

## Q-NORM INEQUALITIES FOR SEQUENCES OF HILBERT SPACE OPERATORS

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*Abstract.* We give some inequalities related to a large class of operator norms, the so-called  $Q$ -norms, for a (not necessary commutative) family of bounded linear operators acting on a Hilbert space that are related to the classical Schwarz inequality. Applications for vector inequalities are also provided.

### 1. Introduction and preliminaries

Let  $(\mathcal{H}; \langle \cdot, \cdot \rangle)$  be a real or complex Hilbert space,  $\mathbb{B}(\mathcal{H})$  the Banach algebra of all bounded linear operators acting on  $\mathcal{H}$  and  $I$  denote the identity operator on  $\mathcal{H}$ .

In many estimates one needs to use upper bounds for the norm of a sum of products  $A_1, \dots, A_n, B_1, \dots, B_n \in \mathbb{B}(\mathcal{H})$ , where separate information for norms of operators are provided. The special case in which  $B_i = \alpha_i I$  ( $\alpha_i \in \mathbb{C}, 1 \leq i \leq n$ ) and the norms in the question are the operator norm has already investigated in [4]. The celebrated Hölder's discrete inequality stating

$$\left| \sum_{i=1}^n \alpha_i \beta_i \right| \leq \left( \sum_{i=1}^n |\alpha_i|^p \right)^{1/p} \left( \sum_{i=1}^n |\beta_i|^q \right)^{1/q} \quad (1)$$

$$\left( 1 < p, \quad 1 < q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \alpha_i, \beta_i \in \mathbb{C} \quad (1 \leq i \leq n) \right)$$

and its variance in the special case  $p = q = 2$  (the so-called Cauchy–Bunyakovsky–Schwarz (CBS) discrete inequality) are of special interest and of larger utility.

To each symmetric gauge functions defined on the sequences of real numbers there corresponds a unitarily invariant norm  $\|\cdot\|$  defined on a two-sided ideal  $\mathcal{C}_{\|\cdot\|}$  of  $\mathbb{B}(\mathcal{H})$  enjoying the invariant property  $\|UAV\| = \|A\|$  for all  $A \in \mathcal{C}_{\|\cdot\|}$  and all unitary operators  $U, V \in \mathbb{B}(\mathcal{H})$ . It is known that  $\mathcal{C}_{\|\cdot\|}$  is a Banach space under the norm  $\|\cdot\|$ . If  $B \in \mathcal{C}_{\|\cdot\|}$ , then  $\|B^*\| = \|B\|$  and  $\|AB\| \leq \|A\| \|B\|$  for all  $A \in \mathbb{B}(\mathcal{H})$ , from which we easily conclude that

$$\| \|ABC\| \leq \|A\| \|B\| \|C\| \quad (A, C \in \mathbb{B}(\mathcal{H})).$$

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Some well-known examples of unitarily invariant norms are the Schatten  $p$ -norms  $\|B\|_p := \text{tr}(|B|^p)^{1/p}$  for  $1 \leq p < \infty$ , operator norm  $\|\cdot\|$ , and the Ky-Fan norms; cf. [1, 6]. A unitarily invariant norm  $\|\cdot\|_Q$  is called a  $Q$ -norm if there is a unitarily invariant norm  $\|\cdot\|_{\hat{Q}}$  such that  $\|BB^*\|_{\hat{Q}} = \|B\|_Q^2$ .

The aim of the present paper is to establish some upper bounds of interest for the quantity  $\|\sum_{i=1}^n A_i B_i\|_Q$  under certain assumptions on  $A_i$ 's and  $B_i$ 's. Our results are significant since we do not use any commutative assumption on the families of operators and as well we provide inequalities for a class of norms which are larger than operator norms (compare the results with those of [4]). Applications for vector inequalities are also given.

We need the following lemmas concerning real numbers (the second lemma is obvious):

LEMMA 1.1. (*Young's inequality; see [5, p. 30 or p. 49]*) If  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for all  $a, b > 0$ .

