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THE GENERALIZED HYPERGEOMETRIC FUNCTION AND ASSOCIATED FAMILIES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

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Abstract. Making use a linear operator, which is defined here by means of the Hadamard product (or convolution) involving the generalized hypergeometric function, we introduce two novel subclasses $Q_{p,q,s}(\alpha_1;A,B,\lambda)$ and $Q^+_{p,q,s}(\alpha_1;A,B,\lambda)$ of meromorphically multivalent functions of order λ $(0 \leqslant \lambda < p)$ in the punctured disc U^*. In this paper we investigate the various important properties and characteristics of these subclasses of meromorphically multivalent functions. We extend the familiar concept of neighborhoods of analytic functions. We also derive many results for the Hadamard products of functions belonging to the class $Q^+_{p,q,s}(\alpha_1;A,B,\lambda)$.

1. Introduction

Let $\sum_p$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in N = \{1,2,...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured disc

$$U^* = \{z : z \in C \quad \text{and} \quad 0 < |z| < 1\} = U \backslash \{0\}.$$

For functions $f(z) \in \sum_p$ given by (1.1) and $g(z) \in \sum_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \quad (p \in N), \tag{1.2}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z). \tag{1.3}$$

For complex parameters

$$\alpha_1,...,\alpha_q \text{ and } \beta_1,...,\beta_s \quad (\beta_j \notin Z_0^- = \{0,-1,-2,...\};\ j = 1,2,...,s),$$

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we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ by

$$
{}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k}\cdot\frac{z^k}{k!}
$$

$$
(q\leqslant s+1;\ q,\ s\in N_0=N\cup\{0\};\ z\in U), \tag{1.4}
$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$
(\theta)_\nu=\frac{\Gamma(\theta+\nu)}{\Gamma(\theta)}=\begin{cases}1 & (\nu=0;\ \theta\in C\backslash\{0\}),\\ \theta(\theta+1)...(\theta+\nu-1), & (\nu\in N;\ \theta\in C).\end{cases} \tag{1.5}
$$

Corresponding to the function $h_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$, defined by

$$
h_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)=z^{-p}\,{}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)\ , \tag{1.6}
$$

we consider a linear operator

$$
H_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s):\Sigma_p\to\Sigma_p\ ,
$$

which is defined by means of the following Hadamard product (or convolution):

$$
H_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z)=h_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)*f(z)\ . \tag{1.7}
$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$
H_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z)=z^{-p}+\sum_{k=1}^{\infty}\frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k}\cdot\frac{a_k}{k!}z^{k-p}\,. \tag{1.8}
$$

If, for convenience, we write

$$
H_{p,q,s}(\alpha_1)=H_p(\alpha_1,...\alpha_q;\beta_1,...,\beta_s)\ , \tag{1.9}
$$

then one can easily verify from the definition (1.7) that

$$
z(H_{p,q,s}(\alpha_1)f(z))'=\alpha_1H_{p,q,s}(\alpha_1+1)f(z)-(\alpha_1+p)H_{p,q,s}(\alpha_1)f(z)\,. \tag{1.10}
$$

The linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [20]. Some interesting subclasses of analytic functions associated with the generalized hypergeometric function, were considered recently by (for example) Dizok and Srivastava ([8] and [9]), Gangadharan et al. [10] and Liu [17].

For fixed parameters A,B and λ $(-1\leqslant B<A\leqslant 1;\ 0\leqslant\lambda<p;\ p\in N)$, we say that a function $f(z)\in\Sigma_p$ is in the class $Q_{p,q,s}(\alpha_1;A,B,\lambda)$ of meromorphically p-valent functions in U if it also satisfies the inequality:

$$
\left|\frac{\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}+p}{B\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}+[pB+(A-B)(p-\lambda)]}\right|<1\ \ (z\in U)\,. \tag{1.11}
$$

Furthermore, we say that a function $f(z) \in Q^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$ wherever $f(z)$ is of the form [cf. Equation. (1.1)]:

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} |a_k| z^k \quad (p \in N)\,. \tag{1.12}$$

We note that:

$$\begin{aligned} Q^{+}_{p,q,s}(\alpha_1; \beta, -\beta, \lambda) &= Q^{+}_{p,q,s}(\alpha_1, \lambda, \beta) \\ &= \left\{ f : f(z) \in \Sigma_p \text{ and } \left| \frac{\frac{z(H_{p,q,s}(\alpha_1) f(z))'}{H_{p,q,s}(\alpha_1) f(z)} + p}{\frac{z(H_{p,q,s}(\alpha_1) f(z))'}{H_{p,q,s}(\alpha_1) f(z)} + 2\lambda - p} \right| < \beta \right. \\ &\qquad \left. (z \in U;\ 0 \leqslant \lambda < p;\ p \in N;\ 0 < \beta \leqslant 1) \right\}. \end{aligned} \tag{1.13}$$

Meromorphic multivalent functions have been extensively studied by (for example) (Mogra [21] and [22]), Uralegaddi and Ganigi [28], Uralegaddi and Somanatha [29], Aouf ([4] and [5]), Aouf and Hossen [6], Srivastava et al. [27], Owa et al. [23], Joshi and Aouf [13], Joshi and Srivastava [14], Aouf et al. [7], Rania and Srivastava [24], Yang ([30] and [31]), Kulkarni et al. [15], Liu [16] and Liu and Srivastava ([18] and [19]).

In this paper we investigate the various important properties and characteristics of the classes $Q_{p,q,s}(\alpha_1; A, B, \lambda)$ and $Q^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$. Following the recent investigations by Altintas et al. [3, p. 1668], we extend the concept of neighborhoods of analytic functions, which was considered earlier by (for example) Goodman [11] and Ruscheweyh [25], to meromorphically multivalent functions, belonging to the classes $Q_{p,q,s}(\alpha_1; A, B, \lambda)$ and $Q^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$. We also derive many results for the Hadamard products of functions belonging to the p-valently meromorphic function class $Q^{+}_{p,q,s}(\alpha_1, A, B, \lambda)$.

2. Inclusion properties of the class $Q_{p,q,s}(\alpha_1; A, B, \lambda)$

We begin by recalling the following result (popularly known as Jack's lemma [12]), which we shall apply in proving our first inclusion theorem (Theorem 1 below).

LEMMA 1. Jack's lemma (see [12]) *Let the (nonconstant) function $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then*

$$z_0 w'(z_0) = \gamma w(z_0)\,, \tag{2.1}$$

where γ is a real number and $\gamma \geqslant 1$.

