

Q-NORM INEQUALITIES FOR SEQUENCES OF HILBERT SPACE OPERATORS

S. S. DRAGOMIR, M. S. MOSLEHIAN AND J. SÁNDOR

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Abstract. We give some inequalities related to a large class of operator norms, the so-called Q -norms, for a (not necessary commutative) family of bounded linear operators acting on a Hilbert space that are related to the classical Schwarz inequality. Applications for vector inequalities are also provided.

1. Introduction and preliminaries

Let $(\mathcal{H}; \langle \cdot, \cdot \rangle)$ be a real or complex Hilbert space, $\mathbb{B}(\mathcal{H})$ the Banach algebra of all bounded linear operators acting on \mathcal{H} and I denote the identity operator on \mathcal{H} .

In many estimates one needs to use upper bounds for the norm of a sum of products $A_1, \dots, A_n, B_1, \dots, B_n \in \mathbb{B}(\mathcal{H})$, where separate information for norms of operators are provided. The special case in which $B_i = \alpha_i I$ ($\alpha_i \in \mathbb{C}, 1 \leq i \leq n$) and the norms in the question are the operator norm has already investigated in [4]. The celebrated Hölder's discrete inequality stating

$$\left| \sum_{i=1}^n \alpha_i \beta_i \right| \leq \left(\sum_{i=1}^n |\alpha_i|^p \right)^{1/p} \left(\sum_{i=1}^n |\beta_i|^q \right)^{1/q} \quad (1)$$

$$\left(1 < p, \quad 1 < q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \alpha_i, \beta_i \in \mathbb{C} \quad (1 \leq i \leq n) \right)$$

and its variance in the special case $p = q = 2$ (the so-called Cauchy–Bunyakovsky–Schwarz (CBS) discrete inequality) are of special interest and of larger utility.

To each symmetric gauge functions defined on the sequences of real numbers there corresponds a unitarily invariant norm $||| \cdot |||$ defined on a two-sided ideal $\mathcal{C}_{||| \cdot |||}$ of $\mathbb{B}(\mathcal{H})$ enjoying the invariant property $||| UAV ||| = ||| A |||$ for all $A \in \mathcal{C}_{||| \cdot |||}$ and all unitary operators $U, V \in \mathbb{B}(\mathcal{H})$. It is known that $\mathcal{C}_{||| \cdot |||}$ is a Banach space under the norm $||| \cdot |||$. If $B \in \mathcal{C}_{||| \cdot |||}$, then $||| B^* ||| = ||| B |||$ and $||| AB ||| \leq ||| A ||| ||| B |||$ for all $A \in \mathbb{B}(\mathcal{H})$, from which we easily conclude that

$$||| ABC ||| \leq ||| A ||| ||| B ||| ||| C ||| \quad (A, C \in \mathbb{B}(\mathcal{H})).$$

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Some well-known examples of unitarily invariant norms are the Schatten p -norms $\|B\|_p := \text{tr}(|B|^p)^{1/p}$ for $1 \leq p < \infty$, operator norm $\|\cdot\|$, and the Ky-Fan norms; cf. [1, 6]. A unitarily invariant norm $\|\cdot\|_Q$ is called a Q -norm if there is a unitarily invariant norm $\|\cdot\|_{\hat{Q}}$ such that $\|BB^*\|_{\hat{Q}} = \|B\|_Q^2$.

The aim of the present paper is to establish some upper bounds of interest for the quantity $\|\sum_{i=1}^n A_i B_i\|_Q$ under certain assumptions on A_i 's and B_i 's. Our results are significant since we do not use any commutative assumption on the families of operators and as well we provide inequalities for a class of norms which are larger than operator norms (compare the results with those of [4]). Applications for vector inequalities are also given.

We need the following lemmas concerning real numbers (the second lemma is obvious):

LEMMA 1.1. (*Young's inequality; see [5, p. 30 or p. 49]*) If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for all $a, b > 0$.

