ON CERTAIN COEFFICIENT INEQUALITIES FOR MULTIVALENT FUNCTIONS

T. N. SHANMUGAM, SHIGEYOSHI OWA, C. RAMACHANDRAN, S. SIVASUBRAMANIAN AND YAYOI NAKAMURA

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Abstract. In the present investigation, the authors obtain sharp upper bounds for certain coefficient inequalities for linear combination of Mocanu $\alpha$-convex $p$-valent functions. The results are extended to functions defined by convolution.

1. Introduction

Let $A_p$ denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

(1.1)

defined on the open unit disk

$$\Delta = \{ z : z \in \mathbb{C} : |z| < 1 \}$$

and let $A_1 := A$. For $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

their convolution (or Hadamard product), denoted by $(f \ast g)$ is defined as

$$(f \ast g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$
such that
\[ f(z) = g(\omega(z)) \quad (z \in \Delta). \]

We denote this subordination by
\[ f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta). \]

In particular, if the function \( g \) is univalent in \( \Delta \), the above subordination is equivalent to
\[ f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta). \]

Let \( \phi(z) \) be an analytic function with positive real part on \( \Delta \) with \( \phi(0) = 1 \), \( \phi'(0) > 0 \) which maps the open unit disk \( \Delta \) onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Ali et al. [1] defined and studied the classes \( S_{b,p}^*(\phi) \) consisting of functions in \( f \in \mathcal{A}_p \) for which
\[
1 + \frac{1}{b} \left( \frac{1}{p} \frac{zf''(z)}{f'(z)} - 1 \right) \prec \phi(z) \quad (z \in \Delta, \ b \in \mathbb{C} \setminus \{0\}),
\]
and the class \( C_{b,p}(\phi) \) of all functions in \( f \in \mathcal{A}_p \) for which
\[
1 - \frac{1}{b} + \frac{1}{bp} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec \phi(z) \quad (z \in \Delta, \ b \in \mathbb{C} \setminus \{0\}).
\]

Note that \( S_{1,1}^*(\phi) = S^*(\phi) \) and \( C_{1,1}(\phi) = C(\phi) \), the classes introduced and studied by Ma and Minda [3]. The familiar class \( S^*(\alpha) \) of starlike functions of order \( \alpha \) and the class \( C(\alpha) \) of convex functions of order \( \alpha \), \( 0 \leq \alpha < 1 \) are the special case of \( S_{1,1}^*(\phi) \) and \( C_{1,1}(\phi) \) respectively when \( \phi(z) = (1 + (1 - 2\alpha)z)/(1 - z) \).

We now define a class of functions which unifies the classes \( S_{b,p}^*(\phi) \) and \( C_{b,p}(\phi) \) in the following:

**Definition 1.1.** Let \( \phi(z) \) be a univalent starlike function with respect to 1 which maps the open unit disk \( \Delta \) onto a region in the right half plane and is symmetric with respect to the real axis, \( \phi(0) = 1 \) and \( \phi'(0) > 0 \). A function \( f \in \mathcal{A}_p \) is in the class \( M_{p,b,\alpha,\lambda}(\phi) \) if
\[
1 + \frac{1}{b} \left[ \frac{1}{p} \left( 1 - \alpha \right) \frac{zf''(z)}{F(z)} + \alpha \left( 1 + \frac{zf''(z)}{F'(z)} \right) \right] - 1 \prec \phi(z) \quad (0 \leq \alpha \leq 1), \quad \text{(1.2)}
\]
where
\[ F(z) := (1 - \lambda)f(z) + \lambda zf'(z). \]

Also, \( M_{p,b,\alpha,\lambda,g}(\phi) \) is the class of all functions \( f \in \mathcal{A}_p \) for which \( f \star g \in M_{p,b,\alpha,\lambda}(\phi) \). The classes \( M_{p,b,\alpha,\lambda}(\phi) \) reduce to the following classes.

1. \( M_{1,1,1,0}(\phi) \equiv C(\phi) \) [3].
2. \( M_{1,1,0,0}(\phi) \equiv S^*(\phi) \) [3].
3. $M_{p,1,0,0}(\phi) \equiv S^*_p(\phi)$ introduced and studied by Ali et al.\cite{1}.

4. $M_{p,1,1,0}(\phi) \equiv C_p(\phi)$ introduced and studied by Ali et al.\cite{1}.