THEOREM 1. *Let $\alpha_1 \in R \backslash \{0\}$. If*

$$\alpha_1 \geqslant \frac{(A-B)(p-\lambda)}{1+B} \quad (-1 < B < A \leqslant 1;\, 0 \leqslant \lambda < p,\ p \in N), \tag{2.2}$$

then

$$Q_{p,q,s}(\alpha_1+1;A,B,\lambda) \subset Q_{p,q,s}(\alpha_1;A,B,\lambda).$$

Proof. Let $f(z) \in Q_{p,q,s}(\alpha_1+1;A,B,\lambda)$ and suppose that

$$\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)} = -\frac{p+[pB+(A-B)(p-\lambda)]\,w(z)}{1+Bw(z)}\,, \tag{2.3}$$

where the function $w(z)$ is either analytic or meromorphic in U, with $w(0)=0$. Then, by using (1.10) and (2.3), we have

$$\alpha_1 \frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} = \frac{\alpha_1+[\alpha_1 B-(A-B)(p-\lambda)]\,w(z)}{1+Bw(z)}\,. \tag{2.4}$$

By differentiating (2.4) with respect to z logarithmically and using (1.10), we have

$$\begin{aligned}\frac{z(H_{p,q,s}(\alpha_1+1)f(z))'}{H_{p,q,s}(\alpha_1+1)f(z)} = &- \frac{p+[pB+(A-B)(p-\lambda)]\,w(z)}{1+Bw(z)} \\ &- \frac{(A-B)(p-\lambda)zw'(z)}{[1+Bw(z)]\{\alpha_1+[\alpha_1 B-(A-B)(p-\lambda)]w(z)\}}.\end{aligned} \tag{2.5}$$

If we suppose now that

$$\max_{|z|\leqslant|z_0|} |w(z)| = |w(z_0)| = 1 \ (z_0 \in U), \tag{2.6}$$

and applying Jack's lemma, we find that

$$z_0 w'(z_0) = \gamma w(z_0) \ (\gamma \geqslant 1)\,. \tag{2.7}$$

Writing $w(z_0) = e^{i\theta}(0 \leqslant \theta \leqslant 2\pi)$ and putting $z=z_0$ in (2.5), we get

$$\begin{aligned}&\left|\frac{\frac{z_0(H_{p,q,s}(\alpha_1+1)f(z_0))'}{H_{p,q,s}(\alpha_1+1)f(z_0)}+p}{B\frac{z_0(H_{p,q,s}(\alpha_1+1)f(z_0))'}{H_{p,q,s}(\alpha_1+1)f(z_0)}+[pB+(A-B)(p-\lambda)]}\right|^2 - 1 \\ &\quad= \left|\frac{(\alpha_1+\gamma)+[\alpha_1 B-(A-B)(p-\lambda)]e^{i\theta}}{\alpha_1+[B(\alpha_1-\gamma)-(A-B)(p-\lambda)]e^{i\theta}}\right|^2 - 1 \\ &\quad= \frac{\gamma^2(1-B^2)+2\gamma[\alpha_1(1+B^2)-B(A-B)(p-\lambda)]+2\gamma[2\alpha_1 B-(A-B)(p-\lambda)]\cos\theta}{|\alpha_1+[B(\alpha_1-\gamma)-(A-B)(p-\lambda)]e^{i\theta}|^2} \\ &\qquad \left(\alpha_1 \geqslant \tfrac{(A-B)(p-\lambda)}{1+B};\, -1<B<A\leqslant 1;\ 0\leqslant\lambda<p;\ p\in N\right).\end{aligned} \tag{2.8}$$

Set

$$\begin{aligned}g(\theta) = {}&\gamma^2(1-B^2)+2\gamma[\alpha_1(1+B^2)-B(A-B)(p-\lambda)] \\ &+2\gamma[2\alpha_1 B-(A-B)(p-\lambda)]\cos\theta \quad (0\leqslant\theta\leqslant 2\pi)\,.\end{aligned} \tag{2.9}$$

Then, by hypothesis, we have

$$g(0) = \gamma^2(1-B^2) + 2\gamma(1+B)[\alpha_1(1+B) - B(A-B)(p-\lambda)] \geqslant 0$$

and

$$g(\pi) = \gamma^2(1-B^2) + 2\gamma(1+B)[\alpha_1(1-B) + (A-B)(p-\lambda)] \geqslant 0,$$

which, together, imply that

$$g(\theta) \geqslant 0 \quad (0 \leqslant \theta \leqslant 2\pi). \tag{2.10}$$

In view of (2.10), (2.8) would obviously contradict our hypothesis that $f(z) \in Q_{p,q,s}(\alpha_1 + 1; A, B, \lambda)$. Hence, we must have $|w(z)| < 1\,(z \in U)$, and we conclude from (2.3) that $f(z) \in Q_{p,q,s}(\alpha_1 + 1; A, B, \lambda)$. The proof of Theorem 1 is thus completed.

Next we prove an inclusion property associated with a certain integral transform introduced below.

THEOREM 2. *If $f(z) \in Q_{p,q,s}(\alpha_1; A, B, \lambda)$, then the function $g(z)$ given by the following integral transform:*

$$H_{p,q,s}(\alpha_1)g(z) = \left(\frac{\mu - p\beta}{z^\mu} \int_0^z t^{\mu-1}[H_{p,q,s}(\alpha_1)f(t)]^\beta dt \right)^{\frac{1}{\beta}}$$

$$\left(\beta > 0;\ \mathrm{Re}\,(\mu) \geqslant \beta \frac{p(A+1) - \lambda(A-B)}{1+B} > 0;\ 0 \leqslant \lambda < p,\ p \in N\right) \tag{2.11}$$

is also in the same class $Q_{p,q,s}(\alpha_1; A, B, \lambda)$.

Proof. Suppose that $f(z) \in Q_{p,q,s}(\alpha_1; A, B, \lambda)$ and put

$$\frac{z(H_{p,q,s}(\alpha_1)g(z))'}{H_{p,q,s}(\alpha_1)g(z)} = -\frac{p + [pB + (A-B)(p-\lambda)]\,w(z)}{1 + Bw(z)}, \tag{2.12}$$

where the function $w(z)$ is either analytic or meromorphic in U, with $w(0) = 0$. Then, by using (2.11) and (2.12), we find after some calculations that

$$\begin{aligned} \frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)} = &- \frac{p + [pB + (A-B)(p-\lambda)]\,w(z)}{1 + Bw(z)} \\ &- \frac{(A-B)(p-\lambda)zw'(z)}{[1+Bw(z)][(\mu - p\beta) + \{\mu B - \beta[pB + (A-B)(p-\lambda)]\}w(z)]}. \end{aligned} \tag{2.13}$$

The remaining part of the proof of Theorem 2 is similar to that of Theorem 1 and so is omitted.