LEMMA 1.2. If $a, b > 0$, then

$$ab \leq \frac{1}{4}(a+b)^2 \leq \frac{1}{2}(a^2 + b^2).$$

LEMMA 1.3. (*Daykin-Eliezer; see [2]*) If $a_k > 1, b_k > 1$ and $\frac{1}{p} + \frac{1}{q} < 1$, then

$$\left(\sum_{k=1}^n a_k b_k \right)^{\frac{1}{p} + \frac{1}{q}} \leq \left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n b_k^q \right)^{\frac{1}{q}}.$$

2. Some General Results

The following result containing 9 different inequalities may be stated:

THEOREM 2.1. Let $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ and $B_1, \dots, B_n \in \mathcal{C}_Q$. Then

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \begin{cases} A \\ B \\ C \end{cases} \quad (2)$$

where

$$A := \begin{cases} \max_{1 \leq k \leq n} \|A_k\|^2 \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}, \\ \max_{1 \leq k \leq n} \|A_k\| \left(\sum_{i=1}^n \|A_i\|^r \right)^{\frac{1}{r}} \left(\sum_{i=1}^n \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right)^s \right)^{\frac{1}{s}}, \\ \quad \text{where } r > 1, \frac{1}{r} + \frac{1}{s} = 1, \\ \max_{1 \leq k \leq n} \|A_k\| \sum_{i=1}^n \|A_i\| \max_{1 \leq i \leq n} \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right), \end{cases}$$

$$B := \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \left(\sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \sum_{i=1}^n \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}}, \\ \left(\sum_{i=1}^n \|A_i\|^t \right)^{\frac{1}{t}} \left(\sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \left[\sum_{i=1}^n \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, \\ \quad \text{where } t > 1, \frac{1}{t} + \frac{1}{u} = 1, \\ \sum_{i=1}^n \|A_i\| \left(\sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \max_{1 \leq i \leq n} \left\{ \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}} \right\}, \end{cases} \quad (3)$$

for $p > 1, \frac{1}{p} + \frac{1}{q} = 1$; and

$$C := \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \sum_{k=1}^n \|A_k\| \sum_{i=1}^n \max_{1 \leq j \leq n} \left\{ \|B_i B_j^*\|_{\widehat{Q}} \right\}, \\ \left(\sum_{i=1}^n \|A_i\|^m \right)^{\frac{1}{m}} \sum_{k=1}^n \|A_k\| \left[\sum_{i=1}^n \left(\max_{1 \leq j \leq n} \left\{ \|B_i B_j^*\|_{\widehat{Q}} \right\} \right)^l \right]^{\frac{1}{l}}, \\ \quad \text{where } m, l > 1, \frac{1}{m} + \frac{1}{l} = 1, \\ \left(\sum_{k=1}^n \|A_k\| \right)^2 \max_{1 \leq i, j \leq n} \left\{ \|B_i B_j^*\|_{\widehat{Q}} \right\}. \end{cases}$$

Proof. In the operator partial order of $\mathbb{B}(\mathcal{H})$, we have

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^n B_i A_i \right)^* \left(\sum_{i=1}^n B_i A_i \right) \\ &= \sum_{i=1}^n A_i^* B_i^* \sum_{j=1}^n B_j A_j = \sum_{i,j=1}^n A_i^* B_i^* B_j A_j. \end{aligned} \quad (4)$$

Taking the norm in (4) and noticing that $\|BB^*\|_{\widehat{Q}} = \|B\|_{\widehat{Q}}^2$ for any $B \in \mathcal{C}_Q$, we have

$$\begin{aligned} \left\| \sum_{i=1}^n A_i B_i \right\|_{\widehat{Q}}^2 &= \left\| \sum_{i=1}^n \sum_{j=1}^n A_i^* B_i^* B_j A_j \right\|_{\widehat{Q}} \\ &\leq \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\widehat{Q}} \\ &= \sum_{i=1}^n \|A_i\| \left(\sum_{j=1}^n \|A_j\| \|B_j B_j^*\|_{\widehat{Q}} \right) =: M. \end{aligned}$$

Utilizing Hölder's discrete inequality we have that

$$\sum_{j=1}^n \|A_j\| \|B_j B_j^*\|_{\widehat{Q}} \leq \begin{cases} \max_{1 \leq k \leq n} \|A_k\| \sum_{j=1}^n \|B_j B_j^*\|_{\widehat{Q}}, \\ \left(\sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n \|B_j B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}}, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \sum_{k=1}^n \|A_k\| \max_{1 \leq j \leq n} \|B_j B_j^*\|_{\widehat{Q}}, \end{cases}$$

for any $i \in \{1, \dots, n\}$.

