x)} \right\|_{L^p} \leqslant C_p(l+1)^2\omega(f, n^{-1/2})_{L^p}$$

holds for $n > N(b)$.

REMARK 3. We improve Theorem M by using $C_p(l+1)^2$ to replace $C_{p,b,l}$, and the method used in this paper is more efficient and simpler.

Without loss of generality, we always assume that $l \geqslant 1$.

We need the following lemmas.

LEMMA 3.3. ([5]) *Let $f(x) \in L^p_{[-1,1]}, 1 \leqslant p \leqslant \infty$, change sign exactly l times in $(0,1)$. Write*

$$f_h(x) = \frac{1}{h}\int_{x-h/2}^{x+h/2} f(u)du$$

as the Steklov function of $f(x)$. Then for sufficiently small $h > 0$, $f_h(x)$ also changes sign exactly l times on $(h/2, 1-h/2)$.

LEMMA 3.4. ([8]) *Let $f \in L^p_{[0,1]}$. Extend f to a function $F_N \in L^p_{[-1,2]}$ on the interval $[-1,2]$ as follows:*

$$F_N(x) := \begin{cases} f(x), & x \in [0,1], \\ N\int_{1-1/N}^{1} f(t)dt, & x \in (1,2], \\ N\int_0^{1/N} f(t)dt, & x \in [-1,0), \end{cases}$$

where N be a positive integer. Then

$$\omega\left(F_N(x), N^{-1}\right)_{L^p_{[-1,2]}} \leqslant C\omega(f, N^{-1})_{L^p}.$$

By the definition, we observe that $F_N(x)$ has the same number of sign change points as $f(x)$ for sufficient large N. Denote by $F_{N,h}$ the corresponding Steklov function of $F_N(x)$, then by Lemma 3.3, we see that $F_{N,h}$ has the same sign change number as that of $f(x)$. Altogether the above observation, with Lemma 3.4, and the well known properties of Steklov functions, leads to

LEMMA 3.5. *Let* $f \in L^p_{[0,1]}, 1 \leqslant p \leqslant \infty$, *then*

$$\|F_N - F_{N,h}\|_{L^p_{[-1+h/2,2-h/2]}} \leqslant C\omega(F_N,h)_{L^p_{[-1,2]}} \leqslant C\omega(f,h)_p, \tag{8}$$

and

$$\|(F_{N,h})'\|_{L^p_{[-1+h/2,2-h/2]}} \leqslant C\omega(F_N,h)_{L^p_{[-1,2]}} \leqslant C\omega(f,h)_p. \tag{9}$$

LEMMA 3.6. ([8]) *Let* $f(x) \in L^p_{[0,1]}, 1 < p \leqslant \infty$, $f(x) \geqslant 0$, $x \in [0,1]$, *and* $f \not\equiv 0$.[1] *Then there exists a polynomial* $Q_n \in \Pi_n(+)$ *such that*

$$\left\| f - \frac{1}{Q_n} \right\|_{L^p} \leqslant C_p\omega(f, n^{-1/2})_{L^p}.$$

Proof of Theorem 3.2. We need to prove Theorem 3.2 in case $1 < p < \infty$ by induction on l, the number of sign changes. Assume that $f(x) \in L^p_{[0,1]}$, $1 < p < \infty$, changes sign l times in $(0,1)$, then as we have pointed out $F_{N,h}(x)$ also changes sign l times in $(0,1)$, say at $0 < b_1 < b_2 < \cdots < b_l < 1$, for sufficient large N and sufficient small $h > 0$. From now on, we will always take $N = h^{-1} = n^{-1/2}$.