LEMMA 1.2. If  $a, b > 0$ , then

$$ab \leq \frac{1}{4}(a+b)^2 \leq \frac{1}{2}(a^2 + b^2).$$

LEMMA 1.3. (*Daykin-Eliezer; see [2]*) If  $a_k > 1, b_k > 1$  and  $\frac{1}{p} + \frac{1}{q} < 1$ , then

$$\left( \sum_{k=1}^n a_k b_k \right)^{\frac{1}{p} + \frac{1}{q}} \leq \left( \sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^n b_k^q \right)^{\frac{1}{q}}.$$

## 2. Some General Results

The following result containing 9 different inequalities may be stated:

THEOREM 2.1. Let  $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$  and  $B_1, \dots, B_n \in \mathcal{C}_Q$ . Then

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \begin{cases} A \\ B \\ C \end{cases} \quad (2)$$

where

$$A := \begin{cases} \max_{1 \leq k \leq n} \|A_k\|^2 \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}, \\ \max_{1 \leq k \leq n} \|A_k\| \left( \sum_{i=1}^n \|A_i\|^r \right)^{\frac{1}{r}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right)^s \right)^{\frac{1}{s}}, \\ \quad \text{where } r > 1, \frac{1}{r} + \frac{1}{s} = 1, \\ \max_{1 \leq k \leq n} \|A_k\| \sum_{i=1}^n \|A_i\| \max_{1 \leq i \leq n} \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right), \end{cases}$$

$$B := \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \left( \sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \sum_{i=1}^n \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}}, \\ \left( \sum_{i=1}^n \|A_i\|^t \right)^{\frac{1}{t}} \left( \sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \left[ \sum_{i=1}^n \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, \\ \quad \text{where } t > 1, \frac{1}{t} + \frac{1}{u} = 1, \\ \sum_{i=1}^n \|A_i\| \left( \sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \max_{1 \leq i \leq n} \left\{ \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}} \right\}, \end{cases} \quad (3)$$

for  $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ ; and

$$C := \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \sum_{k=1}^n \|A_k\| \sum_{i=1}^n \max_{1 \leq j \leq n} \left\{ \|B_i B_j^*\|_{\widehat{Q}} \right\}, \\ \left( \sum_{i=1}^n \|A_i\|^m \right)^{\frac{1}{m}} \sum_{k=1}^n \|A_k\| \left[ \sum_{i=1}^n \left( \max_{1 \leq j \leq n} \left\{ \|B_i B_j^*\|_{\widehat{Q}} \right\} \right)^l \right]^{\frac{1}{l}}, \\ \quad \text{where } m, l > 1, \frac{1}{m} + \frac{1}{l} = 1, \\ \left( \sum_{k=1}^n \|A_k\| \right)^2 \max_{1 \leq i, j \leq n} \left\{ \|B_i B_j^*\|_{\widehat{Q}} \right\}. \end{cases}$$

*Proof.* In the operator partial order of  $\mathbb{B}(\mathcal{H})$ , we have

$$\begin{aligned} 0 &\leq \left( \sum_{i=1}^n B_i A_i \right)^* \left( \sum_{i=1}^n B_i A_i \right) \\ &= \sum_{i=1}^n A_i^* B_i^* \sum_{j=1}^n B_j A_j = \sum_{i,j=1}^n A_i^* B_i^* B_j A_j. \end{aligned} \quad (4)$$

Taking the norm in (4) and noticing that  $\|BB^*\|_{\widehat{Q}} = \|B\|_{\widehat{Q}}^2$  for any  $B \in \mathcal{C}_Q$ , we have

$$\begin{aligned} \left\| \sum_{i=1}^n A_i B_i \right\|_{\widehat{Q}}^2 &= \left\| \sum_{i=1}^n \sum_{j=1}^n A_i^* B_i^* B_j A_j \right\|_{\widehat{Q}} \\ &\leq \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\widehat{Q}} \\ &= \sum_{i=1}^n \|A_i\| \left( \sum_{j=1}^n \|A_j\| \|B_j B_j^*\|_{\widehat{Q}} \right) =: M. \end{aligned}$$

Utilizing Hölder's discrete inequality we have that

$$\sum_{j=1}^n \|A_j\| \|B_j B_j^*\|_{\widehat{Q}} \leq \begin{cases} \max_{1 \leq k \leq n} \|A_k\| \sum_{j=1}^n \|B_j B_j^*\|_{\widehat{Q}}, \\ \left( \sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n \|B_j B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}}, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \sum_{k=1}^n \|A_k\| \max_{1 \leq j \leq n} \|B_j B_j^*\|_{\widehat{Q}}, \end{cases}$$

for any  $i \in \{1, \dots, n\}$ .

