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ON CERTAIN COEFFICIENT INEQUALITIES FOR MULTIVALENT FUNCTIONS

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Abstract. In the present investigation, the authors obtain sharp upper bounds for certain coefficient inequalities for linear combination of Mocanu α-convex $p-$ valent functions. The results are extended to functions defined by convolution.

1. Introduction

Let $\mathscr{A}_p$ denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1.1}$$

defined on the *open* unit disk

$$\Delta = \{z : z \in \mathbb{C} : |z| < 1\}$$

and let $\mathscr{A}_1 := \mathscr{A}$. For $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

their convolution (or Hadamard product), denoted by $(f * g)$ is defined as

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Then we say that the function f is *subordinate* to g if there exists a Schwarz function $\omega(z)$, analytic in Δ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \Delta),$$

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such that

$$f(z) = g(\omega(z)) \quad (z \in \Delta).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta).$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the open unit disk Δ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Ali *et al.*[1] defined and studied the classes $S^*_{b,p}(\phi)$ consisting of functions in $f \in \mathscr{A}_p$ for which

$$1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z) \quad (z \in \Delta, \quad b \in \mathbb{C} \setminus \{0\}),$$

and the class $C_{b,p}(\phi)$ of all functions in $f \in \mathscr{A}_p$ for which

$$1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \phi(z) \quad (z \in \Delta, \quad b \in \mathbb{C} \setminus \{0\}).$$

Note that $S^*_{1,1}(\phi) = S^*(\phi)$ and $C_{1,1}(\phi) = C(\phi)$, the classes introduced and studied by Ma and Minda [3]. The familiar class $S^*(\alpha)$ of *starlike functions* of order α and the class $C(\alpha)$ of *convex functions* of order α, $0 \leqslant \alpha < 1$ are the special case of $S^*_{1,1}(\phi)$ and $C_{1,1}(\phi)$ respectively when $\phi(z) = (1 + (1-2\alpha)z)/(1-z)$.

We now define a class of functions which unifies the classes $S^*_{b,p}(\phi)$ and $C_{b,p}(\phi)$ in the following:

DEFINITION 1.1. Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the open unit disk Δ onto a region in the right half plane and is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in \mathscr{A}_p$ is in the class $M_{p,b,\alpha,\lambda}(\phi)$ if

$$1 + \frac{1}{b}\left[\frac{1}{p}\left((1-\alpha)\frac{zF'(z)}{F(z)} + \alpha\left(1 + \frac{zF''(z)}{F'(z)}\right)\right) - 1\right] \prec \phi(z) \quad (0 \leqslant \alpha \leqslant 1), \qquad (1.2)$$

where

$$F(z) := (1-\lambda)f(z) + \lambda z f'(z).$$

Also, $M_{p,b,\alpha,\lambda,g}(\phi)$ is the class of all functions $f \in \mathscr{A}_p$ for which $f * g \in M_{p,b,\alpha,\lambda}(\phi)$. The classes $M_{p,b,\alpha,\lambda}(\phi)$ reduce to the following classes.

1. $M_{1,1,1,0}(\phi) \equiv C(\phi)$ [3].
2. $M_{1,1,0,0}(\phi) \equiv S^*(\phi)$ [3].

3. $M_{p,1,0,0}(\phi) \equiv S_p^*(\phi)$ introduced and studied by Ali *et al.*[1].

4. $M_{p,1,1,0}(\phi) \equiv C_p(\phi)$ introduced and studied by Ali *et al.*[1].

5. $M_{p,b,0,0}(\phi) \equiv S_{b,p}^*(\phi)$ introduced and studied by Ali *et al.*[1].

6. $M_{p,b,1,0}(\phi) \equiv C_{b,p}(\phi)$ introduced and studied by Ali *et al.*[1].

7. $M_{1,1,\alpha,0}(\phi) \equiv M_\alpha(\phi)$ introduced and studied by Shanmugam and Sivasubramanian [6].

Very recently Ali et al. [1] obtained the sharp coefficient inequality for functions in the class $S_{b,p}^*(\phi)$ and some other subclasses of $\mathscr{A}_p$.