5. $M_{p,b,0,0}(\phi) \equiv S^*_{b,p}(\phi)$ introduced and studied by Ali et al.\cite{1}.

6. $M_{p,b,1,0}(\phi) \equiv C_{b,p}(\phi)$ introduced and studied by Ali et al.\cite{1}.

7. $M_{1,1,\alpha,0}(\phi) \equiv M_{\alpha}(\phi)$ introduced and studied by Shanmugam and Sivasubramanian\cite{6}.

Very recently Ali et al.\cite{1} obtained the sharp coefficient inequality for functions in the class $S^*_{b,p}(\phi)$ and some other subclasses of $A_p$.

In the present paper, we prove the sharp coefficient inequality in Theorem 2.1 for a more general class of analytic functions which we have defined above in Definition 1.1. Also we give applications of our results to certain functions defined through Hadamard product. The results obtained in this paper generalizes the results obtained by Ali et al.\cite{1}, Shanmugam and Sivasubramanian\cite{6}, Ravichandran et al.\cite{5} and Srivastava and Mishra\cite{7}.

Let $\Omega$ be the class of analytic functions of the form
\[ w(z) = w_1z + w_2z^2 + \cdots \tag{1.3} \]
in the open unit disk $\Delta$ satisfying $|w(z)| < 1$.

To prove our main result, we need the following:

**Lemma 1.2.** \cite{1} If $w \in \Omega$, then
\[ |w_2 - tw_1^2| \leq \begin{cases} -t & \text{if } t \leq -1 \\ 1 & \text{if } -1 \leq t \leq 1 \\ t & \text{if } t \geq 1 \end{cases} \]

When $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations.

If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations.

Equality holds for $t = -1$ if and only if
\[ w(z) = \frac{z(z + \lambda)}{1 + \lambda z} \quad (0 \leq \lambda \leq 1) \]
or one of its rotations while for $t = 1$, the equality holds if and only if
\[ w(z) = -\frac{z(z + \lambda)}{1 + \lambda z} \quad (0 \leq \lambda \leq 1) \]
or one of its rotations.

Although the above upper bound is sharp, it can be improved as follows when $-1 < t < 1$:
\[ |w_2 - tw_1^2| + (t + 1)|w_1|^2 \leq 1 \quad (-1 < t \leq 0) \]
and
\[ |w_2 - tw_1^2| + (1 - t)|w_1|^2 \leq 1 \quad (0 < t < 1). \]
LEMMA 1.3. [2] If \( w \in \Omega \), then for any complex number \( t \)

\[
|w_2 - tw_1^2| \leq \max\{1, |t|\}.
\]

The result is sharp for the functions \( w(z) = z \) or \( w(z) = z^2 \).

LEMMA 1.4. [4] If \( w \in \Omega \), then for any real numbers \( q_1 \) and \( q_2 \) the following sharp estimate holds:

\[
|w_3 + q_1w_1w_2 + q_2w_1^3| \leq H(q_1, q_2) \tag{1.4}
\]

where

\[
H(q_1, q_2) = \begin{cases} 
1 & \text{for } (q_1, q_2) \in D_1 \cup D_2 \\
|q_2| & \text{for } (q_1, q_2) \in \bigcup_{k=3}^7 D_k \\
\frac{2}{3}(|q_1| + 1) \left( \frac{|q_1| + 1}{3(|q_1| + 1 + q_2)} \right)^\frac{1}{2} & \text{for } (q_1, q_2) \in D_8 \cup D_9 \\
\frac{q_2}{3} \left( \frac{q_1^2 - 4}{q_1^2 - 4q_2} \right) \left( \frac{q_1^2 - 4}{3(q_2 - 1)} \right)^\frac{1}{2} & \text{for } (q_1, q_2) \in D_{10} \cup D_{11} \setminus \{ \pm 2, 1 \} \\
\frac{2}{3}(|q_1| - 1) \left( \frac{|q_1| - 1}{3(|q_1| - 1 - q_2)} \right)^\frac{1}{2} & \text{for } (q_1, q_2) \in D_{12}.
\end{cases}
\]