This provides the following inequalities:

$$M \leq \begin{cases} \max_{1 \leq k \leq n} \|A_k\| \sum_{i=1}^n \|A_i\| \left( \sum_{j=1}^n \|B_j B_j^*\|_{\widehat{Q}} \right) =: M_1, \\ \left( \sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \sum_{i=1}^n \|A_i\| \left( \sum_{j=1}^n \|B_j B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}} := M_p, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \sum_{k=1}^n \|A_k\| \sum_{i=1}^n \|A_i\| \left( \max_{1 \leq j \leq n} \|B_j B_j^*\|_{\widehat{Q}} \right) := M_\infty. \end{cases}$$

Utilizing Hölder's inequality for  $r, s > 1$ ,  $\frac{1}{r} + \frac{1}{s} = 1$ , we have:

$$\sum_{i=1}^n \|A_i\| \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right) \leq \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}, \\ \left( \sum_{i=1}^n \|A_i\|^r \right)^{\frac{1}{r}} \left[ \sum_{i=1}^n \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right)^s \right]^{\frac{1}{s}}, \\ \text{where } r > 1, \frac{1}{r} + \frac{1}{s} = 1, \\ \sum_{i=1}^n \|A_i\| \max_{1 \leq i \leq n} \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right), \end{cases}$$

and thus we can state that

$$M_1 \leq \begin{cases} \max_{1 \leq k \leq n} \|A_k\|^2 \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}, \\ \max_{1 \leq k \leq n} \|A_k\| \left( \sum_{i=1}^n \|A_i\|^r \right)^{\frac{1}{r}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right)^s \right)^{\frac{1}{s}}, \\ \text{where } r > 1, \frac{1}{r} + \frac{1}{s} = 1, \\ \max_{1 \leq k \leq n} \|A_k\| \sum_{i=1}^n \|A_i\| \max_{1 \leq i \leq n} \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right), \end{cases}$$

and the first part of the theorem is proved.

By Hölder's inequality we can also have that (for  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ )

$$M_p \leq \left( \sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \times \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \sum_{i=1}^n \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}}, \\ \left( \sum_{i=1}^n \|A_i\|^t \right)^{\frac{1}{t}} \left[ \sum_{i=1}^n \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, \\ \text{where } t > 1, \frac{1}{t} + \frac{1}{u} = 1, \\ \sum_{i=1}^n \|A_i\| \max_{1 \leq i \leq n} \left\{ \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}} \right\}, \end{cases}$$

and the second part of (2) is proved.

Finally, we may state that

$$M_\infty \leq \sum_{k=1}^n \|A_k\| \times \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \sum_{i=1}^n \max_{1 \leq j \leq n} \left\{ \|B_i B_j^*\|_{\hat{Q}} \right\}, \\ \left( \sum_{i=1}^n \|A_i\|^m \right)^{\frac{1}{m}} \left[ \sum_{i=1}^n \left( \max_{1 \leq j \leq n} \left\{ \|B_i B_j^*\|_{\hat{Q}} \right\} \right)^l \right]^{\frac{1}{l}}, \\ \text{where } m, l > 1, \frac{1}{m} + \frac{1}{l} = 1, \\ \sum_{i=1}^n \|A_i\| \max_{1 \leq i, j \leq n} \left\{ \|B_i B_j^*\|_{\hat{Q}} \right\}, \end{cases}$$

giving the last part of (2).

REMARK 2.2. One can obtain various particular inequalities as well. For instance, by using the usual Cauchy–Bunyakovsky–Schwarz inequality and norm properties of the space  $\mathcal{C}_Q$  we get

$$\begin{aligned} \left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 &\leq \left( \sum_{i=1}^n \|A_i B_i\|_Q \right)^2 \\ &\leq \left( \sum_{i=1}^n \|A_i\| \|B_i\|_Q \right)^2 \\ &\leq \sum_{i=1}^n \|A_i\|^2 \sum_{i=1}^n \|B_i\|_Q^2. \end{aligned} \quad (5)$$

If we consider again the Cauchy–Bunyakovsky–Schwarz inequality

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \|A_i\|_Q^2 \sum_{i=1}^n \|B_i\|_Q^2 \quad (6)$$

then we can conclude that (5) is a refinement of the (CBS) inequality (6) whenever the operator norm  $\|\cdot\|$  is majorized by  $\|\cdot\|_Q$ .