In the present paper, we prove the sharp coefficient inequality in Theorem 2.1 for a more general class of analytic functions which we have defined above in Definition 1.1. Also we give applications of our results to certain functions defined through Hadamard product. The results obtained in this paper generalizes the results obtained by Ali *et al.*[1], Shanmugam and Sivasubramanian [6], Ravichandran *et al.*[5] and Srivastava and Mishra [7].

Let Ω be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + \cdots \tag{1.3}$$

in the open unit disk Δ satisfying $|w(z)| < 1$.

To prove our main result, we need the following:

LEMMA 1.2. [1] *If $w \in \Omega$, then*

$$|w_2 - t w_1^2| \leqslant \begin{cases} -t & \text{if} \quad t \leqslant -1 \\ 1 & \text{if} \quad -1 \leqslant t \leqslant 1 \\ t & \text{if} \quad t \geqslant 1 \end{cases}$$

When $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if

$$w(z) = \frac{z(z+\lambda)}{1+\lambda z} \quad (0 \leqslant \lambda \leqslant 1)$$

or one of its rotations while for $t = 1$, the equality holds if and only if

$$w(z) = -\frac{z(z+\lambda)}{1+\lambda z} \quad (0 \leqslant \lambda \leqslant 1)$$

or one of its rotations.

Although the above upper bound is sharp, it can be improved as follows when $-1 < t < 1$:

$$|w_2 - t w_1^2| + (t+1)|w_1|^2 \leqslant 1 \quad (-1 < t \leqslant 0)$$

and

$$|w_2 - t w_1^2| + (1-t)|w_1|^2 \leqslant 1 \quad (0 < t < 1).$$

LEMMA 1.3. [2] *If $w \in \Omega$, then for any complex number t*

$$|w_2 - tw_1^2| \leqslant \max\{1; |t|\}.$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

LEMMA 1.4. [4] *If $w \in \Omega$, then for any real numbers q_1 and q_2 the following sharp estimate holds:*

$$\left|w_3 + q_1 w_1 w_2 + q_2 w_1^3\right| \leqslant H(q_1, q_2) \tag{1.4}$$

where

$$H(q_1,q_2) = \begin{cases} 1 & \text{for } (q_1,q_2) \in D_1 \cup D_2 \\ |q_2| & \text{for } (q_1,q_2) \in \cup_{k=3}^{7} D_k \\ \dfrac{2}{3}(|q_1|+1)\left(\dfrac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_8 \cup D_9 \\ \dfrac{q_2}{3}\left(\dfrac{q_1^2-4}{q_1^2-4q_2}\right)\left(\dfrac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{10} \cup D_{11} \setminus \{\pm 2, 1\} \\ \dfrac{2}{3}(|q_1|-1)\left(\dfrac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{12}. \end{cases}$$

The extremal functions, up to rotations, are of the form

$$w(z) = z^3, \qquad w(z) = z, \qquad w(z) = w_0(z) = \frac{(z[(1-\lambda)\varepsilon_2 + \lambda\varepsilon_1] - \varepsilon_1\varepsilon_2 z)}{1 - [(1-\lambda)\varepsilon_1 + \lambda\varepsilon_2]z},$$

$$w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z}, \qquad w(z) = w_2(z) = \frac{z(t_2 + z)}{1 + t_2 z},$$

$$|\varepsilon_1| = |\varepsilon_2| = 1, \qquad \varepsilon_1 = t_0 - e^{\frac{-i\theta_0}{2}}(a \mp b), \qquad \varepsilon_2 = -e^{\frac{-i\theta_0}{2}}(ia \pm b),$$

$$a = t_0 \cos\frac{\theta_0}{2}, \qquad b = \sqrt{1 - t_0^2 \sin^2\frac{\theta_0}{2}}, \qquad \lambda = \frac{b \pm a}{2b},$$

$$t_0 = \left[\frac{2q_2(q_1^2+2) - 3q_1^2}{3(q_2-1)(q_1^2-4q_2)}\right]^{\frac{1}{2}}, \qquad t_1 = \left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}},$$

$$t_2 = \left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}}, \qquad \cos\frac{\theta_0}{2} = \frac{q_1}{2}\left[\frac{q_2(q_1^2+8) - 2(q_1^2+2)}{2q_2(q_1^2+2) - 3q_1^2}\right].$$