This provides the following inequalities:

$$M \leq \begin{cases} \max_{1 \leq k \leq n} \|A_k\| \sum_{i=1}^n \|A_i\| \left(\sum_{j=1}^n \|B_j B_j^*\|_{\widehat{Q}} \right) =: M_1, \\ \left(\sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \sum_{i=1}^n \|A_i\| \left(\sum_{j=1}^n \|B_j B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}} := M_p, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \sum_{k=1}^n \|A_k\| \sum_{i=1}^n \|A_i\| \left(\max_{1 \leq j \leq n} \|B_j B_j^*\|_{\widehat{Q}} \right) := M_\infty. \end{cases}$$

Utilizing Hölder's inequality for $r, s > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, we have:

$$\sum_{i=1}^n \|A_i\| \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right) \leq \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}, \\ \left(\sum_{i=1}^n \|A_i\|^r \right)^{\frac{1}{r}} \left[\sum_{i=1}^n \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right)^s \right]^{\frac{1}{s}}, \\ \text{where } r > 1, \frac{1}{r} + \frac{1}{s} = 1, \\ \sum_{i=1}^n \|A_i\| \max_{1 \leq i \leq n} \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right), \end{cases}$$

and thus we can state that

$$M_1 \leq \begin{cases} \max_{1 \leq k \leq n} \|A_k\|^2 \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}, \\ \max_{1 \leq k \leq n} \|A_k\| \left(\sum_{i=1}^n \|A_i\|^r \right)^{\frac{1}{r}} \left(\sum_{i=1}^n \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right)^s \right)^{\frac{1}{s}}, \\ \text{where } r > 1, \frac{1}{r} + \frac{1}{s} = 1, \\ \max_{1 \leq k \leq n} \|A_k\| \sum_{i=1}^n \|A_i\| \max_{1 \leq i \leq n} \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right), \end{cases}$$

and the first part of the theorem is proved.

By Hölder's inequality we can also have that (for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$)

$$M_p \leq \left(\sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \times \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \sum_{i=1}^n \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}}, \\ \left(\sum_{i=1}^n \|A_i\|^t \right)^{\frac{1}{t}} \left[\sum_{i=1}^n \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, \\ \text{where } t > 1, \frac{1}{t} + \frac{1}{u} = 1, \\ \sum_{i=1}^n \|A_i\| \max_{1 \leq i \leq n} \left\{ \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}} \right\}, \end{cases}$$

and the second part of (2) is proved.

Finally, we may state that

$$M_\infty \leq \sum_{k=1}^n \|A_k\| \times \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \sum_{i=1}^n \max_{1 \leq j \leq n} \left\{ \|B_i B_j^*\|_{\hat{Q}} \right\}, \\ \left(\sum_{i=1}^n \|A_i\|^m \right)^{\frac{1}{m}} \left[\sum_{i=1}^n \left(\max_{1 \leq j \leq n} \left\{ \|B_i B_j^*\|_{\hat{Q}} \right\} \right)^l \right]^{\frac{1}{l}}, \\ \text{where } m, l > 1, \frac{1}{m} + \frac{1}{l} = 1, \\ \sum_{i=1}^n \|A_i\| \max_{1 \leq i, j \leq n} \left\{ \|B_i B_j^*\|_{\hat{Q}} \right\}, \end{cases}$$

giving the last part of (2).

REMARK 2.2. One can obtain various particular inequalities as well. For instance, by using the usual Cauchy–Bunyakovsky–Schwarz inequality and norm properties of the space \mathcal{C}_Q we get

$$\begin{aligned} \left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 &\leq \left(\sum_{i=1}^n \|A_i B_i\|_Q \right)^2 \\ &\leq \left(\sum_{i=1}^n \|A_i\| \|B_i\|_Q \right)^2 \\ &\leq \sum_{i=1}^n \|A_i\|^2 \sum_{i=1}^n \|B_i\|_Q^2. \end{aligned} \quad (5)$$

If we consider again the Cauchy–Bunyakovsky–Schwarz inequality

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \|A_i\|_Q^2 \sum_{i=1}^n \|B_i\|_Q^2 \quad (6)$$

then we can conclude that (5) is a refinement of the (CBS) inequality (6) whenever the operator norm $\|\cdot\|$ is majorized by $\|\cdot\|_Q$.