COROLLARY 2.3. Let  $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$  and  $B_1, \dots, B_n \in \mathcal{C}_Q$  so that  $B_i B_j^* = 0$  with  $i \neq j$ . Then

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \begin{cases} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{cases} \quad (7)$$

where

$$\tilde{A} := \begin{cases} \max_{1 \leq k \leq n} \|A_k\|^2 \sum_{i=1}^n \|B_i\|_Q^2, \\ \max_{1 \leq k \leq n} \|A_k\| \left( \sum_{i=1}^n \|A_i\|^r \right)^{\frac{1}{r}} \left( \sum_{i=1}^n \|B_i\|_Q^{2s} \right)^{\frac{1}{s}}, \\ \quad \text{where } r > 1, \frac{1}{r} + \frac{1}{s} = 1, \\ \max_{1 \leq k \leq n} \|A_k\| \sum_{i=1}^n \|A_i\| \max_{1 \leq i \leq n} \left\{ \|B_i\|_Q^2 \right\}, \end{cases}$$

$$\tilde{B} := \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \left( \sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \sum_{i=1}^n \|B_i\|_Q^2, \\ \left( \sum_{i=1}^n \|A_i\|^t \right)^{\frac{1}{t}} \left( \sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \left[ \sum_{i=1}^n \|B_i\|_Q^{2u} \right]^{\frac{1}{u}}, \\ \quad \text{where } t > 1, \frac{1}{t} + \frac{1}{u} = 1, \\ \sum_{i=1}^n \|A_i\| \left( \sum_{k=1}^n \|A_k\|^p \right)^{\frac{1}{p}} \max_{1 \leq i \leq n} \left\{ \|B_i\|_Q^2 \right\}, \end{cases}$$

where  $p > 1$  and

$$\tilde{C} := \begin{cases} \max_{1 \leq i \leq n} \|A_i\| \sum_{k=1}^n \|A_k\| \sum_{i=1}^n \|B_i\|_Q^2, \\ \left( \sum_{i=1}^n \|A_i\|^m \right)^{\frac{1}{m}} \sum_{k=1}^n \|A_k\| \left( \sum_{i=1}^n \|B_i\|_Q^{2l} \right)^{\frac{1}{l}}, \\ \quad \text{where } m, l > 1, \frac{1}{m} + \frac{1}{l} = 1, \\ \left( \sum_{k=1}^n \|A_k\| \right)^2 \max_{1 \leq i, j \leq n} \left\{ \|B_i\|_Q^2 \right\}. \end{cases}$$

As in the proof of the Theorem 2.1 one has

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \|A_i\| \left( \sum_{j=1}^n \|A_j\| \|B_i B_j^*\|_{\hat{Q}} \right) =: M$$

Using Hölder's generalized inequality from Lemma 1.3, and assuming that  $\|A_j\| > 1$ ,  $\|B_i B_j^*\|_{\hat{Q}} > 1$ , we get

$$\left( \sum_{j=1}^n \|A_j\| \|B_i B_j^*\|_{\hat{Q}} \right) \leq \left( \sum_{k=1}^n \|A_k\|^p \right)^{\frac{q}{p+q}} \left( \sum_{j=1}^n \|B_i B_j^*\|_{\hat{Q}}^q \right)^{\frac{p}{p+q}}.$$

Thus the following result is true:

**THEOREM 2.4.** *Let  $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$  and  $B_1, \dots, B_n \in \mathcal{C}_Q$ . Assume that  $\|A_j\| > 1$  and  $\|B_i B_j^*\|_{\widehat{Q}} > 1$  for all  $i, j \in \{1, 2, \dots, n\}$ . Then*

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \left( \sum_{k=1}^n \|A_k\|^p \right)^{\frac{q}{p+q}} \sum_{i=1}^n \|A_i\| \left( \sum_{j=1}^n \|B_i B_j^*\|_Q^q \right)^{\frac{p}{p+q}},$$

where  $\frac{1}{p} + \frac{1}{q} < 1$ .