The extremal functions, up to rotations, are of the form

\[
w(z) = z^3, \quad w(z) = z, \quad w(z) = w_0(z) = \frac{(z[(1 - \lambda)e_2 + \lambda e_1] - e_1e_2z)}{1 - [(1 - \lambda)e_1 + \lambda e_2]z},
\]

\[
w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1z}, \quad w(z) = w_2(z) = \frac{z(t_2 + z)}{1 + t_2z},
\]

\[
|e_1| = |e_2| = 1, \quad e_1 = t_0 - e^{-\frac{ib}{2}}(a \mp b), \quad e_2 = -e^{-\frac{ib}{2}}(ia \pm b),
\]

\[
a = t_0 \cos \frac{\theta_0}{2}, \quad b = \sqrt{1 - t_0^2 \sin^2 \frac{\theta_0}{2}}, \quad \lambda = \frac{b \pm a}{2b},
\]

\[
t_0 = \left[ \frac{2q_2(q_1^2 + 2) - 3q_1^2}{3(q_2 - 1)(q_1^2 - 4q_2)} \right]^\frac{1}{2}, \quad t_1 = \left( \frac{|q_1| + 1}{3(|q_1| + 1 + q_2)} \right)^\frac{1}{2},
\]

\[
t_2 = \left( \frac{|q_1| - 1}{3(|q_1| - 1 - q_2)} \right)^\frac{1}{2}, \quad \cos \frac{\theta_0}{2} = \frac{q_1}{2} \left[ \frac{q_2(q_1^2 + 8) - 2(q_1^2 + 2)}{2q_2(q_1^2 + 2) - 3q_1^2} \right].
\]

The sets \( D_k, k = 1, 2, \ldots, 12 \), are defined as follows:

\[
D_1 = \left\{ (q_1, q_2) : |q_1| < \frac{1}{2}, |q_2| < 1 \right\},
\]
By making use of the Lemmas 1.2–1.4, we prove the following:

**Theorem 2.1.** Let \( \phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots \), where \( B_n \)'s are real with \( B_1 > 0 \) and \( B_2 \geq 0 \). Let \( 0 \leq \alpha \leq 1, \ 0 \leq \lambda \leq 1, \ 0 \leq \mu \leq 1, \) and

\[
\sigma_1 := \frac{(p + \alpha)^2}{2B_1^2p^2(p + 2\alpha)} \left( \frac{1 + \lambda p}{1 + \lambda p - \lambda^2} \right)^2 \left( B_2 - B_1 + pB_1^2 \left( \frac{p^2 + 2\alpha p + \alpha}{(p + \alpha)^2} \right) \right),
\]

\[
\sigma_2 := \frac{(p + \alpha)^2}{2B_1^2p^2(p + 2\alpha)} \left( \frac{1 + \lambda p}{1 + \lambda p - \lambda^2} \right)^2 \left( B_2 + B_1 + pB_1^2 \left( \frac{p^2 + 2\alpha p + \alpha}{(p + \alpha)^2} \right) \right),
\]

\[
\sigma_3 := \frac{(p + \alpha)^2}{2B_1^2p^2(p + 2\alpha)} \left( \frac{1 + \lambda p}{1 + \lambda p - \lambda^2} \right)^2 \left( B_2 + pB_1^2 \left( \frac{p^2 + 2\alpha p + \alpha}{(p + \alpha)^2} \right) \right),
\]

\[
\Lambda(p, \alpha, \lambda, \mu) := \frac{p^2 + 2\alpha p + \alpha}{(p + \alpha)^2} - 2p\mu \left( \frac{p + 2\alpha}{(p + \alpha)^2} \right) \left( 1 - \left( \frac{\lambda}{1 + \lambda p} \right)^2 \right).
\]
If $f(z)$ given by (1.1) belongs to $M_{p, 1, \alpha, \lambda}(\phi)$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \left\{ \begin{array}{ll}
\frac{p^2}{2(p+2\alpha)} \left( \frac{1+\lambda(p-1)}{1+\lambda(p+1)} \right) \{B_2 + pB_1^2 \Lambda(p, \alpha, \lambda, \mu)\} & \text{if } \mu \leq \sigma_1 \\
\frac{p^2B_1}{2(p+2\alpha)} \left( \frac{1+\lambda(p-1)}{1+\lambda(p+1)} \right) & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\
-\frac{p^2}{2(p+2\alpha)} \left( \frac{1+\lambda(p-1)}{1+\lambda(p+1)} \right) \{B_2 + pB_1^2 \Lambda(p, \alpha, \lambda, \mu)\} & \text{if } \mu \geq \sigma_2.
\end{array} \right. \quad (2.1)$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(p+\alpha)^2}{2B_1^2p^2(p+2\alpha)} \left( \frac{1+\lambda p}{1+\lambda p - \lambda^2} \right) \times \left[ B_1 - B_2 - pB_1^2 \Lambda(p, \alpha, \lambda, \mu) \right] |a_{p+1}|^2 \leq \frac{p^2B_1}{2(p+2\alpha)} \left( \frac{1+\lambda(p-1)}{1+\lambda(p+1)} \right). \quad (2.2)$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(p+\alpha)^2}{2B_1^2p^2(p+2\alpha)} \left( \frac{1+\lambda p}{1+\lambda p - \lambda^2} \right) \times \left[ B_1 + B_2 + pB_1^2 \Lambda(p, \alpha, \lambda, \mu) \right] |a_{p+1}|^2 \leq \frac{p^2B_1}{2(p+2\alpha)} \left( \frac{1+\lambda(p-1)}{1+\lambda(p+1)} \right). \quad (2.3)$$