The sets D_k, $k = 1, 2, \cdots, 12$, are defined as follows:

$$D_1 = \left\{(q_1, q_2) : |q_1| \leqslant \frac{1}{2}, |q_2| \leqslant 1\right\},$$

$$D_2 = \left\{(q_1, q_2) : \frac{1}{2} \leqslant |q_1| \leqslant 2, \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1) \leqslant q_2 \leqslant 1\right\},$$
$$D_3 = \left\{(q_1, q_2) : |q_1| \leqslant \frac{1}{2}, q_2 \leqslant -1\right\},$$
$$D_4 = \left\{(q_1, q_2) : |q_1| \geqslant \frac{1}{2}, q_2 \leqslant -\frac{2}{3}(|q_1|+1)\right\},$$
$$D_5 = \{(q_1, q_2) : |q_1| \leqslant 2, q_2 \geqslant 1\},$$
$$D_6 = \left\{(q_1, q_2) : 2 \leqslant |q_1| \leqslant 4, q_2 \geqslant \frac{1}{12}(q_1^2+8)\right\},$$
$$D_7 = \left\{(q_1, q_2) : |q_1| \geqslant 4, q_2 \geqslant \frac{2}{3}(|q_1|-1)\right\},$$
$$D_8 = \left\{(q_1, q_2) : \frac{1}{2} \leqslant |q_1| \leqslant 2, -\frac{2}{3}(|q_1|+1) \leqslant q_2 \leqslant \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1)\right\},$$
$$D_9 = \left\{(q_1, q_2) : |q_1| \geqslant 2, -\frac{2}{3}(|q_1|+1) \leqslant q_2 \leqslant \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\right\},$$
$$D_{10} = \left\{(q_1, q_2) : 2 \leqslant |q_1| \leqslant 4, \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \leqslant q_2 \leqslant \frac{1}{12}(q_1^2+8)\right\},$$
$$D_{11} = \left\{(q_1, q_2) : |q_1| \geqslant 4, \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \leqslant q_2 \leqslant \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\right\},$$
$$D_{12} = \left\{(q_1, q_2) : |q_1| \geqslant 4, \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4} \leqslant q_2 \leqslant \frac{2}{3}(|q_1|-1)\right\}.$$

2. Coefficient Bounds

By making use of the Lemmas 1.2–1.4, we prove the following:

THEOREM 2.1. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, *where* B_n*'s are real with* $B_1 > 0$ *and* $B_2 \geqslant 0$. *Let* $0 \leqslant \alpha \leqslant 1,\ 0 \leqslant \lambda \leqslant 1,\ 0 \leqslant \mu \leqslant 1$, *and*

$$\sigma_1 := \frac{(p+\alpha)^2}{2B_1^2 p^2 (p+2\alpha)} \left(\frac{(1+\lambda p)^2}{(1+\lambda p)^2 - \lambda^2}\right) \left(B_2 - B_1 + pB_1^2 \left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right)\right),$$

$$\sigma_2 := \frac{(p+\alpha)^2}{2B_1^2 p^2 (p+2\alpha)} \left(\frac{(1+\lambda p)^2}{(1+\lambda p)^2 - \lambda^2}\right) \left(B_2 + B_1 + pB_1^2 \left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right)\right),$$

$$\sigma_3 := \frac{(p+\alpha)^2}{2B_1^2 p^2 (p+2\alpha)} \left(\frac{(1+\lambda p)^2}{(1+\lambda p)^2 - \lambda^2}\right) \left(B_2 + pB_1^2 \left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right)\right),$$

$$\Lambda(p, \alpha, \lambda, \mu) := \frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2} - 2p\mu \left(\frac{p+2\alpha}{(p+\alpha)^2}\right) \left(1 - \left(\frac{\lambda}{1+\lambda p}\right)^2\right).$$