### 3. Other Results

A different approach is embodied in the following theorem:

**THEOREM 3.1.** *If  $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$  and  $B_1, \dots, B_n \in \mathcal{C}_Q$ , then*

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \|A_i\|^2 \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \tag{8}$$

$$\leq \begin{cases} \sum_{i=1}^n \|A_i\|^2 \max_{1 \leq i \leq n} \left[ \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right], \\ \left( \sum_{i=1}^n \|A_i\|^{2p} \right)^{\frac{1}{p}} \left[ \sum_{i=1}^n \left( \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right)^q \right]^{\frac{1}{q}}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}. \end{cases}$$

*Proof.* From the proof of Theorem 2.1 we have that

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\widehat{Q}}.$$

Using the simple observation that

$$\|A_i\| \|A_j\| \leq \frac{1}{2} \left( \|A_i\|^2 + \|A_j\|^2 \right), \quad i, j \in \{1, \dots, n\},$$



we get

$$\begin{aligned}
 \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\widehat{Q}} &\leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left[ \|A_i\|^2 + \|A_j\|^2 \right] \|B_i B_j^*\|_{\widehat{Q}} \\
 &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left[ \|A_i\|^2 \|B_i B_j^*\|_{\widehat{Q}} + \|A_j\|^2 \|B_j B_i^*\|_{\widehat{Q}} \right] \\
 &= \sum_{i=1}^n \|A_i\|^2 \sum_{j=1}^n \|B_i B_j^*\|_{\widehat{Q}},
 \end{aligned}$$

which proves the first inequality in (8).

The second part follows by Hölder's inequality and the details are omitted.

REMARK 3.2. If in (8) we choose  $A_1 = \dots = A_n = I$ , then we get

$$\left\| \sum_{i=1}^n B_i \right\|_Q \leq \left( \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}} \right)^{1/2} \leq \left( \sum_{i,j=1}^n \|B_i\|_{\widehat{Q}} \|B_j^*\|_{\widehat{Q}} \right)^{1/2} = \left\| \sum_{i=1}^n B_i \right\|_{\widehat{Q}},$$

which is a refinement for the generalized triangle inequality for  $\|\cdot\|_{\widehat{Q}}$ , whenever  $\|\cdot\|_Q$  is majorized by  $\|\cdot\|_{\widehat{Q}}$ .

The following corollary may be stated:

COROLLARY 3.3. If  $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$  and  $B_1, \dots, B_n \in \mathcal{C}_Q$  are such that  $B_i B_j^* = 0$  for  $i \neq j$ ,  $i, j \in \{1, \dots, n\}$ , then

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \|A_i\|^2 \|B_i\|_Q^2 \tag{9}$$

$$\leq \begin{cases} \sum_{i=1}^n \|A_i\|^2 \max_{1 \leq i \leq n} \|B_i\|_Q^2, \\ \left( \sum_{i=1}^n \|A_i\|^{2p} \right)^{\frac{1}{p}} \left[ \sum_{j=1}^n \|B_j\|_Q^{2q} \right]^{\frac{1}{q}}, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i=1}^n \|B_i\|_Q^2. \end{cases}$$

THEOREM 3.4. If  $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$  and  $B_1, \dots, B_n \in \mathcal{C}_Q$ , then

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \begin{cases} \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}, \\ \left( \sum_{i=1}^n \|A_i\|^p \right)^{\frac{2}{p}} \left( \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}}, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left( \sum_{i=1}^n \|A_i\| \right)^2 \max_{1 \leq i,j \leq n} \left\{ \|B_i B_j^*\|_{\widehat{Q}} \right\}. \end{cases} \quad (10)$$

*Proof.* We know that

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\widehat{Q}} =: P.$$

Firstly, we obviously have that

$$P \leq \max_{1 \leq i,j \leq n} \{ \|A_i\| \|A_j\| \} \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}} = \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}.$$

Secondly, by the Hölder inequality for double sums, we obtain

$$\begin{aligned} P &\leq \left[ \sum_{i,j=1}^n (\|A_i\| \|A_j\|)^p \right]^{\frac{1}{p}} \left( \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}} \\ &= \left( \sum_{i=1}^n \|A_i\|^p \sum_{j=1}^n \|A_j\|^p \right)^{\frac{1}{p}} \left( \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}} \\ &= \left( \sum_{i=1}^n \|A_i\|^p \right)^{\frac{2}{p}} \left( \sum_{i,j=1}^n \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{1}{q}}, \end{aligned}$$

where  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

Finally, we have

$$\begin{aligned} P &\leq \max_{1 \leq i,j \leq n} \left\{ \|B_i B_j^*\|_{\widehat{Q}} \right\} \sum_{i,j=1}^n \|A_i\| \|A_j\| \\ &= \left( \sum_{i=1}^n \|A_i\| \right)^2 \max_{1 \leq i,j \leq n} \left\{ \|B_i B_j^*\|_{\widehat{Q}} \right\} \end{aligned}$$

and the theorem is proved.