For any complex number $\mu$,

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{p^2B_1}{2(p+2\alpha)} \left( \frac{1+\lambda(p-1)}{1+\lambda(p+1)} \right) \max\left\{ 1, \frac{B_2}{B_1} + pB_1^2 \Lambda(p, \alpha, \lambda, \mu) \right\}. \quad (2.4)$$

Further,

$$|a_{p+3}| \leq \frac{p^2B_1}{3(p+3\alpha)} \left( \frac{1+\lambda(p-1)}{1+\lambda(p+2)} \right) H(q_1, q_2) \quad (2.5)$$

where $H(q_1, q_2)$ is as defined in Lemma 1.4,

$$q_1 := \frac{1}{2B_1} \left( 4B_2 + 3pB_1^2 \left( \frac{p^2 + 3\alpha p + \alpha}{(p+\alpha)(p+2\alpha)} \right) \right)$$

$$q_2 := \left\{ \begin{array}{l}
\frac{1}{2B_1} \left( 2B_3 + 3pB_1 \left( B_2 + pB_1^2 \left( \frac{p^2 + 2\alpha p + \alpha}{(p+\alpha)^2} \right) \left( \frac{p^3 + 3\alpha p + \alpha}{(p+\alpha)(p+2\alpha)} \right) \right) \right) \\
-\frac{p^2B_1^2}{(p+\alpha)^3} \left( \frac{p^3 + 3\alpha p^2 + 3\alpha p + \alpha}{(p+\alpha)^3} \right)
\end{array} \right. $$
These results are sharp.

Proof. If \( f(z) \in M_{p,1,\alpha,\lambda}(\phi) \), then there is a Schwarz function

\[
w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega
\]

such that

\[
\frac{1}{p} \left( (1-\alpha) \frac{zf'(z)}{F(z)} + \alpha \left( 1 + \frac{zf''(z)}{F'(z)} \right) \right) = \phi(w(z)). \tag{2.6}
\]

Since,

\[
\frac{1}{p} \left( (1-\alpha) \frac{zf'(z)}{F(z)} + \alpha \left( 1 + \frac{zf''(z)}{F'(z)} \right) \right)
\]

\[
= \left\{ 1 + \frac{1}{p} \left( \frac{p + \alpha}{p} \right) A_{p+1} z^2 + \frac{2}{p^2} \left( \frac{p + 2\alpha}{p} \right) A_{p+2} - \frac{1}{p} \left( \frac{p^2 + 2\alpha p + \alpha}{p^2} \right) A_{p+1}^2 \right\} z^2
\]

\[
+ \left( \frac{3}{p} \left( \frac{p + 3\alpha}{p} \right) A_{p+3} - \frac{3}{p} \left( \frac{p^2 + 3\alpha p + 2\alpha}{p^2} \right) A_{p+1} A_{p+2} \right) z^3
\]

\[
+ \left( \frac{1}{p} \left( \frac{p^3 + 3\alpha p^2 + 3\alpha p + \alpha}{p^3} \right) A_{p+1}^3 \right) z^3 + \cdots,
\]

where

\[
A_{p+n} = \left( \frac{1 + \lambda (p + n - 1)}{1 + \lambda (p - 1)} \right) a_{p+n}. \tag{2.7}
\]