3. Properties of the class $Q^{+}_{p,q,s}(\alpha_1;A,B,\lambda)$

In this section we assume further that

$$\alpha_j > 0\ (j=1,...,q),\ \beta_j > 0\ (j=1,...,s),\ A+B \leqslant 0\ (-1 \leqslant B < A \leqslant 1),\ 0 \leqslant \lambda < p$$

and $p \in N$.

We first determine a necessary and sufficient condition for a function $f(z) \in \Sigma_p$ of the form (1.12) to be in the class $Q^{+}_{p,q,s}(\alpha_1;A,B,\lambda)$ of meromorphically p-valent functions with positive coefficients.

THEOREM 3. *Let $f(z) \in \Sigma_p$ be given by (1.12). Then $f(z) \in Q^{+}_{p,q,s}(\alpha_1;A,B,\lambda)$ if and only if*

$$\sum_{k=p}^{\infty} [(k+p)(1-B)-(A-B)(p-\lambda)]\Gamma_{k+p}(\alpha_1)|a_k| \leqslant (A-B)(p-\alpha), \tag{3.1}$$

where, for convenience,

$$\Gamma_m(\alpha_1) = \frac{(\alpha_1)_m...(\alpha_q)_m}{m!(\beta_1)_m...(\beta_s)_m} \quad (m \in N). \tag{3.2}$$

Proof. Let $f(z) \in Q^{+}_{p,q,s}(\alpha_1;A,B,\lambda)$ is given by (1.12). Then, from (1.11) and (1.12), we have

$$\left| \frac{\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)} + p}{B\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)} + [pB+(A-B)(p-\lambda)]} \right|$$
$$= \left| \frac{\sum\limits_{k=p}^{\infty}(k+p)\Gamma_{k+p}(\alpha_1)|a_k|z^{k+p}}{(A-B)(p-\lambda)+\sum\limits_{k=p}^{\infty}[(A-B)(p-\lambda)+(k+p)B]\Gamma_{k+p}(\alpha_1)|a_k|z^{k+p}} \right| < 1 \quad (z \in U). \tag{3.3}$$

Since $|\mathrm{Re}\,(z)| \leqslant |z| (z \in C)$, choosing z to be real and letting $z \to 1^-$ through real values, (3.3) yields

$$\sum_{k=p}^{\infty}(k+p)\Gamma_{k+p}(\alpha_1)|a_k| \leqslant (A-B)(p-\lambda) + \sum_{k=p}^{\infty}[(A-B)(p-\lambda)+(k+p)B]\Gamma_{k+p}(\alpha_1)|a_k|, \tag{3.4}$$

which leads us at once to (3.1).

In order to prove the converse, we assume that the inequality (3.1) holds true.

Then, if we let $z \in \partial U$, we find from (1.12) and (3.1) that

$$\left| \frac{\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)} + p}{B\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)} + [pB + (A-B)(p-\lambda)]} \right|$$

$$\leqslant \frac{\sum_{k=p}^{\infty} (k+p)\Gamma_{k+p}(\alpha_1)|a_k|}{(A-B)(p-\lambda) + \sum_{k=p}^{\infty} [(A-B)(p-\lambda) + (k+p)B]\Gamma_{k+p}(\alpha_1)|a_k|}$$

$$< 1 \quad (z \in \partial U = \{z : z \in C \text{ and } |z| = 1\}) . \tag{3.5}$$

Hence, by the maximum modulus theorem, we have $f(z) \in Q^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$. This completes the proof of Theorem 3.

COROLLARY 1. *Let $f(z) \in \Sigma_p$ be given by (1.12). If $f(z) \in Q^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$, then*

$$|a_k| \leqslant \frac{(A-B)(p-\lambda)}{[(k+p)(1-B) - (A-B)(p-\lambda)]\Gamma_{k+p}(\alpha_1)} \quad (k \geqslant p; p \in N). \tag{3.6}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(p-\lambda)}{[(k+p)(1-B) - (A-B)(p-\lambda)]\Gamma_{k+p}(\alpha_1)} z^k \quad (k \geqslant p; p \in N). \tag{3.7}$$

THEOREM 4. *The following inclusion relationship holds true:*

$$Q^{+}_{p,q,s}(\alpha_1 + 1; A, B, \lambda) \subset Q^{+}_{p,q,s}(\alpha_1; A, B, \lambda^*) ,$$

where

$$\lambda^* = p - \frac{(1-B)(p-\lambda)\alpha_1}{(1-B)(\alpha_1 + 2p) - (A-B)(p-\lambda)} . \tag{3.8}$$

The result is sharp.

Proof. We first assume that the function $f(z)$, given by (1.12), belongs to the class $Q^{+}_{p,q,s}(\alpha_1 + 1; A, B, \lambda)$. Then, by using Theorem 3, we have

$$\sum_{k=p}^{\infty} \frac{[(k+p)(1-B) - (A-B)(p-\lambda)]\Gamma_{k+p}(\alpha_1 + 1)}{(A-B)(p-\lambda)} |a_k| \leqslant 1 . \tag{3.9}$$

In order to prove that $f(z) \in Q^{+}_{p,q,s}(\alpha_1; A, B, \lambda^*)$, we must have

$$\sum_{k=p}^{\infty} \frac{[(k+p)(1-B) - (A-B)(p-\lambda^*)]\Gamma_{k+p}(\alpha_1)}{(A-B)(p-\lambda^*)} |a_k| \leqslant 1 . \tag{3.10}$$

Thus, in view of (3.9), (3.10) will be satisfied if

$$\frac{[(k+p)(1-B)-(A-B)(p-\lambda^*)]\Gamma_{k+p}(\alpha_1)}{(A-B)(p-\lambda^*)} \leqslant \frac{[(k+p)(1-B)-(A-B)(p-\lambda)]\Gamma_{k+p}(\alpha_1+1)}{(A-B)(p-\lambda)} \quad (k \geqslant p).$$

This is equivalent to

$$\lambda^* = p - \frac{(1-B)(p-\lambda)\alpha_1}{(1-B)(\alpha_1+k+p)-(A-B)(p-\lambda)} \quad (k \geqslant p). \tag{3.11}$$

Since the right-hand side of (3.11) is an increasing function of k, putting $k = p$ in (3.11), we get (3.8).

Finally the result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(p-\lambda)}{[2p(1-B)-(A-B)(p-\lambda)]\Gamma_{2p}(\alpha_1+1)} z^p \quad (p \in N). \tag{3.12}$$

Next we prove the following growth and distortion properties for the class $Q^+_{p,q,s}(\alpha_1; A,B,\lambda)$.