COROLLARY 3.5. *If  $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$  and  $B_1, \dots, B_n \in \mathcal{C}_Q$  are such that  $B_i B_j^* = 0$  for  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ , then*

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \begin{cases} \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i=1}^n \|B_i\|_Q^2, \\ \left( \sum_{i=1}^n \|A_i\|^p \right)^{\frac{2}{p}} \left( \sum_{i=1}^n \|B_i\|_Q^{2q} \right)^{\frac{1}{q}}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left( \sum_{i=1}^n \|A_i\| \right)^2 \max_{1 \leq i \leq n} \left\{ \|B_i\|_Q^2 \right\}. \end{cases} \quad (11)$$

THEOREM 3.6. *If  $\frac{1}{p} + \frac{1}{q} < 1$  and  $\|A_i\| > 1$  and  $\|B_i B_j^*\| > 1$  for all  $1 \leq i, j \leq n$  then*

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \left( \sum_{i=1}^n \|A_i\|^p \right)^{\frac{2q}{p+q}} \left( \sum_{i,j=1}^n \|B_i B_j^*\|_{\hat{Q}}^q \right)^{\frac{p}{p+q}}.$$

*Proof.* Using Lemma 1.3 for double sums, we have, as in the proof of Theorem 3.4, that

$$p = \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\hat{Q}} \leq \left( \sum_{i,j=1}^n (\|A_i\| \|A_j\|)^p \right)^{\frac{q}{p+q}} \left( \sum_{i,j=1}^n \|B_i B_j^*\|_{\hat{Q}}^q \right)^{\frac{p}{p+q}},$$

if we assume that  $\|A_i\| > 1$  and  $\|B_i B_j^*\| > 1$  for all  $1 \leq i, j \leq n$ .

Thus, we get

$$p \leq \left( \sum_{i=1}^n \|A_i\|^p \right)^{\frac{2q}{p+q}} \left( \sum_{i,j=1}^n \|B_i B_j^*\|_{\hat{Q}}^q \right)^{\frac{p}{p+q}},$$

if  $\frac{1}{p} + \frac{1}{q} < 1$ .

Finally, the following result may be stated as well:

THEOREM 3.7. *If  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$  and  $B_1, \dots, B_n \in \mathcal{C}_Q$ , then*

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \left( \frac{1}{p} \sum_{i=1}^n \|A_i\|^p + \frac{1}{q} \sum_{i=1}^n \|A_i\|^q \right) \max_{1 \leq i \leq n} \left[ \sum_{j=1}^n \|B_i B_j^*\|_{\hat{Q}} \right].$$

*Proof.* As in the proof of Theorem 3.1 we have

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\hat{Q}}.$$

Now, using Lemma 1.1, we can write

$$\|A_i\| \|A_j\| \leq \frac{1}{p} \|A_i\|^p + \frac{1}{q} \|A_j\|^q.$$

Thus, we have

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\hat{Q}} &\leq \sum_{i=1}^n \sum_{j=1}^n \left[ \frac{1}{p} \|A_i\|^p + \frac{1}{q} \|A_j\|^q \right] \|B_i B_j^*\|_{\hat{Q}} \\ &= \frac{1}{p} \sum_{i=1}^n \sum_{j=1}^n \|A_i\|^p \|B_i B_j^*\|_{\hat{Q}} + \frac{1}{q} \sum_{i=1}^n \sum_{j=1}^n \|A_j\|^q \|B_i B_j^*\|_{\hat{Q}} \\ &= \frac{1}{p} \sum_{i=1}^n \|A_i\|^p \sum_{j=1}^n \|B_i B_j^*\|_{\hat{Q}} + \frac{1}{q} \sum_{j=1}^n \|A_j\|^q \sum_{i=1}^n \|B_i B_j^*\|_{\hat{Q}} \\ &\leq \left( \frac{1}{p} \sum_{i=1}^n \|A_i\|^p + \frac{1}{q} \sum_{j=1}^n \|A_j\|^q \right) \max_{1 \leq i \leq n} \left[ \sum_{j=1}^n \|B_i B_j^*\|_{\hat{Q}} \right]. \end{aligned}$$