We have from (2.6),

\[
a_{p+1} = \frac{p^2 B_1 \omega_1}{p + \alpha} \left( \frac{1 + \lambda (p - 1)}{1 + \lambda p} \right) \tag{2.8}
\]

\[
a_{p+2} = \frac{p^2 B_1}{2(p+2\alpha)} \left( \frac{1 + \lambda (p - 1)}{1 + \lambda (p + 1)} \right) \left( w_2 + w_1 \frac{B_2}{B_1} + p B_1 \left( \frac{p^2 + 2\alpha p + \alpha}{(p+\alpha)^2} \right) \right) \tag{2.9}
\]

and

\[
a_{p+3} = \frac{p^2 B_1}{3(p+3\alpha)} \left( \frac{1 + \lambda (p - 1)}{1 + \lambda (p + 2)} \right) H(q_1,q_2), \tag{2.10}
\]

where \( q_1 \) and \( q_2 \) is given as in Theorem 2.1. Therefore, we have

\[
a_{p+2} - \mu a_{p+1}^2 = \frac{p^2 B_1}{2(p+2\alpha)} \left( \frac{1 + \lambda (p - 1)}{1 + \lambda (p + 1)} \right) \left\{ w_2 - v w_1^2 \right\}, \tag{2.11}
\]

where

\[
v := 2\mu B_1 (p + 2\alpha) \left( \frac{p}{p + \alpha} \right)^2 \left( 1 - \left( \frac{\lambda}{1 + \lambda p} \right)^2 \right) - p B_1 \left( \frac{p^2 + 2\alpha p + \alpha}{(p+\alpha)^2} \right) - \frac{B_2}{B_1}.
\]
The results (2.1)–(2.3) are established by an application of Lemma 1.2, inequality (2.4) by Lemma (1.3) and (2.5) follows from Lemma 1.4. To show that the bounds in (2.1)–(2.3) are sharp, we define the functions $K_{\phi n}$ ($n = 2, 3, \ldots$) by

$$
\frac{1}{p} \left( (1 - \alpha) \frac{z(K_{\phi n})'(z)}{K_{\phi n}(z)} + \alpha \left( 1 + \frac{z(K_{\phi n})''(z)}{(K_{\phi n})'(z)} \right) \right) = \phi(z^{n - 1}),
$$

$$
K_{\phi n}(0) = 0 = [K_{\phi n}]'(0) - 1
$$

and the function $F_{\lambda}$ and $G_{\lambda}$ ($0 \leq \lambda \leq 1$) by

$$
\frac{1}{p} \left( (1 - \alpha) \frac{z(F_{\lambda})'(z)}{F_{\lambda}(z)} + \alpha \left( 1 + \frac{z(F_{\lambda})''(z)}{(F_{\lambda})'(z)} \right) \right) = \phi \left( \frac{z(1 + \lambda z)}{1 + \lambda z} \right),
$$

$$
F_{\lambda}(0) = 0 = F_{\lambda}'(0) - 1
$$

and

$$
\frac{1}{p} \left( (1 - \alpha) \frac{z(G_{\lambda})'(z)}{G_{\lambda}(z)} + \alpha \left( 1 + \frac{z(G_{\lambda})''(z)}{(G_{\lambda})'(z)} \right) \right) = \phi \left( \frac{z(1 + \lambda z)}{1 + \lambda z} \right),
$$

$$
G_{\lambda}(0) = 0 = G_{\lambda}'(0) - 1.
$$

Clearly the functions $K_{\phi n}, F_{\lambda}, G_{\lambda} \in M_{p,1,\alpha,\lambda}(\phi)$. Also we write $K_{\phi} := K_{\phi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if $f$ is $K_{\phi}$ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if $f$ is $K_{\phi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if $f$ is $F_{\lambda}$ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if $f$ is $G_{\lambda}$ or one of its rotations.

**Remark 2.2.** For $\alpha = 0, \lambda = 0$, results (2.1)–(2.5) coincides with the results obtained for the class $S_p'(\phi)$ by Ali et al. [1].

**Remark 2.3.** For $\alpha = 0, \lambda = 0, p = 1$ results (2.1)–(2.5) coincides with the results obtained for the class $S'(\phi)$ by Ma and Minda [3].

### 3. Applications to functions defined by convolution

We define $M_{p,b,\alpha,\lambda,g}(\phi)$ to be the class of all functions $f \in \mathcal{A}_p$ for which $f \ast g \in M_{p,b,\alpha,\lambda}(\phi)$, where $g$ is a fixed function with positive coefficients and the class $M_{p,b,\alpha,\lambda}(\phi)$ is as Definition 1.1. In Theorem 2.1 we obtained the coefficient estimate for the class $M_{p,1,\alpha,\lambda}(\phi)$. Now, we obtain the coefficient estimate for the class $M_{p,b,\alpha,\lambda,g}(\phi)$.