COROLLARY 2.3. Let $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ and $B_1, \dots, B_n \in \mathcal{C}_Q$ so that $B_i B_j^* = 0$ with $i \neq j$. Then

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \begin{cases} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{cases} \quad (7)$$

where

$$\tilde{A} := \begin{cases} \max_{1 \leq k \leq n} \|A_k\|^2 \sum_{i=1}^n \|B_i\|_Q^2, \\ \max_{1 \leq k \leq n} \|A_k\| \left(\sum_{i=1}^n \|A_i\|^r \right)^{\frac{1}{r}} \left(\sum_{i=1}^n \|B_i\|_Q^{2s} \right)^{\frac{1}{s}}, \\ \quad \text{where } r > 1, \frac{1}{r} + \frac{1}{s} = 1, \\ \max_{1 \leq k \leq n} \|A_k\| \sum_{i=1}^n \|A_i\| \max_{1 \leq i \leq n} \left\{ \|B_i\|_Q^2 \right\}, \end{cases}$$

$$\tilde{B} := \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \left(\sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \sum_{i=1}^n \|B_i\|_Q^2, \\ \left(\sum_{i=1}^n \|A_i\|^t \right)^{\frac{1}{t}} \left(\sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \left[\sum_{i=1}^n \|B_i\|_Q^{2u} \right]^{\frac{1}{u}}, \\ \quad \text{where } t > 1, \frac{1}{t} + \frac{1}{u} = 1, \\ \sum_{i=1}^n \|A_i\| \left(\sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \max_{1 \leq i \leq n} \left\{ \|B_i\|_Q^2 \right\}, \end{cases}$$

where $p > 1$ and

$$\tilde{C} := \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \sum_{k=1}^n \|A_k\| \sum_{i=1}^n \|B_i\|_Q^2, \\ \left(\sum_{i=1}^n \|A_i\|^m \right)^{\frac{1}{m}} \sum_{k=1}^n \|A_k\| \left(\sum_{i=1}^n \|B_i\|_Q^{2l} \right)^{\frac{1}{l}}, \\ \quad \text{where } m, l > 1, \frac{1}{m} + \frac{1}{l} = 1, \\ \left(\sum_{k=1}^n \|A_k\| \right)^2 \max_{1 \leq i, j \leq n} \left\{ \|B_i\|_Q^2 \right\}. \end{cases}$$

As in the proof of the Theorem 2.1 one has

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \|A_i\| \left(\sum_{j=1}^n \|A_j\| \|B_i B_j^*\|_{\hat{Q}} \right) =: M$$

Using Hölder's generalized inequality from Lemma 1.3, and assuming that $\|A_j\| > 1$, $\|B_i B_j^*\|_{\hat{Q}} > 1$, we get

$$\left(\sum_{j=1}^n \|A_j\| \|B_i B_j^*\|_{\hat{Q}} \right) \leq \left(\sum_{k=1}^n \|A_k\|^p \right)^{\frac{q}{p+q}} \left(\sum_{j=1}^n \|B_i B_j^*\|_{\hat{Q}}^q \right)^{\frac{p}{p+q}}.$$

Thus the following result is true:

THEOREM 2.4. *Let $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ and $B_1, \dots, B_n \in \mathcal{C}_Q$. Assume that $\|A_j\| > 1$ and $\|B_i B_j^*\|_{\widehat{Q}} > 1$ for all $i, j \in \{1, 2, \dots, n\}$. Then*

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \left(\sum_{k=1}^n \|A_k\|^p \right)^{\frac{q}{p+q}} \sum_{i=1}^n \|A_i\| \left(\sum_{j=1}^n \|B_i B_j^*\|_Q^q \right)^{\frac{p}{p+q}},$$

where $\frac{1}{p} + \frac{1}{q} < 1$.