When $l = 1$, according to Theorem 2.1, there exists a polynomial $B_n(x) \in \Pi_n(+)$ such that

$$0 \leqslant 1 - \frac{|x-b_1|}{B_n(x)} \leqslant \min\left\{1, \frac{C\varphi(x)}{\sqrt{n}|x-b_1|}\right\}. \tag{10}$$

We restrict $F_{N,h}(x)$ on $[0,1]$. By Lemma 3.6, there exists a polynomial $Q_n(x) \in \Pi_n(+)$ such that (by (8))

$$\begin{aligned}
\left\| |F_{N,h}| - \frac{1}{Q_n} \right\|_{L^p} &\leqslant C_p\omega(|F_{N,h}|, n^{-1/2})_{L^p} \leqslant C_p\omega(F_{N,h}, n^{-1/2})_{L^p} \\
&\leqslant C_p\left(\|F_N - F_{N,h}\|_{L^p} + \omega(F_N, n^{-1/2})_{L^p}\right) \\
&\leqslant C_p\omega(f, n^{-1/2})_{L^p}.
\end{aligned} \tag{11}$$

[1] $f \not\equiv 0$ means $\operatorname{mes}(x : f \neq 0) > 0$.

Let $P_n(x) = B_n(x)Q_n(x)$, then

$$\begin{aligned}
\left|F_{N,h}(x) - \frac{(x-b_1)}{P_n(x)}\right| &= \left||F_{N,h}(x)| - \frac{|x-b_1|}{P_n(x)}\right| \\
&= \left||F_{N,h}(x)|\left(1 - \frac{|x-b_1|}{B_n(x)}\right) + \frac{|x-b_1|}{B_n(x)}\left(|F_{N,h}(x)| - \frac{1}{Q_n(x)}\right)\right| \\
&\leqslant |F_{N,h}(x)| \min\left\{1, \frac{C}{\sqrt{n}|x-b_1|}\right\} + \frac{|x-b_1|}{B_n(x)}\left(|F_{N,h}(x)| - \frac{1}{Q_n(x)}\right) \\
&=: I_1 + I_2. \qquad (12)
\end{aligned}$$

By (11) and the inequality (see (10))

$$\frac{|x-b_1|}{B_n(x)} \leqslant 1, x \in [0,1],$$

we have

$$\|I_2\|_{L^p} \leqslant C_p \omega(f, n^{-1/2})_{L^p}. \qquad (13)$$

Define the Hardy-Littlewood maximum function $M(f,x)$ by

$$M(f,x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(t)| dt,$$

then it is well-known that

$$\|M(f)\|_{L^p} \leqslant C_p \|f\|_{L^p}, \ p > 1. \qquad (14)$$

Since $F_{N,h}(x) \in C_{[0,1]}$, then $F_{N,h}(b_1) = 0$. For any $x \neq b_1$ and $p > 1$, it holds that

$$\begin{aligned}
|F_{N,h}(x)|^p \min\left\{1, \frac{1}{\sqrt{n}|x-b_1|}\right\}^p &= \left(|F_{N,h}(x) - F_{N,h}(b_1)| \frac{1}{\sqrt{n}|x-b_1|}\right)^p \\
&\leqslant \frac{1}{(\sqrt{n})^p}(M((F_{N,h})'^p. \qquad (15)
\end{aligned}$$

From (9), (10), (14) and (15), we have

$$\begin{aligned}
\|I_1\|_{L^p} &\leqslant \tfrac{1}{\sqrt{n}} \|M((F_{N,h})')\|_{L^p} \leqslant \tfrac{C_p}{\sqrt{n}} \|(F_{N,h})'\|_{L^p} \\
&\leqslant C_p \omega(f, n^{-1/2})_{L^p}. \qquad (16)
\end{aligned}$$

Combining (12), (13) and (16) yields that

$$\left\|F_{N,h}(x) - \frac{(x-b_1)}{P_n(x)}\right\|_{L^p} \leqslant C_p \omega(f, n^{-1/2})_{L^p}. \qquad (17)$$

With (8) and (17), we get

$$\begin{aligned}\left\|f(x)-\frac{(x-b_1)}{P_n(x)}\right\|_{L^p} &\leqslant \|f-F_{N,h}\|_{L^p}+\left\|F_{N,h}(x)-\frac{(x-b_1)}{P_n(x)}\right\|_{L^p}\\ &=\|F_N-F_{N,h}\|_{L^p}+\left\|F_{N,h}(x)-\frac{(x-b_1)}{P_n(x)}\right\|\\ &\leqslant C_p\omega(f,n^{-1/2})_{L^p}.\end{aligned}$$

Thus Theorem 3.2 holds for $l=1$.