THEOREM 5. *If a function $f(z)$ defined by (1.12) is in the class $Q^+_{p,q,s}(\alpha_1;A,B,\lambda)$. If the sequence $\{C_k\}$ is nondecreasing, then*

$$\left(\frac{(p+m-1)!}{(p-1)!} - \frac{(A-B)(p-\lambda)}{C_p} \cdot \frac{p!}{(p-m)!} r^{2p}\right) r^{-(p+m)} \leqslant \left|f^{(m)}(z)\right| \leqslant \left(\frac{(p+m-1)!}{(p-1)!} + \frac{(A-B)(p-\lambda)}{C_p} \cdot \frac{p!}{(p-m)!} r^{2p}\right) r^{-(p+m)} \tag{3.13}$$

$$(0 < |z| = r < 1;\ 0 \leqslant \lambda < p;\ m \in N_0 = N \cup \{0\};\ p \in N;\ p > m),$$

where

$$C_k = [(k+p)(1-B)-(A-B)(p-\lambda)]\Gamma_{k+p}(\alpha_1)\ (k \geqslant p; p \in N). \tag{3.14}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(p-\lambda)}{[2p(1-B)-(A-B)(p-\lambda)]\Gamma_{2p}(\alpha_1)} z^p\ (p \in N). \tag{3.15}$$

Proof. In view of Theorem 3, we have

$$\frac{C_p}{p!} \sum_{k=p}^{\infty} k!|a_k| \leqslant \sum_{k=p}^{\infty} C_k|a_k| \leqslant (A-B)(p-\lambda),$$

which yields

$$\sum_{k=p}^{\infty} k!|a_k| \leqslant \frac{(A-B)(p-\lambda)p!}{C_p} \quad (p \in N). \tag{3.16}$$

Now, by differentiating both sides of (1.12) m times with respect to z, we have

$$f^{(m)}(z) = (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-(p+m)} + \sum_{k=p}^{\infty} \frac{k!}{(k-m)!} |a_k| z^{k-m},$$

$$(m \in N_0;\ p \in N;\ p > m), \tag{3.17}$$

and Theorem 5 follows easily from (3.16) and (3.17).

Finally, it is easy to see that the bounds in (3.13) are attained for the function $f(z)$ given by (3.15).

Next we determine the radii of meromorphically p-valent starlikeness of order $\delta\,(0 \leqslant \delta < p)$ and meromorphically p-valent convexity of order $\delta\,(0 \leqslant \delta < p)$ for functions in the class $Q^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$.

THEOREM 6. *Let the function $f(z)$ defined by (1.12) be in the class $Q^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$ Then*

(i) $f(z)$ is meromorphically p-valent starlike of order δ $(0 \leqslant \delta < p)$ in the disc $|z| < r_1$, that is,

$$\operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right) > \delta \quad (|z| < r_1;\ 0 \leqslant \delta < p;\ p \in N), \tag{3.18}$$

where

$$r_1 = \inf_{k \geqslant p} \left\{ \frac{(p-\delta)[(k+p)(1-B) - (A-B)(p-\lambda)]\Gamma_{k+p}(\alpha_1)}{(k+\delta)(A-B)(p-\lambda)} \right\}^{\frac{1}{k+p}}. \tag{3.19}$$

(ii) $f(z)$ is meromorphically p-valent convex of order δ $(0 \leqslant \delta < p)$ in the disc $|z| < r_2$, that is,

$$\operatorname{Re}\left\{-(1+\frac{zf''(z)}{f'(z)})\right\} > \delta \quad (|z| < r_2;\ 0 \leqslant \delta < p;\ p \in N), \tag{3.20}$$

where

$$r_2 = \inf_{k \geqslant p} \left\{ \frac{p(p-\delta)[(k+p)(1-B) - (A-B)(p-\lambda)]\Gamma_{k+p}(\alpha_1)}{k(k+\delta)(A-B)(p-\lambda)} \right\}^{\frac{1}{k+p}}. \tag{3.21}$$

Each of these results is sharp for the function $f(z)$ given by (3.7).

Proof. (i) From the definition (1.12), we easily get

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \leqslant \frac{\sum\limits_{k=p}^{\infty}(k+p)|a_k||z|^{k+p}}{2(p-\delta)-\sum\limits_{k=p}^{\infty}(k-p+2\delta)|a_k||z|^{k+p}}. \tag{3.22}$$

Thus, we have the desired inequality:

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \leqslant 1\ (0\leqslant\delta<p;\ p\in N) \tag{3.23}$$

if

$$\sum_{k=p}^{\infty}\left(\frac{k+\delta}{p-\delta}\right)|a_k||z|^{k+p}\leqslant 1\,. \tag{3.24}$$

Hence, by Theorem 3, (3.24) will be true if

$$\left(\frac{k+\delta}{p-\delta}\right)|z|^{k+p}\leqslant\frac{(k+p)(1-B)-(A-B)(p-\lambda)}{(A-B)(p-\lambda)}\quad(k\geqslant p;\ p\in N). \tag{3.25}$$

The last inequality (3.25) leads us immediately to the disc $|z|<r_1$, where r_1 is given by (3.19).

(ii) In order to prove the second assertion of Theorem 6, we find from the definition (1.12) that

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\delta}\right| \leqslant \frac{\sum\limits_{k=p}^{\infty}k(k+p)|a_k||z|^{k+p}}{2p(p-\delta)-\sum\limits_{k=p}^{\infty}k(k-p+2\delta)|a_k||z|^{k+p}}. \tag{3.26}$$

Thus we have the desired inequality:

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\delta}\right| \leqslant 1\quad(0\leqslant\delta<p;\ p\in N), \tag{3.27}$$

if

$$\sum_{k=p}^{\infty}\frac{k(k+\delta)}{p(p-\delta)}|a_k||z|^{k+p}\leqslant 1\,. \tag{3.28}$$

Hence, by Theorem 3, (3.28) will be true if

$$\frac{k(k+\delta)}{p(p-\delta)}|z|^{k+p}\leqslant\frac{[(k+p)(1-B)-(A-B)(p-\lambda)]\Gamma_{k+p}(\alpha_1)}{(A-B)(p-\lambda)}\ (k\geqslant p;\ p\in N). \tag{3.29}$$

The last inequality (3.21) readily yields the disc $|z|<r_2$ with r_2 defined by (3.21), and the proof of Theorem 6 is completed by merely verifying that each assertion is sharp for the function $f(z)$ given by (3.7).