3. Other Results

A different approach is embodied in the following theorem:

THEOREM 3.1. *If $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ and $B_1, \dots, B_n \in \mathcal{C}_Q$, then*

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \|A_i\|^2 \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \tag{8}$$

$$\leq \begin{cases} \sum_{i=1}^n \|A_i\|^2 \max_{1 \leq i \leq n} \left[\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right], \\ \left(\sum_{i=1}^n \|A_i\|^{2p} \right)^{\frac{1}{p}} \left[\sum_{i=1}^n \left(\sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right)^q \right]^{\frac{1}{q}}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}. \end{cases}$$

Proof. From the proof of Theorem 2.1 we have that

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\widehat{Q}}.$$

Using the simple observation that

$$\|A_i\| \|A_j\| \leq \frac{1}{2} \left(\|A_i\|^2 + \|A_j\|^2 \right), \quad i, j \in \{1, \dots, n\},$$

we get

$$\begin{aligned}
 \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\widehat{Q}} &\leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left[\|A_i\|^2 + \|A_j\|^2 \right] \|B_i B_j^*\|_{\widehat{Q}} \\
 &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left[\|A_i\|^2 \|B_i B_j^*\|_{\widehat{Q}} + \|A_j\|^2 \|B_j B_i^*\|_{\widehat{Q}} \right] \\
 &= \sum_{i=1}^n \|A_i\|^2 \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}},
 \end{aligned}$$

which proves the first inequality in (8).

The second part follows by Hölder's inequality and the details are omitted.

REMARK 3.2. If in (8) we choose $A_1 = \dots = A_n = I$, then we get

$$\left\| \sum_{i=1}^n B_i \right\|_Q \leq \left(\sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right)^{1/2} \leq \left(\sum_{i,j=1}^n \|B_i\|_{\widehat{Q}} \|B_j^*\|_{\widehat{Q}} \right)^{1/2} = \left\| \sum_{i=1}^n B_i \right\|_{\widehat{Q}},$$

which is a refinement for the generalized triangle inequality for $\|\cdot\|_{\widehat{Q}}$, whenever $\|\cdot\|_Q$ is majorized by $\|\cdot\|_{\widehat{Q}}$.

The following corollary may be stated:

COROLLARY 3.3. If $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ and $B_1, \dots, B_n \in \mathcal{C}_Q$ are such that $B_i B_j^* = 0$ for $i \neq j$, $i, j \in \{1, \dots, n\}$, then

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \|A_i\|^2 \|B_i\|_Q^2 \tag{9}$$

$$\leq \begin{cases} \sum_{i=1}^n \|A_i\|^2 \max_{1 \leq i \leq n} \|B_i\|_Q^2, \\ \left(\sum_{i=1}^n \|A_i\|^{2p} \right)^{\frac{1}{p}} \left[\sum_{j=1}^n \|B_j\|_Q^{2q} \right]^{\frac{1}{q}}, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i=1}^n \|B_i\|_Q^2. \end{cases}$$

THEOREM 3.4. If $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ and $B_1, \dots, B_n \in \mathcal{C}_Q$, then

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \begin{cases} \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}, \\ \left(\sum_{i=1}^n \|A_i\|^p \right)^{\frac{2}{p}} \left(\sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}}, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left(\sum_{i=1}^n \|A_i\| \right)^2 \max_{1 \leq i,j \leq n} \left\{ \|B_i B_j^*\|_{\widehat{Q}} \right\}. \end{cases} \quad (10)$$

Proof. We know that

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\widehat{Q}} =: P.$$

Firstly, we obviously have that

$$P \leq \max_{1 \leq i,j \leq n} \{ \|A_i\| \|A_j\| \} \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}} = \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}.$$

Secondly, by the Hölder inequality for double sums, we obtain

$$\begin{aligned} P &\leq \left[\sum_{i,j=1}^n (\|A_i\| \|A_j\|)^p \right]^{\frac{1}{p}} \left(\sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^n \|A_i\|^p \sum_{j=1}^n \|A_j\|^p \right)^{\frac{1}{p}} \left(\sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^n \|A_i\|^p \right)^{\frac{2}{p}} \left(\sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}}, \end{aligned}$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$.

Finally, we have

$$\begin{aligned} P &\leq \max_{1 \leq i,j \leq n} \left\{ \|B_i B_j^*\|_{\widehat{Q}} \right\} \sum_{i,j=1}^n \|A_i\| \|A_j\| \\ &= \left(\sum_{i=1}^n \|A_i\| \right)^2 \max_{1 \leq i,j \leq n} \left\{ \|B_i B_j^*\|_{\widehat{Q}} \right\} \end{aligned}$$

and the theorem is proved.