Assume that Theorem 3.2 holds in case $f(x)$ changes sign $l-1$ times, that is, there exists a polynomial $A_n(x)\in\Pi_n(+)$ such that

$$\left\|f(x)-\frac{\prod_{j=1}^{l-1}(x-b_j)}{A_n(x)}\right\|_{L^p}\leqslant C_pl^2\omega(f,n^{-1/2})_{L^p}.$$

Set $\tilde{F}_{N,h}(x)=F_{N,h}(x)\operatorname{sgn}(x-b_l)$, For $\tilde{F}_{N,h}(x)$, we obviously have

$$\omega(\tilde{F}_{N,h},t)_{L^p}\leqslant\omega(F_{N,h},t)_{L^p}\leqslant\omega(f,t)_{L^p}. \tag{18}$$

Now, $\tilde{F}_{N,h}(x)$ change sign $l-1$ times in $(0,1)$. By the assumption and (18), there exists a polynomial $C_n(x)\in\Pi_n(+)$ such that

$$\left\|\tilde{F}_{N,h}(x)-\frac{\prod_{j=1}^{l-1}(x-b_j)}{C_n(x)}\right\|_{L^p}\leqslant C_pl^2\omega(f,n^{-1/2})_{L^p}.$$

Employing Theorem 2.1 again, we see that there exists a polynomial $D_n(x)\in\Pi_n(+)$ such that

$$\left|1-\frac{|x-b_l|}{D_n(x)}\right|\leqslant\min\left\{1,\frac{C}{\sqrt{n}|x-b_l|}\right\}.$$

Define

$$E_n(x)=C_n(x)D_n(x),$$

then

$$\begin{aligned}\left\|F_{N,h}(x)-\frac{\prod_{j=1}^{l}(x-b_j)}{E_n(x)}\right\|_{L^p} &= \left\|\tilde{F}_{N,h}(x)\operatorname{sgn}(x-b_l)-\frac{\prod_{j=1}^{l}(x-b_j)}{C_n(x)D_n(x)}\right\|_{L^p}\\ &\leqslant \left\|\tilde{F}_{N,h}\left(\operatorname{sgn}(x-b_l)-\frac{(x-b_l)}{D_n(x)}\right)\right\|_{L^p}\\ &\quad+\left\|\left(\tilde{F}_{N,h}(x)-\frac{\prod_{j=1}^{l-1}(x-b_j)}{C_n(x)}\right)\frac{(x-b_l)}{D_n(x)}\right\|_{L^p}\\ &\leqslant \left\||\tilde{F}_{N,h}(x)|\left|1-\frac{|x-b_l|}{D_n(x)}\right|\right\|_{L^p}\\ &\quad+\left\|\left|\tilde{F}_{N,h}(x)-\frac{\prod_{j=1}^{l-1}(x-b_j)}{C_n(x)}\right|\frac{|x-b_l|}{D_n(x)}\right\|_{L^p}\\ &:=J_1+J_2.\end{aligned}$$

Note that $|\tilde{F}_{N,h}(x)| = |F_{N,h}(x)|$ and $F_{N,h}(b_l) = 0$, then repeat the proof of the case $l = 1$, we can easily deduce that

$$J_1 \leqslant C_p \omega(f, n^{-1/2})_{L^p},$$

and

$$J_2 \leqslant C_p l^2 \omega(f, n^{-1/2})_{L^p}.$$

Finally, we achieve that

$$\begin{aligned}\left\| f(x) - \frac{\prod_{j=1}^{l}(x-b_j)}{E_n(x)} \right\|_{L^p} &\leqslant \|f - F_{N,h}\|_{L^p} + \left\| F_{N,h}(x) - \frac{\prod_{j=1}^{l}(x-b_j)}{E_n(x)} \right\|_{L^p} \\ &\leqslant C_p (l+1)^2 \omega(f, n^{-1/2})_{L^p}\end{aligned}$$

to complete Theorem 1. □

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