REMARK 3.8. Another variant may be obtained via Lemma 1.2:

$$\|A_i\| \|A_j\| \leq \frac{1}{4} (\|A_i\| + \|A_j\|)^2, \text{ so}$$

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \|B_i B_j^*\|_{\hat{Q}} &\leq \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n (\|A_i\| + \|A_j\|)^2 \|B_i B_j^*\|_{\hat{Q}} \\ &\leq \sup_{1 \leq i \leq n} \sum_{j=1}^n \|B_i B_j^*\|_{\hat{Q}} \left( \sum_{i,j=1}^n (\|A_i\| + \|A_j\|)^2 \right). \end{aligned}$$

#### 4. Applications

Throughout this section we assume  $\|\cdot\|_Q$  (and hence  $\|\cdot\|_{\hat{Q}}$ ) to be the operator norm. If by  $M(\mathbf{B}, \mathbf{A})$  we denote any of the bounds provided by (2), (5), (8) or (10) for the quantity  $\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2$ , then we may state the following general fact:

*Under the assumptions of Theorem 2.1, we have:*

$$\left\| \sum_{i=1}^n A_i B_i x \right\|_Q^2 \leq \|x\|^2 M(\mathbf{B}, \mathbf{A}). \quad (12)$$

for any  $x \in \mathcal{H}$  and

$$\left| \sum_{i=1}^n \langle A_i B_i x, y \rangle \right|^2 \leq \|x\|^2 \|y\|^2 M(\mathbf{B}, \mathbf{A}). \quad (13)$$

for any  $x, y \in \mathcal{H}$ , respectively.

The proof follows by the Schwarz inequality in the Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ .

Now, we consider the non zero vectors  $y_1, \dots, y_n \in \mathcal{H}$ . Define  $B_i : \mathcal{H} \rightarrow \mathcal{H}$  by  $B_i = \frac{y_i \otimes y_i}{\|y_i\|} (1 \leq i \leq n)$ . Then

$$\|B_i B_j^*\| = \frac{\|\langle y_j, y_i \rangle y_i \otimes y_j\|}{\|y_i\| \|y_j\|} = |\langle y_i, y_j \rangle| \quad (1 \leq i, j \leq n).$$

If  $(y_i)_{1 \leq i \leq n}$  is an orthogonal family on  $\mathcal{H}$ , then  $\|B_i\| = 1$  and  $B_i B_j = 0$  for  $i, j \in \{1, \dots, n\}$ ,  $i \neq j$ .

Now, utilizing, for instance, the inequalities in Theorem 3.1 we may state that:

$$\left\| \sum_{i=1}^n A_i \frac{\langle x, y_i \rangle}{\|y_i\|} y_i \right\|^2 \leq \|x\|^2 \sum_{i=1}^n \|A_i\|^2 \sum_{j=1}^n |\langle y_i, y_j \rangle| \quad (14)$$

$$\leq \|x\|^2 \times \begin{cases} \sum_{i=1}^n \|A_i\|^2 \max_{1 \leq i \leq n} \left[ \sum_{j=1}^n |\langle y_i, y_j \rangle| \right], \\ \left( \sum_{i=1}^n \|A_i\|^{2p} \right)^{\frac{1}{p}} \left[ \sum_{i=1}^n \left( \sum_{j=1}^n |\langle y_i, y_j \rangle| \right)^q \right]^{\frac{1}{q}}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i,j=1}^n |\langle y_i, y_j \rangle|. \end{cases}$$

for any  $x, y_1, \dots, y_n \in \mathcal{H}$  and  $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ .

The choice  $A_i = \|y_i\| I (i = 1, \dots, n)$  will produce some interesting bounds for the norm of the Fourier series  $\|\sum_{i=1}^n \langle x, y_i \rangle\|$ . Notice the vectors  $y_i (i = 1, \dots, n)$  are not necessarily orthonormal.

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