**Theorem 3.1.** Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ where $B_n$’s are real with $B_1 > 0$ and $B_2 \leq 0$. Let $0 \leq \alpha \leq 1, 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1$ and

$$
\sigma_1 := \frac{g_{p+1}^2}{g_{p+2}} \frac{(p + \alpha)^2}{2B_1^2 p^2(p + 2\alpha)} \left( \frac{(1 + \lambda p)^2}{(1 + \lambda p)^2 - \lambda^2} \right) \left( B_2 - B_1 + pB_1^2 \frac{p^2 + 2\alpha p + \alpha}{(p + \alpha)^2} \right),
$$

where $B_n$’s are real with $B_1 > 0$ and $B_2 \leq 0$. Let $0 \leq \alpha \leq 1, 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1$ and
\[
\sigma_2 := \frac{g_{p+1}^2}{g_{p+2}} \frac{(p + \alpha)^2}{2B_1^2 p^2(p + 2\alpha)} \left( \frac{(1 + \lambda p)^2}{(1 + \lambda p)^2 - \lambda^2} \right) \left( B_2 + B_1 + pB_1^2 \left( \frac{p^2 + 2\alpha p + \alpha}{(p + \alpha)^2} \right) \right),
\]

\[
\sigma_3 := \frac{g_{p+1}^2}{g_{p+2}} \frac{(p + \alpha)^2}{2B_1^2 p^2(p + 2\alpha)} \left( \frac{(1 + \lambda p)^2}{(1 + \lambda p)^2 - \lambda^2} \right) \left( B_2 + pB_1^2 \left( \frac{p^2 + 2\alpha p + \alpha}{(p + \alpha)^2} \right) \right),
\]

\[
\Lambda(p, \alpha, \lambda, \mu, g) := \frac{p^2 + 2\alpha p + \alpha}{(p + \alpha)^2} - 2p\mu \frac{g_{p+2}}{g_{p+1}^2} \left( \frac{p + 2\alpha}{(p + \alpha)^2} \right) \left( 1 - \left( \frac{\lambda}{1 + \lambda p} \right)^2 \right).
\]

If \( f(z) \) given by (1.1) belongs to \( M_{p, 1, \alpha, \lambda}(\phi) \), then

\[
|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} 
\frac{p^2 B_1}{2(p + 2\alpha) g_{p+2}} \left( \frac{1 + \lambda(p - 1)}{1 + \lambda(p + 1)} \right) \left\{ B_2 + pB_1^2 \Lambda(p, \alpha, \lambda, \mu, g) \right\} & \text{if } \mu \leq \sigma_1 \\
\frac{p^2 B_1}{2(p + 2\alpha) g_{p+2}} \left( \frac{1 + \lambda(p - 1)}{1 + \lambda(p + 1)} \right) & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\
-\frac{p^2}{2(p + 2\alpha) g_{p+2}} \left( \frac{1 + \lambda(p - 1)}{1 + \lambda(p + 1)} \right) \left\{ B_2 + pB_1^2 \Lambda(p, \alpha, \lambda, \mu, g) \right\} & \text{if } \mu \geq \sigma_2.
\end{cases}
\]

Further, if \( \sigma_1 \leq \mu \leq \sigma_3 \), then

\[
|a_{p+2} - \mu a_{p+1}^2| + \frac{g_{p+1}^2}{g_{p+2}} \frac{(p + \alpha)^2}{2B_1^2 p^2(p + 2\alpha)} \left( \frac{(1 + \lambda p)^2}{(1 + \lambda p)^2 - \lambda^2} \right) \times
\]

\[
\times \left\{ B_1 - B_2 - pB_1^2 \Lambda(p, \alpha, \lambda, \mu, g) \right\} |a_{p+1}|^2
\]

\[
\leq \frac{p^2 B_1}{2(p + 2\alpha)} \left( \frac{1 + \lambda(p - 1)}{1 + \lambda(p + 1)} \right) .
\]