4. The concept of neighborhoods and associated partial sums

In this section, we also assume that

$$\alpha_j > (j=1,...,q) \text{ and } \beta_j > 0\,(j=1,...,s)\,.$$

Following the earlier works (based upon the familiar concept of neighorhoods of analytic functions) by Goodman [11] and Ruscheweyh [25], and (more recenlty) by Altintas et al. ([1], [2] and [3]), Liu [16], and Liu and Srivastava ([18], [19] and [20]), we begin by introducing here the δ-neighborhood of a function $f(z) \in \Sigma_p$ of the form (1.1) by means of the definition given below:

$$N_\delta(f) = \Bigg\{ g : g \in \Sigma_p : g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \quad \text{and} \\ \sum_{k=1}^{\infty} \frac{[(A-B)(p-\lambda)+k(1+|B|)]\Gamma_k(\alpha_1)}{(A-B)(p-\lambda)} |a_k - b_k| \leqslant \delta \\ (-1 \leqslant B < A \leqslant 1;\ \delta > 0;\ 0 \leqslant \lambda < p; p \in N) \Bigg\}\,. \tag{4.1}$$

Making use of the definition (4.1), we now prove Theorem 7 below.

THEOREM 7. *Let the function $f(z)$ defined by (1.1) be in the class $Q_{p,q,s}(\alpha_1; A,B,\lambda)$. If $f(z)$ satisfies the following condition:*

$$\frac{f(z)+\varepsilon z^{-p}}{1+\varepsilon} \in Q_{p,q,s}(\alpha_1;A,B,\lambda) \quad (\varepsilon \in C,\ |\varepsilon| < \delta,\ \delta > 0)\,, \tag{4.2}$$

then

$$N_\delta(f) \subset Q_{p,q,s}(\alpha_1;A,B,\lambda)\,. \tag{4.3}$$

Proof. It is easily seen from (1.11) that $g(z) \in Q_{p,q,s}(\alpha_1;A,B,\lambda)$ if and only if, for any complex σ with $|\sigma| = 1$,

$$\frac{\frac{z(H_{p,q,s}(\alpha_1)g(z))'}{H_{p,q,s}(\alpha_1)g(z)} + p}{B\frac{z(H_{p,q,s}(\alpha_1)g(z))'}{H_{p,q,s}(\alpha_1)g(z)} + [pB+(A-B)(p-\lambda)]} \neq \sigma \quad (z \in U;\ \sigma \in C;\ |\sigma| = 1)\,, \tag{4.4}$$

which is equivalent to

$$\frac{(g*h)(z)}{z^{-p}} \neq 0 \quad (z \in U)\,, \tag{4.5}$$

where, for convenience,

$$h(z) = z^{-p} + \sum_{k=1}^{\infty} c_k z^{k-p} \\ = z^{-p} + \sum_{k=1}^{\infty} \frac{[k(1-\sigma B)-(A-B)(p-\lambda)\sigma]\Gamma_k(\alpha_1)}{(B-A)(p-\lambda)\sigma} z^{k-p}\,. \tag{4.6}$$

From (4.6), we have

$$|c_k| = \left| \frac{[k(1-\sigma B)-(A-B)(p-\lambda)\sigma]\Gamma_k(\alpha_1)}{(B-A)(p-\lambda)\sigma} \right|$$
$$\leqslant \frac{[k(1+|B|)+(A-B)(p-\lambda)]\Gamma_k(\alpha_1)}{(A-B)(p-\lambda)} \quad (k;p \in N). \tag{4.7}$$

Now if $f(z) \in \Sigma_p$, given by (1.1), satisfies the condition (4.2), then (4.5) yields

$$\left| \frac{(f*h)(z)}{z^{-p}} \right| \geqslant \delta \quad (z \in U;\ \delta > 0). \tag{4.8}$$

By letting

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \in N_\delta(f), \tag{4.9}$$

so that

$$\left| \frac{[f(z)-g(z)]*h(z)}{z^{-p}} \right| = \left| \sum_{k=1}^{\infty} (a_k - b_k) c_k z^k \right|$$
$$\leqslant \sum_{k=1}^{\infty} \frac{[(A-B)(p-\lambda)+k(1+|B|)]\Gamma_k(\alpha_1)}{(A-B)(p-\lambda)} |a_k - b_k|$$
$$< \delta \quad (z \in U; \delta > 0), \tag{4.10}$$

which leads us to (4.5), and hence also (4.4) for any $\sigma \in C$ such that $|\sigma| = 1$. This implies that $g(z) \in Q_{p,q,s}(\alpha_1; A, B, \lambda)$, which evidenlty completes the proof of the assertion (4.3) of Theorem 7.

We now define the δ-neighorhood of a function $f(z) \in \Sigma_p$ of the form (1.12) as follows

$$N_\delta^+(f) = \Bigg\{ g : g \in \Sigma_p : g(z) = z^{-p} + \sum_{k=p}^{\infty} |b_k| z^k \text{ and }$$
$$\sum_{k=p}^{\infty} \frac{[(k+p)(1-B)-(A-B)(p-\lambda)]\Gamma_{k+p}(\alpha_1)}{(A-B)(p-\lambda)} ||a_k| - |b_k|| \leqslant \delta,$$
$$(-1 \leqslant B < A \leqslant 1;\ \delta > 0;\ 0 \leqslant \lambda < p;\ p \in N) \Bigg\}. \tag{4.11}$$

THEOREM 8. *Let the function $f(z)$ defined by (1.12) be in the class $Q_{p,q,s}^+(\alpha_1 + 1; A, B, \lambda)(-1 \leqslant B < A \leqslant 1; A+B \leqslant 0;\ 0 \leqslant \lambda < p; p \in N)$. Then*

$$N_\delta^+(f) \subset Q_{p,q,s}^+(\alpha_1; A, B, \lambda) \quad (\delta = \frac{2p}{\alpha_1 + 2p}). \tag{4.12}$$

The result is sharp.

Proof. Making use of the same method as in the proof of Theorem 7, we can show that [cf. Equation (4.6)]

$$\begin{aligned} h(z) &= z^{-p} + \sum_{k=p}^{\infty} c_k z^k \\ &= z^{-p} + \sum_{k=p}^{\infty} \frac{\{(k+p) - \sigma[(A-B)(p-\lambda) + (k+p)B]\}\Gamma_{k+p}(\alpha_1)}{(B-A)(p-\lambda)\sigma} z^k . \end{aligned} \tag{4.13}$$

Thus, under the constraints: $A+B \leqslant 0, 0 \leqslant \lambda < p$ and $p \in N$, which are provided by the hypothesis of Theorem 8, if $f(z) \in Q^{+}_{p,q,s}(\alpha_1+1;A,B,\lambda)$ is given by (1.12), we obtain

$$\begin{aligned} \left| \frac{(f*h)(z)}{z^{-p}} \right| &= \left| 1 + \sum_{k=p}^{\infty} c_k |a_k| z^{k+p} \right| \\ &\geqslant 1 - \frac{\alpha_1}{\alpha_1+2p} \sum_{k=p}^{\infty} \frac{[(k+p)(1-B) - (A-B)(p-\lambda)]\Gamma_{k+p}(\alpha_1+1)}{(A-B)(p-\alpha)} |a_k| . \end{aligned}$$

Also, from Theorem 3, we get

$$\left| \frac{(f*h)(z)}{z^{-p}} \right| \geqslant 1 - \frac{\alpha_1}{\alpha_1+2p} = \frac{2p}{\alpha_1+2p} = \delta .$$

The remaining part of the proof of Theorem 8 is similar to that of Theorem 7, and we skip the details involved.