If $f(z)$ given by (1.1) belongs to $M_{p,1,\alpha,\lambda}(\phi)$, then

$$|a_{p+2}-\mu a_{p+1}^2| \leqslant \begin{cases} \dfrac{p^2}{2(p+2\alpha)}\left(\dfrac{1+\lambda(p-1)}{1+\lambda(p+1)}\right)\left\{B_2+pB_1^2\Lambda(p,\alpha,\lambda,\mu)\right\} & \text{if } \mu \leqslant \sigma_1 \\ \dfrac{p^2B_1}{2(p+2\alpha)}\left(\dfrac{1+\lambda(p-1)}{1+\lambda(p+1)}\right) & \text{if } \sigma_1 \leqslant \mu \leqslant \sigma_2 \\ -\dfrac{p^2}{2(p+2\alpha)}\left(\dfrac{1+\lambda(p-1)}{1+\lambda(p+1)}\right)\left\{B_2+pB_1^2\Lambda(p,\alpha,\lambda,\mu)\right\} & \text{if } \mu \geqslant \sigma_2. \end{cases} \tag{2.1}$$

Further, if $\sigma_1 \leqslant \mu \leqslant \sigma_3$, then

$$\begin{aligned} |a_{p+2}-\mu a_{p+1}^2| + \frac{(p+\alpha)^2}{2{B_1}^2p^2(p+2\alpha)}\left(\frac{(1+\lambda p)^2}{(1+\lambda p)^2-\lambda^2}\right) \times \\ \times\left[B_1-B_2-pB_1^2\Lambda(p,\,\alpha,\,\lambda,\,\mu)\right]|a_{p+1}|^2 \\ \leqslant \frac{p^2B_1}{2(p+2\alpha)}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right). \end{aligned} \tag{2.2}$$

If $\sigma_3 \leqslant \mu \leqslant \sigma_2$, then

$$\begin{aligned} |a_{p+2}-\mu a_{p+1}^2| + \frac{(p+\alpha)^2}{2{B_1}^2p^2(p+2\alpha)}\left(\frac{(1+\lambda p)^2}{(1+\lambda p)^2-\lambda^2}\right) \times \\ \times\left[B_1+B_2+pB_1^2\Lambda(p,\,\alpha,\,\lambda,\,\mu)\right]|a_{p+1}|^2 \\ \leqslant \frac{p^2B_1}{2(p+2\alpha)}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right). \end{aligned} \tag{2.3}$$

For any complex number μ,

$$|a_{p+2}-\mu a_{p+1}^2| \leqslant \frac{p^2B_1}{2(p+2\alpha)}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right)\max\left\{1,\left|\frac{B_2}{B_1}+pB_1^2\Lambda(p,\,\alpha,\,\lambda,\,\mu)\right|\right\}. \tag{2.4}$$

Further,

$$|a_{p+3}| \leqslant \frac{p^2B_1}{3(p+3\alpha)}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+2)}\right)H(q_1,q_2) \tag{2.5}$$

where $H(q_1,q_2)$ is as defined in Lemma 1.4,

$$q_1 := \frac{1}{2B_1}\left(4B_2+3pB_1^2\left(\frac{p^2+3\alpha p+\alpha}{(p+\alpha)(p+2\alpha)}\right)\right)$$

$$q_2 := \begin{cases} \dfrac{1}{2B_1}\left(2B_3+3pB_1\left(B_2+pB_1^2\left(\dfrac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right)\left(\dfrac{p^3+3\alpha p+\alpha}{(p+\alpha)(p+2\alpha)}\right)\right)\right) \\ -p^2B_1^2\left(\dfrac{p^3+3\alpha p^2+3\alpha p+\alpha}{(p+\alpha)^3}\right) \end{cases}$$

These results are sharp.