If \( \sigma_3 \leq \mu \leq \sigma_2 \), then

\[
|a_{p+2} - \mu a_{p+1}^2| + \frac{g_{p+1}^2}{g_{p+2}} \frac{(p + \alpha)^2}{2B_1^2 p^2(p + 2\alpha)} \left( \frac{(1 + \lambda p)^2}{(1 + \lambda p)^2 - \lambda^2} \right) \times
\]

\[
\times \left\{ B_1 + B_2 + pB_1^2 \Lambda(p, \alpha, \lambda, \mu, g) \right\} |a_{p+1}|^2
\]

\[
\leq \frac{p^2 B_1}{2(p + 2\alpha)} \left( \frac{1 + \lambda(p - 1)}{1 + \lambda(p + 1)} \right) .
\]

For any complex number \( \mu \),

\[
|a_{p+2} - \mu a_{p+1}^2| \leq \frac{p^2 B_1}{2(p + 2\alpha)} \left( \frac{1 + \lambda(p - 1)}{1 + \lambda(p + 1)} \right) \max \left\{ 1, \left| \frac{B_2}{B_1} + pB_1^2 \Lambda(p, \alpha, \lambda, \mu, g) \right| \right\} .
\]
Further,\[|a_{p+3}| \leq \frac{p^2 B_1}{3(p+3\alpha)} \left(1 + \lambda \left(\frac{p-1}{p+1}\right)\right) H(q_1, q_2)\quad (3.5)\]

where \(H(q_1, q_2)\) is as defined in Lemma 1.4,
\[q_1 := \frac{1}{2B_1} \left(4B_2 + 3pB_1^2 \left(\frac{p^2 + 3\alpha p + \alpha}{(p+\alpha)(p+2\alpha)}\right)\right)\]
\[q_2 := \begin{cases} \frac{1}{2B_1} \left(2B_3 + 3pB_1 \left(B_2 + pB_1^2 \left(\frac{p^2 + 2\alpha p + \alpha}{(p+\alpha)^2}\right)\right)\right) \\ -p^2B_1^2 \left(\frac{p^3 + 3\alpha p^2 + 3\alpha p + \alpha}{(p+\alpha)^3}\right) \end{cases}\]

These results are sharp.

**Proof.** The proof is similar to the proof of Theorem 2.1 and hence omitted.

**Remark 3.2.** For \(p = 1, \alpha = 0\),
\[g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda}\]
and
\[g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)},\]
in inequality (3.1), we get the result obtained by Srivastava and Mishra [7].

**Theorem 3.3.** Let \(\phi(z)\) be as in Theorem 2.1. If \(f(z)\) given by (1.1) belongs to \(M_{p,b,\alpha,\lambda,g}(\phi)\), then for any complex number \(\mu\),
\[|a_{p+2} - \mu a_{p+1}^2| \leq \frac{p^2|b|B_1}{2(p+2\alpha)g_{p+2}} \left(1 + \lambda \left(\frac{p-1}{p+1}\right)\right) \times \]
\[\max \left\{1, \left|bpB_1^2 \Lambda_2(p, b, \alpha, \lambda, \mu, g) + \frac{B_2}{B_1}\right|\right\}, \quad (3.6)\]

where
\[\Lambda_2(p, b, \alpha, \lambda, \mu, g) = \frac{p^2 + 2\alpha p + \alpha}{(p+\alpha)^2} - 2p\mu \frac{g_{p+2}}{g_{p+1}} \left(\frac{p+2\alpha}{(p+\alpha)^2}\right) \left(1 - \left(\frac{\lambda}{1+\lambda p}\right)^2\right)\]

**Proof.** The proof is similar to the proof of Theorem 2.1 and hence omitted.

**Remark 3.4.** For \(p = 1, \alpha = 0\) and \(\lambda = 0\), the result in (3.6) coincides with the results obtained by Ravichandran et al. [5].
REFERENCES


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T. N. Shanmugam
Department of mathematics
Anna University
Chennai-25
Tamilnadu
India
e-mail: drtns2001@yahoo.com

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka
Osaka 577-8502
Japan
e-mail: owa@math.kindai.ac.jp

C. Ramachandran
Department of Mathematics
Anna University
Chennai-25, Tamilnadu
India
e-mail: crjsp2004@yahoo.com

S. Sivasubramanian
Department of Mathematics
Waswari Engineering College
Ramaparam
Chennai-600 089
Tamilnadu
India
e-mail: drsiva2006@yahoo.com

Yayoi Nakamura
Department of Mathematics
Kinki University
Higashi-Osaka
Osaka 577-8502
Japan
e-mail: yayoi@math.kindai.ac.jp