To show the sharpness of the assertion of Theorem 8, we consider the functions $f(z)$ and $g(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(p-\lambda)}{[2p(1-B) - (A-B)(p-\lambda)]\Gamma_{2p}(\alpha_1+1)} z^p \in Q^{+}_{p,q,s}(\alpha_1+1;A,B,\lambda) \tag{4.14}$$

and

$$\begin{aligned} g(z) = z^{-p} + \Bigg[& \frac{(A-B)(p-\lambda)}{[2p(1-B) - (A-B)(p-\lambda)]\Gamma_{2p}(\alpha_1+1)} \\ & + \frac{(A-B)(p-\lambda)\delta'}{[2p(1-B) - (A-B)(p-\lambda)]\Gamma_{2p}(\alpha_1)} \Bigg] z^p, \end{aligned} \tag{4.15}$$

where

$$\delta' > \delta = \frac{2p}{\alpha_1+2p} .$$

Clearly, the function $g(z)$ belongs to $N^{+}_{\delta'}(f)$. On the other hand, we find from Theorem 3 that $g(z)$ is not in the class $Q^{+}_{p,q,s}(\alpha_1;A,B,\lambda)$. Thus the proof of Theorem 8 is completed.

Next, we prove the following result.

THEOREM 9. *Let $f(z) \in \Sigma_p$ be given by (1.1) and define the partial sums $s_1(z)$ and $s_n(z)$ as follows:*

$$s_1(z) = z^{-p} \text{ and } s_n(z) = z^{-p} + \sum_{k=1}^{n-1} a_k z^{k-p} \quad (n \in N), \tag{4.16}$$

Suppose also that

$$\sum_{k=1}^{\infty} d_k |a_k| \leqslant 1 \quad (d_k = \frac{[(A-B)(p-\lambda)+k(1+|B|)]\Gamma_k(\alpha_1)}{(A-B)(p-\lambda)}). \tag{4.17}$$

then

(i) $f(z) \in Q_{p,q,s}(\alpha_1; A, B, \lambda)$,

(ii) If $\{\Gamma_k(\alpha_1)\}$ $(k \in N)$ is nondecreasing and

$$\Gamma_1(\alpha_1) > \frac{(A-B)(p-\lambda)}{[(A-B)(p-\lambda)+(1+|B|)]}. \tag{4.18}$$

Then

$$\mathrm{Re}\left\{\frac{f(z)}{s_n(z)}\right\} > 1 - \frac{1}{d_n} \quad (z \in U; n \in N), \tag{4.19}$$

and

$$\mathrm{Re}\left\{\frac{s_n(z)}{f(z)}\right\} > \frac{d_n}{1+d_n} \quad (z \in U; n \in N). \tag{4.20}$$

Each of the bounds in (4.19) and (4.20) is the best possible for each $n \in N$.

Proof. (i) It is not difficult to see that $z^{-p} \in Q_{p,q,s}(\alpha_1; A, B, \lambda)(p \in N)$. Thus, from Theorem 7 and the hypothesis (4.17) of Theorem 9, we have

$$N_1(z^{-p}) \subset Q_{p,q,s}(\alpha_1; A, B, \lambda), \tag{4.21}$$

which shows that $f(z) \in Q_{p,q,s}(\alpha_1; A, B, \lambda)$ as asserted by Theorem 9.

(ii) Under the hypothesis in Part (ii) of Theorem 9, we can see from (4.17) that

$$d_{k+1} > d_k > 1 \quad (k \in N). \tag{4.22}$$

Therefore, we have

$$\sum_{k=1}^{n-1} |a_k| + d_n \sum_{k=n}^{\infty} |a_k| \leqslant \sum_{k=1}^{\infty} d_k |a_k| \leqslant 1, \tag{4.23}$$

by using the hypothesis (4.17) of Theorem 9 again.

By setting

$$g_1(z) = d_n \left[\frac{f(z)}{s_n(z)} - \left(1 - \frac{1}{d_n}\right)\right] = 1 + \frac{d_n \sum\limits_{k=n}^{\infty} a_k z^k}{1 + \sum\limits_{k=1}^{n-1} a_k z^k}, \tag{4.24}$$

and applying (4.23), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leqslant \frac{d_n \sum\limits_{k=n}^{\infty} |a_k|}{2-2\sum\limits_{k=1}^{n-1}|a_k| - d_n \sum\limits_{k=n}^{\infty}|a_k|} \leqslant 1 \quad (z \in U), \tag{4.25}$$

which readily yields the assertion (4.19) of Theorem 9. If we take

$$f(z) = z^{-p} - \frac{z^{n-p}}{d_n}, \tag{4.26}$$

then

$$\frac{f(z)}{s_n(z)} = 1 - \frac{z^n}{d_n} \to 1 - \frac{1}{d_n} \ (z \to 1^-),$$

which shows that the bound in (4.19) is the best possible for each $n \in N$.

Similarly, if we put

$$g_2(z) = (1+d_n)\left(\frac{s_n(z)}{f(z)} - \frac{d_n}{1+d_n}\right) = 1 - \frac{(1+d_n)\sum\limits_{k=n}^{\infty} a_k z^k}{1+\sum\limits_{k=1}^{\infty} a_k z^k} \tag{4.27}$$

and make use of (4.23), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leqslant \frac{(1+d_n)\sum\limits_{k=n}^{\infty}|a_k|}{2-2\sum\limits_{k=1}^{n-1}|a_k| + (1-d_n)\sum\limits_{k=n}^{\infty}|a_k|} \leqslant 1 \quad (z \in U), \tag{4.28}$$

which leads us immediately to the assertion (4.20) of Theorem 9.

The bound in (4.20) is sharp for each $n \in N$, with the extremal function $f(z)$ given by (4.26). The proof of Theorem 9 is thus completed.