Proof. If $f(z) \in M_{p,1,\alpha,\lambda}(\phi)$, then there is a Schwarz function

$$w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$$

such that

$$\frac{1}{p}\left((1-\alpha)\frac{zF'(z)}{F(z)} + \alpha\left(1 + \frac{zF''(z)}{F'(z)}\right)\right) = \phi(w(z)). \tag{2.6}$$

Since,

$$\frac{1}{p}\left((1-\alpha)\frac{zF'(z)}{F(z)} + \alpha\left(1 + \frac{zF''(z)}{F'(z)}\right)\right)$$

$$= \begin{cases} 1 + \frac{1}{p}\left(\frac{p+\alpha}{p}\right)A_{p+1}z + \left(\frac{2}{p}\left(\frac{p+2\alpha}{p}\right)A_{p+2} - \frac{1}{p}\left(\frac{p^2+2\alpha p+\alpha}{p^2}\right)A_{p+1}^2\right)z^2 \\ + \left(\frac{3}{p}\left(\frac{p+3\alpha}{p}\right)A_{p+3} - \frac{3}{p}\left(\frac{p^2+3\alpha p+2\alpha}{p^2}\right)A_{p+1}A_{p+2}\right)z^3 \\ + \left(\frac{1}{p}\left(\frac{p^3+3\alpha p^2+3\alpha p+\alpha}{p^3}\right)A_{p+1}^3\right)z^3 + \cdots, \end{cases}$$

where

$$A_{p+n} = \left(\frac{1+\lambda(p+n-1)}{1+\lambda(p-1)}\right)a_{p+n}. \tag{2.7}$$

We have from (2.6),

$$a_{p+1} = \frac{p^2 B_1 \omega_1}{p+\alpha}\left(\frac{1+\lambda(p-1)}{1+\lambda p}\right) \tag{2.8}$$

$$a_{p+2} = \frac{p^2 B_1}{2(p+2\alpha)}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right)\left(w_2 + w_1^2\left(\frac{B_2}{B_1} + pB_1\left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right)\right)\right) \tag{2.9}$$

and

$$a_{p+3} = \frac{p^2 B_1}{3(p+3\alpha)}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+2)}\right)H(q_1,q_2), \tag{2.10}$$

where q_1 and q_2 is given as in Theorem 2.1. Therefore, we have

$$a_{p+2} - \mu a_{p+1}^2 = \frac{p^2 B_1}{2(p+2\alpha)}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right)\left\{w_2 - v w_1^2\right\}, \tag{2.11}$$

where

$$v := 2\mu B_1(p+2\alpha)\left(\frac{p}{p+\alpha}\right)^2\left(1 - \left(\frac{\lambda}{1+\lambda p}\right)^2\right) - pB_1\left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right) - \frac{B_2}{B_1}.$$

The results(2.1)–(2.3) are established by an application of Lemma 1.2, inequality (2.4) by Lemma (1.3) and (2.5) follows from Lemma 1.4. To show that the bounds in (2.1)-(2.3) are sharp, we define the functions $K_{\phi n}$ $(n = 2,3,\dots)$ by

$$\frac{1}{p}\left((1-\alpha)\frac{z(K_{\phi n})'(z)}{K_{\phi n}(z)} + \alpha\left(1+\frac{z(K_{\phi n})''(z)}{(K_{\phi n})'(z)}\right)\right) = \phi(z^{n-1}),$$

$$K_{\phi n}(0) = 0 = [K_{\phi n}]'(0) - 1$$

and the function F_λ and G_λ $(0 \leqslant \lambda \leqslant 1)$ by

$$\frac{1}{p}\left((1-\alpha)\frac{z(F_\lambda)'(z)}{F_\lambda(z)} + \alpha\left(1+\frac{z(F_\lambda)''(z)}{(F_\lambda)'(z)}\right)\right) = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right),$$

$$F_\lambda(0) = 0 = F'_\lambda(0) - 1$$

and

$$\frac{1}{p}\left((1-\alpha)\frac{z(G_\lambda)'(z)}{G_\lambda(z)} + \alpha\left(1+\frac{z(G_\lambda)''(z)}{(G_\lambda)'(z)}\right)\right) = \phi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right),$$

$$G_\lambda(0) = 0 = G'_\lambda(0) - 1.$$

Clearly the functions $K_{\phi n}, F_\lambda, G_\lambda \in M_{p,1,\alpha,\lambda}(\phi)$. Also we write $K_\phi := K_{\phi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_λ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_λ or one of its rotations.