COROLLARY 3.5. *If $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ and $B_1, \dots, B_n \in \mathcal{C}_Q$ are such that $B_i B_j^* = 0$ for $i, j \in \{1, \dots, n\}$ with $i \neq j$, then*

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \begin{cases} \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i=1}^n \|B_i\|_Q^2, \\ \left(\sum_{i=1}^n \|A_i\|^p \right)^{\frac{2}{p}} \left(\sum_{i=1}^n \|B_i\|_Q^{2q} \right)^{\frac{1}{q}}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left(\sum_{i=1}^n \|A_i\| \right)^2 \max_{1 \leq i \leq n} \left\{ \|B_i\|_Q^2 \right\}. \end{cases} \quad (11)$$

THEOREM 3.6. *If $\frac{1}{p} + \frac{1}{q} < 1$ and $\|A_i\| > 1$ and $\|B_i B_j^*\| > 1$ for all $1 \leq i, j \leq n$ then*

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \left(\sum_{i=1}^n \|A_i\|^p \right)^{\frac{2q}{p+q}} \left(\sum_{i,j=1}^n \|B_i B_j^*\|_{\hat{Q}}^q \right)^{\frac{p}{p+q}}.$$

Proof. Using Lemma 1.3 for double sums, we have, as in the proof of Theorem 3.4, that

$$p = \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\hat{Q}} \leq \left(\sum_{i,j=1}^n (\|A_i\| \|A_j\|)^p \right)^{\frac{q}{p+q}} \left(\sum_{i,j=1}^n \|B_i B_j^*\|_{\hat{Q}}^q \right)^{\frac{p}{p+q}},$$

if we assume that $\|A_i\| > 1$ and $\|B_i B_j^*\| > 1$ for all $1 \leq i, j \leq n$.

Thus, we get

$$p \leq \left(\sum_{i=1}^n \|A_i\|^p \right)^{\frac{2q}{p+q}} \left(\sum_{i,j=1}^n \|B_i B_j^*\|_{\hat{Q}}^q \right)^{\frac{p}{p+q}},$$

if $\frac{1}{p} + \frac{1}{q} < 1$.

Finally, the following result may be stated as well:

THEOREM 3.7. *If $\frac{1}{p} + \frac{1}{q} = 1$, $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ and $B_1, \dots, B_n \in \mathcal{C}_Q$, then*

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \left(\frac{1}{p} \sum_{i=1}^n \|A_i\|^p + \frac{1}{q} \sum_{i=1}^n \|A_i\|^q \right) \max_{1 \leq i \leq n} \left[\sum_{j=1}^n \|B_i B_j^*\|_{\hat{Q}} \right].$$

Proof. As in the proof of Theorem 3.1 we have

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\hat{Q}}.$$

Now, using Lemma 1.1, we can write

$$\|A_i\| \|A_j\| \leq \frac{1}{p} \|A_i\|^p + \frac{1}{q} \|A_j\|^q.$$

Thus, we have

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\hat{Q}} &\leq \sum_{i=1}^n \sum_{j=1}^n \left[\frac{1}{p} \|A_i\|^p + \frac{1}{q} \|A_j\|^q \right] \|B_i B_j^*\|_{\hat{Q}} \\ &= \frac{1}{p} \sum_{i=1}^n \sum_{j=1}^n \|A_i\|^p \|B_i B_j^*\|_{\hat{Q}} + \frac{1}{q} \sum_{i=1}^n \sum_{j=1}^n \|A_j\|^q \|B_i B_j^*\|_{\hat{Q}} \\ &= \frac{1}{p} \sum_{i=1}^n \|A_i\|^p \sum_{j=1}^n \|B_i B_j^*\|_{\hat{Q}} + \frac{1}{q} \sum_{j=1}^n \|A_j\|^q \sum_{i=1}^n \|B_i B_j^*\|_{\hat{Q}} \\ &\leq \left(\frac{1}{p} \sum_{i=1}^n \|A_i\|^p + \frac{1}{q} \sum_{j=1}^n \|A_j\|^q \right) \max_{1 \leq i \leq n} \left[\sum_{j=1}^n \|B_i B_j^*\|_{\hat{Q}} \right]. \end{aligned}$$