5. Convolution properties for the class $Q^+_{p,q,s}(\alpha_1;A,B,\lambda)$

For the functions

$$f_j(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,j}| z^k \quad (j = 1,2; p \in N), \tag{5.1}$$

we denote by $(f_1 * f_2)(z)$ the Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,1}||a_{k,2}| z^k. \tag{5.2}$$

Throughout this section, we assume further that

$$C(p,\lambda,A,B,k) = (k+p)(1-B) - (A-B)(p-\lambda)\ (k \geqslant p) \tag{5.3}$$

and

$$D(p,\lambda,A,B) = (A-B)(p-\lambda). \tag{5.4}$$

THEOREM 10. *Let the functions* $f_j(z)$ $(j=1,2)$ *defined by (5.1) be in the class* $Q^{+}_{p,q,s}(\alpha_1;A,B,\lambda)$. *Then* $(f_1 * f_2)(z) \in Q^{+}_{p,q,s}(\alpha_1;A,B,\gamma)$, *where*

$$\gamma = p\left\{1 - \frac{2(1-B)(A-B)(p-\lambda)^2}{[2p(1-B)-(A-B)(p-\lambda)]^2\Gamma_{2p}(\alpha_1)+(A-B)^2(p-\lambda)^2}\right\}. \tag{5.5}$$

The result is sharp for the functions $f_j(z)(j=1,2)$ *given by*

$$f_j(z) = z^{-p} + \frac{(A-B)(p-\lambda)}{[2p(1-B)-(A-B)(p-\lambda)]\Gamma_{2p}(\alpha_1)} z^p \quad (j=1,2;\ p\in N). \tag{5.6}$$

Proof. Employing the technique used earlier by Schild and Silverman [26], we need to find the largest γ such that

$$\sum_{k=p}^{\infty} \frac{C(p,\gamma,A,B,k)\Gamma_{p+k}(\alpha_1)}{D(p,\gamma,A,B)} |a_{k,1}||a_{k,2}| \leqslant 1 \tag{5.7}$$

for $f_j(z) \in Q^{+}_{p,q,s}(\alpha_1;A,B,\lambda)$ $(j=1,2)$. Since $f_j(z) \in Q^{+}_{p,q,s}(\alpha_1;A,B,\lambda)$ $(j=1,2)$, we readily see that

$$\sum_{k=p}^{\infty} \frac{C(p,\lambda,A,B,k)\Gamma_{p+k}(\alpha_1)}{D(p,\lambda,A,B)} |a_{k,j}| \leqslant 1 \quad (j=1,2). \tag{5.8}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=p}^{\infty} \frac{C(p,\lambda,A,B,k)\Gamma_{p+k}(\alpha_1)}{D(p,\lambda,A,B)} \sqrt{|a_{k,1}||a_{k,2}|} \leqslant 1. \tag{5.9}$$

This implies that we only need to show that

$$\frac{C(p,\gamma,A,B,k)}{(p-\gamma)} |a_{k,1}||a_{k,2}| \leqslant \frac{C(p,\lambda,A,B,k)}{(p-\lambda)} \sqrt{|a_{k,1}||a_{k,2}|} \quad (k \geqslant p) \tag{5.10}$$

or, equivalently, that

$$\sqrt{|a_{k,1}||a_{k,2}|} \leqslant \frac{(p-\gamma)C(p,\lambda,A,B,k)}{(p-\lambda)C(p,\gamma,A,B,k)} \quad (k \geqslant p). \tag{5.11}$$

Hence, by the inequality (5.9), it is sufficient to prove that

$$\frac{D(p,\lambda,A,B)}{C(p,\lambda,A,B,k)\Gamma_{p+k}(\alpha_1)} \leqslant \frac{(p-\gamma)C(p,\lambda,A,B,k)}{(p-\lambda)C(p,\gamma,A,B,k)} \quad (k \geqslant p). \tag{5.12}$$

It follows from (5.12) that

$$\gamma \leqslant p - \frac{(k+p)(1-B)(A-B)(p-\lambda)^2}{[C(p,\lambda,A,B,k)]^2\Gamma_{p+k}(\alpha_1)+[D(p,\lambda,A,B)]^2} \quad (k \geqslant p). \tag{5.13}$$

Now, defining the function $\varphi(k)$ by

$$\varphi(k) = p - \frac{(k+p)(1-B)(A-B)(p-\lambda)^2}{[C(p,\lambda,A,B,k)]^2\Gamma_{p+k}(\alpha_1) + [D(p,\lambda,A,B)]^2} \quad (k \geqslant p), \tag{5.14}$$

we see that $\varphi(k)$ is an increasing function of k. Therefore, we conclude that

$$\gamma \leqslant \varphi(p) = p\left\{1 - \frac{2(1-B)(A-B)(p-\lambda)^2}{[2p(1-B)-(A-B)(p-\lambda)]^2\Gamma_{2p}(\alpha_1) + (A-B)^2(p-\lambda)^2}\right\}, \tag{5.15}$$

which evidently completes the proof of Theorem 10.

Putting $A = \beta$ and $B = -\beta (0 < \beta \leqslant 1)$ in Theorem 10, we obtain the following consequence.

COROLLARY 2. *Let the functions* $f_j(z)$ $(j = 1,2)$ *defined by (5.1) be in the class* $Q^+_{p,q,s}(\alpha_1,\lambda,\beta)$. *Then* $(f_1 * f_2)(z) \in Q^+_{p,q,s}(\alpha_1,\gamma,\beta)$, *where*

$$\gamma = p\left(1 - \frac{\beta(1+\beta)(p-\lambda)^2}{(p+\alpha\beta)^2\Gamma_{2p}(\alpha_1) + \beta^2(p-\lambda)^2}\right). \tag{5.16}$$

The result is sharp for the functions $f_j(z)(j = 1,2)$ *given by*

$$f_j(z) = z^{-p} + \frac{\beta(p-\lambda)}{(p+\alpha\beta)\Gamma_{2p}(\alpha_1)} z^p \quad (j = 1,2;\ p \in N). \tag{5.17}$$

Using arguments similar to these in the proof of Theorem 10, we obtain the following result:

THEOREM 11. *Let the function* $f_1(z)$ *defined by (5.1) be in the class* $Q^+_{p,q,s}(\alpha_1;A,B,\lambda)$. *Suppose also that the function* $f_2(z)$ *defined by (5.1) be in the class* $Q^+_{p,q,s}(\alpha_1;A,B,\gamma)$. *Then* $(f_1 * f_2)(z) \in Q^+_{p,q,s}(\alpha_1,A,B,\xi)$, *where*

$$\xi = p\left(1 - \frac{2(1-B)(A-B)(p-\lambda)(p-\gamma)}{[2p(1-B)-(A-B)(p-\lambda)][2p(1-B)-(A-B)(p-\gamma)]\Gamma_{2p}(\alpha_1) + \Omega}\right).$$

$$(\Omega = (A-B)^2(p-\lambda)(p-\gamma)). \tag{5.18}$$

The result is sharp for the functions $f_j(z)(j = 1,2)$ *given by*

$$f_1(z) = z^{-p} + \frac{(A-B)(p-\lambda)}{[2p(1-B)-(A-B)(p-\lambda)]\Gamma_{2p}(\alpha_1)} z^p \quad (p \in N) \tag{5.19}$$

and

$$f_2(z) = z^{-p} + \frac{(A-B)(p-\gamma)}{[2p(1-B)-(A-B)(p-\gamma)]\Gamma_{2p}(\alpha_1)} z^p \quad (p \in N). \tag{5.20}$$

Putting $A = \beta$ and $B = -\beta (0 < \beta \leqslant 1)$ in Theorem 11, we obtain Corollary 3 below.

COROLLARY 3. *Let the function* $f_1(z)$ *defined by (5.1) be in the class* $Q^+_{p,q,s}(\alpha_1, \lambda,\beta)$. *Suppose also that the function* $f_2(z)$ *defined by (5.1) be in the class* $Q^+_{p,q,s}(\alpha_1,\gamma,\beta)$. *Then* $(f_1 * f_2)(z) \in Q^+_{p,q,s}(\alpha_1,\eta,\beta)$, *where*

$$\eta = p\left(1 - \frac{\beta(1+\beta)(p-\lambda)(p-\gamma)}{(p+\lambda\beta)(p+\gamma\beta)\Gamma_{2p}(\alpha_1)+\beta^2(p-\lambda)(p-\gamma)}\right). \tag{5.21}$$

The result is the best possible for the functions $f_j(z)(j=1,2)$ *given by*

$$f_1(z) = z^{-p} + \frac{\beta(p-\lambda)}{(p+\lambda\beta)\Gamma_{2p}(\alpha_1)} z^p \quad (p \in N) \tag{5.22}$$

and

$$f_2(z) = z^{-p} + \frac{\beta(p-\gamma)}{(p+\gamma\beta)\Gamma_{2p}(\alpha_1)} z^p \quad (p \in N). \tag{5.23}$$

THEOREM 12. *Let the functions* $f_j(z)$ $(j=1,2)$ *defined by (5.1) be in the class* $Q^+_{p,q,s}(\alpha_1;A,B,\lambda)$. *Then the function* $h(z)$ *defined by*

$$h(z) = z^{-p} + \sum_{k=p}^{\infty}(|a_{k,1}|^2+|a_{k,2}|^2)z^k \tag{5.24}$$

belongs to the class $Q^+_{p,q,s}(\alpha_1;A,B,\zeta)$, *where*

$$\zeta = p\left(1 - \frac{4(1-B)(A-B)(p-\lambda)^2}{[2p(1-B)-(A-B)(p-\lambda)]^2\Gamma_{2p}(\alpha_1)+2(A-B)^2(p-\lambda)^2}\right). \tag{5.25}$$

This result is sharp for the functions $f_j(z)(j=1,2)$ *defined by (5.6).*

Proof. Noting that

$$\sum_{k=p}^{\infty}\frac{[C(p,\lambda,A,B,k)]^2}{[D(p,\lambda,A,B)]^2}\left[\Gamma_{k+p}(\alpha_1)\right]^2|a_{k,j}|^2$$
$$\leqslant \left(\sum_{k=p}^{\infty}\frac{C(p,\lambda,A,B,k)\Gamma_{k+p}(\alpha_1)}{D(p,\lambda,A,B)}|a_{k,j}|\right)^2 \leqslant 1 \quad (j=1,2), \tag{5.26}$$

for $f_j(z) \in Q^+_{p,q,s}(\alpha_1;A,B,\lambda)$ $(j=1,2)$, we have

$$\sum_{k=p}^{\infty}\frac{[C(p,\lambda,A,B,k)]^2}{2[D(p,\lambda,A,B)]^2}\left[\Gamma_{k+p}(\alpha_1)\right]^2(|a_{k,1}|^2+|a_{k,2}|^2) \leqslant 1. \tag{5.27}$$

Therefore, we have to find the largest ζ such that

$$\frac{C(p,\zeta,A,B,k)}{(p-\zeta)} \leqslant \frac{[C(p,\lambda,A,B,k)]^2\Gamma_{k+p}(\alpha_1)}{2(A-B)(p-\lambda)^2} \quad (k \geqslant p), \tag{5.28}$$

that is, that

$$\zeta \leqslant p - \frac{2(k+p)(1-B)(A-B)(p-\lambda)^2}{[C(p,\lambda,A,B,k)]^2\Gamma_{k+p}(\alpha_1)+2[D(p,\lambda,A,B)]^2} \quad (k \geqslant p). \tag{5.29}$$

Now, defining a function $\Psi(k)$ by

$$\Psi(k) = p - \frac{2(k+p)(1-B)(A-B)(p-\lambda)^2}{[C(p,\lambda,A,B,k)]^2\Gamma_{k+p}(\alpha_1)+2[D(p,\lambda,A,B)]^2} \quad (k \geqslant p), \tag{5.30}$$

we observe that $\Psi(k)$ is an increasing function of k. We thus conclude that

$$\zeta \leqslant \Psi(p) = p\left(1 - \frac{4(1-B)(A-B)(p-\lambda)^2}{[2(1-B)-(A-B)(p-\lambda)]^2 1 - \Gamma_{2p}(\alpha_1)+2(A-B)^2(p-\lambda)^2}\right), \tag{5.31}$$

which completes the proof of Theorem 12.

Setting $A = \beta$ and $B = -\beta (0 < \beta \leqslant 1)$ in Theorem 12, we obtain the following corollary.

COROLLARY 4. *Let the functions* $f_j(z)$ $(j = 1,2)$ *defined by (5.1) be in the class* $Q^+_{p,q,s}(\alpha_1,\lambda,\beta)$. *Then the function* $h(z)$ *defined by (5.24) belongs to the class* $Q^+_{p,q,s}(\alpha_1,\Im,\beta)$, *where*

$$\Im = p\left(1 - \frac{2\beta(1+\beta)(p-\lambda)^2}{(p+\lambda\beta)^2\Gamma_{2p}(\alpha_1)+2\beta^2(p-\lambda)^2}\right). \tag{5.32}$$

The result is sharp for the functions $f_j(z)$ $(j = 1,2)$ *given already by (5.17).*

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