REMARK 2.2. For $\alpha = 0, \lambda = 0,$ results (2.1)–(2.5) coincides with the results obtained for the class $S^*_p(\phi)$ by Ali *et al.*[1].

REMARK 2.3. For $\alpha = 0, \lambda = 0, p = 1$ results (2.1)–(2.5) coincides with the results obtained for the class $S^*(\phi)$ by Ma and Minda [3].

3. Applications to functions defined by convolution

We define $M_{p,b,\alpha,\lambda,g}(\phi)$ to be the class of all functions $f \in \mathscr{A}_p$ for which $f * g \in M_{p,b,\alpha,\lambda}(\phi)$, where g is a fixed function with positive coefficients and the class $M_{p,b,\alpha,\lambda}(\phi)$ is as Definition 1.1. In Theorem 2.1 we obtained the coefficient estimate for the class $M_{p,1,\alpha,\lambda}(\phi)$. Now, we obtain the coefficient estimate for the class $M_{p,1,\alpha,\lambda,g}(\phi)$.

THEOREM 3.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ where B_n's are real with $B_1 > 0$ and $B_2 \leqslant 0$. Let $0 \leqslant \alpha \leqslant 1,\ 0 \leqslant \lambda \leqslant 1,\ 0 \leqslant \mu \leqslant 1$ and*

$$\sigma_1 := \frac{g_{p+1}^2}{g_{p+2}}\frac{(p+\alpha)^2}{2B_1^2 p^2(p+2\alpha)}\left(\frac{(1+\lambda p)^2}{(1+\lambda p)^2-\lambda^2}\right)\left(B_2 - B_1 + pB_1^2\left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right)\right),$$

$$\sigma_2 := \frac{g_{p+1}^2}{g_{p+2}} \frac{(p+\alpha)^2}{2B_1^2 p^2 (p+2\alpha)} \left(\frac{(1+\lambda p)^2}{(1+\lambda p)^2 - \lambda^2} \right) \left(B_2 + B_1 + pB_1^2 \left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2} \right) \right),$$

$$\sigma_3 := \frac{g_{p+1}^2}{g_{p+2}} \frac{(p+\alpha)^2}{2B_1^2 p^2 (p+2\alpha)} \left(\frac{(1+\lambda p)^2}{(1+\lambda p)^2 - \lambda^2} \right) \left(B_2 + pB_1^2 \left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2} \right) \right),$$

$$\Lambda(p,\alpha,\lambda,\mu,g) := \frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2} - 2p\mu \frac{g_{p+2}}{g_{p+1}^2} \left(\frac{p+2\alpha}{(p+\alpha)^2} \right) \left(1 - \left(\frac{\lambda}{1+\lambda p} \right)^2 \right).$$

If $f(z)$ given by (1.1) belongs to $M_{p,1,\alpha,\lambda}(\phi)$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leqslant \begin{cases} \dfrac{p^2}{2(p+2\alpha) g_{p+2}} \left(\dfrac{1+\lambda(p-1)}{1+\lambda(p+1)} \right) \left\{ B_2 + pB_1^2 \Lambda(p,\alpha,\lambda,\mu,g) \right\} & \text{if} \quad \mu \leqslant \sigma_1 \\ \dfrac{p^2 B_1}{2(p+2\alpha) g_{p+2}} \left(\dfrac{1+\lambda(p-1)}{1+\lambda(p+1)} \right) & \text{if} \quad \sigma_1 \leqslant \mu \leqslant \sigma_2 \\ -\dfrac{p^2}{2(p+2\alpha) g_{p+2}} \left(\dfrac{1+\lambda(p-1)}{1+\lambda(p+1)} \right) \left\{ B_2 + pB_1^2 \Lambda(p,\alpha,\lambda,\mu,g) \right\} & \text{if} \quad \mu \geqslant \sigma_2. \end{cases} \tag{3.1}$$