REMARK 3.8. Another variant may be obtained via Lemma 1.2:

$$\|A_i\| \|A_j\| \leq \frac{1}{4} (\|A_i\| + \|A_j\|)^2, \text{ so}$$

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\hat{Q}} &\leq \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n (\|A_i\| + \|A_j\|)^2 \|B_i B_j^*\|_{\hat{Q}} \\ &\leq \sup_{1 \leq i \leq n} \sum_{j=1}^n \|B_i B_j^*\|_{\hat{Q}} \left(\sum_{i,j=1}^n (\|A_i\| + \|A_j\|)^2 \right). \end{aligned}$$

4. Applications

Throughout this section we assume $\|\cdot\|_Q$ (and hence $\|\cdot\|_{\hat{Q}}$) to be the operator norm. If by $M(\mathbf{B}, \mathbf{A})$ we denote any of the bounds provided by (2), (5), (8) or (10) for the quantity $\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2$, then we may state the following general fact:

Under the assumptions of Theorem 2.1, we have:

$$\left\| \sum_{i=1}^n A_i B_i x \right\|_Q^2 \leq \|x\|^2 M(\mathbf{B}, \mathbf{A}). \quad (12)$$

for any $x \in \mathcal{H}$ and

$$\left| \sum_{i=1}^n \langle A_i B_i x, y \rangle \right|^2 \leq \|x\|^2 \|y\|^2 M(\mathbf{B}, \mathbf{A}). \quad (13)$$

for any $x, y \in \mathcal{H}$, respectively.

The proof follows by the Schwarz inequality in the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$.

Now, we consider the non zero vectors $y_1, \dots, y_n \in \mathcal{H}$. Define $B_i : \mathcal{H} \rightarrow \mathcal{H}$ by $B_i = \frac{y_i \otimes y_i}{\|y_i\|} (1 \leq i \leq n)$. Then

$$\|B_i B_j^*\| = \frac{\|\langle y_j, y_i \rangle y_i \otimes y_j\|}{\|y_i\| \|y_j\|} = |\langle y_i, y_j \rangle| \quad (1 \leq i, j \leq n).$$

If $(y_i)_{1 \leq i \leq n}$ is an orthogonal family on \mathcal{H} , then $\|B_i\| = 1$ and $B_i B_j = 0$ for $i, j \in \{1, \dots, n\}$, $i \neq j$.

Now, utilizing, for instance, the inequalities in Theorem 3.1 we may state that:

$$\left\| \sum_{i=1}^n A_i \frac{\langle x, y_i \rangle}{\|y_i\|} y_i \right\|^2 \leq \|x\|^2 \sum_{i=1}^n \|A_i\|^2 \sum_{j=1}^n |\langle y_i, y_j \rangle| \quad (14)$$

$$\leq \|x\|^2 \times \begin{cases} \sum_{i=1}^n \|A_i\|^2 \max_{1 \leq i \leq n} \left[\sum_{j=1}^n |\langle y_i, y_j \rangle| \right], \\ \left(\sum_{i=1}^n \|A_i\|^{2p} \right)^{\frac{1}{p}} \left[\sum_{i=1}^n \left(\sum_{j=1}^n |\langle y_i, y_j \rangle| \right)^q \right]^{\frac{1}{q}}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i,j=1}^n |\langle y_i, y_j \rangle|. \end{cases}$$

for any $x, y_1, \dots, y_n \in \mathcal{H}$ and $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$.

The choice $A_i = \|y_i\| I (i = 1, \dots, n)$ will produce some interesting bounds for the norm of the Fourier series $\|\sum_{i=1}^n \langle x, y_i \rangle\|$. Notice the vectors $y_i (i = 1, \dots, n)$ are not necessarily orthonormal.

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S. S. Dragomir
 Research Group in Mathematical Inequalities & Applications
 School of Engineering & Science
 Victoria University
 P. O. Box 14428
 Melbourne City
 Victoria 8001
 Australia
 e-mail: sever.dragomir@vu.edu.au

M. S. Moslehian
 Department of Pure Mathematics
 Ferdowsi University of Mashhad
 P.O. Box 1159
 Mashhad 91775
 Iran
 Centre of Excellence in Analysis on Algebraic Structures (CEAAS)
 Ferdowsi University of Mashhad
 Iran
 e-mail: moslehian@ferdowsi.um.ac.ir and moslehian@ams.org

J. Sándor
 Department of Mathematics
 Babes-Bolyai University
 Str. Kogalniceanu
 400084 Cluj-Napoca
 Romania
 e-mail: jsandor@math.ubbcluj.ro