Further, if $\sigma_1 \leqslant \mu \leqslant \sigma_3$, then

$$\begin{aligned} |a_{p+2} - \mu a_{p+1}^2| + \frac{g_{p+1}^2}{g_{p+2}} \frac{(p+\alpha)^2}{2{B_1}^2 p^2 (p+2\alpha)} \left(\frac{(1+\lambda p)^2}{(1+\lambda p)^2 - \lambda^2} \right) \times \\ \times \left[B_1 - B_2 - pB_1^2 \Lambda(p,\alpha,\lambda,\mu,g) \right] |a_{p+1}|^2 \\ \leqslant \frac{p^2 B_1}{2(p+2\alpha)} \left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)} \right). \end{aligned} \tag{3.2}$$

If $\sigma_3 \leqslant \mu \leqslant \sigma_2$, then

$$\begin{aligned} |a_{p+2} - \mu a_{p+1}^2| + \frac{g_{p+1}^2}{g_{p+2}} \frac{(p+\alpha)^2}{2{B_1}^2 p^2 (p+2\alpha)} \left(\frac{(1+\lambda p)^2}{(1+\lambda p)^2 - \lambda^2} \right) \times \\ \times \left[B_1 + B_2 + pB_1^2 \Lambda(p,\alpha,\lambda,\mu,g) \right] |a_{p+1}|^2 \\ \leqslant \frac{p^2 B_1}{2(p+2\alpha)} \left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)} \right). \end{aligned} \tag{3.3}$$

For any complex number μ,

$$|a_{p+2} - \mu a_{p+1}^2| \leqslant \frac{p^2 B_1}{2(p+2\alpha)} \left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)} \right) \max\left\{ 1, \left| \frac{B_2}{B_1} + pB_1^2 \Lambda(p,\alpha,\lambda,\mu,g) \right| \right\}. \tag{3.4}$$

Further,

$$|a_{p+3}| \leqslant \frac{p^2 B_1}{3(p+3\alpha)} \left(\frac{1+\lambda(p-1)}{1+\lambda(p+2)}\right) H(q_1,q_2) \tag{3.5}$$

where $H(q_1,q_2)$ *is as defined in Lemma* 1.4*,*

$$q_1 := \frac{1}{2B_1}\left(4B_2 + 3pB_1^2\left(\frac{p^2+3\alpha p+\alpha}{(p+\alpha)(p+2\alpha)}\right)\right)$$

$$q_2 := \begin{cases} \frac{1}{2B_1}\left(2B_3 + 3pB_1\left(B_2 + pB_1^2\left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right)\left(\frac{p^3+3\alpha p+\alpha}{(p+\alpha)(p+2\alpha)}\right)\right)\right) \\ -p^2B_1^2\left(\frac{p^3+3\alpha p^2+3\alpha p+\alpha}{(p+\alpha)^3}\right) \end{cases}$$

These results are sharp.

Proof. The proof is similar to the proof of Theorem 2.1 and hence omitted.

REMARK 3.2. For $p=1,\ \alpha=0,$

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)},$$

in inequality (3.1), we get the result obtained by Srivastava and Mishra [7].

THEOREM 3.3. *Let* $\phi(z)$ *be as in Theorem* 2.1*. If* $f(z)$ *given by (*1.1*) belongs to* $M_{p,b,\alpha,\lambda,g}(\phi)$*, then for any complex number* μ*,*

$$|a_{p+2} - \mu a_{p+1}^2| \leqslant \frac{p^2|b|B_1}{2(p+2\alpha)g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right) \times$$
$$\times \max\left\{1, \left|bpB_1^2\Lambda_2(p,b,\alpha,\lambda,\mu,g) + \frac{B_2}{B_1}\right|\right\}, \tag{3.6}$$

where

$$\Lambda_2(p,b,\alpha,\lambda,\mu,g) = \frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2} - 2p\mu\frac{g_{p+2}}{g_{p+1}^2}\left(\frac{p+2\alpha}{(p+\alpha)^2}\right)\left(1-\left(\frac{\lambda}{1+\lambda p}\right)^2\right).$$

Proof. The proof is similar to the proof of Theorem 2.1 and hence omitted.

REMARK 3.4. For $p=1, \alpha=0$ and $\lambda=0,$ the result in (3.6) coincides with the results obtained by Ravichandran et al. [5].

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