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GENERALIZED HYERS–ULAM–RASSIAS STABILITY OF FUNCTIONAL INEQUALITIES AND FUNCTIONAL EQUATIONS

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Abstract. In this paper, the definitions of the stability of functional inequalities and functional equations are given. Also we prove the generalized Hyers-Ulam-Rassias stability of the following functional inequality and functional equation

$$\|f(x)+f(y)+2f(z)\| \leqslant \left\|2f\left(\frac{x+y}{2}+z\right)\right\|,$$

$$f(x)+f(y)+2f(z) = 2f\left(\frac{x+y}{2}+z\right),$$

in the spirit of the Hyers' direct method for approximately additive mappings.

1. Introduction

The stability problem of equations originated from a question of Ulam [1] concerning the stability of group homomorphisms.

ULAM'S QUESTION. Let G_1 be a group and G_2 a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) \leqslant \delta$ for all $x, y \in G_1$, then there is a homomorphism $g : G_1 \to G_2$ with $d(f(x), g(x)) \leqslant \varepsilon$ for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it.

In 1941, Hyers [2] considered the case of approximately additive mappings between Banach spaces and proved the following result.

THEOREM 1.1 (D. H. HYERS). *Suppose that E_1 and E_2 are Banach spaces and $f : E_1 \to E_2$ satisfies the following condition: there is an $\varepsilon \geqslant 0$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon, \qquad \forall x, y \in E_1.$$

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Then the limit

$$h(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x), \qquad \forall x \in E_1$$

exists and there exists a unique additive mapping $h : E_1 \to E_2$ *such that*

$$\|f(x) - h(x)\| \leqslant \varepsilon, \qquad \forall x \in E_1.$$

Moreover, if $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each* $x \in E_1$, *then the* h *is linear.*

Taking this famous result into consideration, the additive Cauchy equation

$$f(x+y) = f(x) + f(y) \tag{1.0}$$

is said to have the Hyers-Ulam stability (HU-stability, shortly) on (E_1, E_2) if for each $f : E_1 \to E_2$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon, \qquad \forall x, y \in E_1$$

for some $\varepsilon \geqslant 0$, there exists an additive $h : E_1 \to E_2$ such that $f - h$ is bounded on E_1.

The method which was provided by Hyers, and which produces the additive h, was called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations.

In 1978, Th. M. Rassias [3] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded.

THEOREM 1.2 (TH. M. RASSIAS). *Let* E_1 *and* E_2 *be two Banach spaces and* $f : E_1 \to E_2$ *be a mapping such that* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* x. *Assume that there exists* $\theta \geqslant 0$ *and* $p \in [0, 1)$ *such that*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta(\|x\|^p + \|y\|^p), \qquad \forall x, y \in E_1. \tag{1.1}$$

Then there exists a unique linear mapping $T : E_1 \to E_2$ *such that*

$$\|f(x) - T(x)\| \leqslant \frac{2\theta}{2 - 2^p} \|x\|^p, \qquad \forall x \in E_1.$$

This phenomenon of stability was introduced by Rassias and called the Hyers-Ulam-Rassias stability (HUR-stability, shortly). Clearly, the Hyers' theorem is the special case of the Rassias' theorem.

In 1990, Th. M. Rassias [4] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. In 1991, Z. Gajda [5] following the same approach as in [3], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [5], as well as by Th. M. Rassias and P. Šemrl [6], that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. The counterexamples of Z. Gajda [5], as well as of Th. M. Rassias and P. Šemrl [6], have stimulated several mathematicians to invent new definitions of approximately additive or approximately linear mappings, cf. P. Găvruta [7] and S. Jung [8], who among others studied the HUR-stability of functional equations. The inequality (1.1) that was introduced for the first time by Th. M. Rassias [3] provided a lot of influence in the development of a generalization of the HU-stability concept. This new concept is known as HUR-stability of functional equations (cf. the books of P. Czerwik [9] and D. H. Hyers, G. Isac and Th. M. Rassias [10]).

J. M. Rassias [11] following the spirit of the innovative approach of Th. M. Rassias [3] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^p$ for $p,q \in \mathbb{R}$ with $p+q \neq 1$.

In 1992, a generalized of Rassias' theorem was obtained by Gãvruta in [7].

THEOREM 1.3 (GÃVRUTA). *Suppose that $(G,+)$ is an abelian group, E is a Banach space and that there is a function $\varphi : G \times G \to [0,\infty)$ satisfying*

$$\tilde{\varphi}(x,y) := \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\varphi(2^n x, 2^n y) < \infty, \qquad \forall x,y \in G.$$

If $f : G \to E$ is a mapping with

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varphi(x,y), \qquad \forall x,y \in G,$$

then there exists a unique additive mapping $T : G \to E$ such that

$$\|f(x) - T(x)\| \leqslant \tilde{\varphi}(x,x), \qquad \forall x,y \in G.$$

We call this phenomenon of stability the generalized Hyers-Ulam-Rassias stability (GHUR-stability, shortly).

In 1996, G. Isac and Th. M. Rassias [12] applied the HUR-stability theory to prove fixed point theorems and study some new applications in nonlinear analysis. In [13], D. H. Hyers, G. Isac and Th. M. Rassias studied the asymptoticity aspect of HU-stability of mappings. During the past few years several mathematicians have published on various generalizations and applications of HU-stability and HUR-stability to a number of functional equations and mappings, for example: quadratic functional equation, invariant means, multiplicative mappings-superstability, bounded nth differences, convex functions, generalized orthogonality functional equation, Euler-Lagrange functional equation, Navier-Stokes equations. Several mathematicians have contributed their papers on these subjects; see C. Park [14-16], Th. M. Rassias [17-19] and F. Skof [20].

In the period 1982-1994 further generalizations were obtained by J. M. Rassias [11, 21-23].

J. M. Rassias and M. J. Rassias [24] considered and investigated quadratic equations involving a product of powers of norms following the innovative approach of Th. M. Rassias who had introduced the concept of the unbounded Cauchy difference in the year 1978 and he had treated the subject for the sum of powers of norms. They studied the problem in which an approximate quadratic mapping degenerates to a genuine quadratic mapping. Analogous results could be investigated with additive type equations involving a product of powers of norms. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 14, 15, 18, 25]).

Let G be an n-divisible abelian group where $n \in \mathbb{N}$ (i.e., $a \mapsto na : G \to G$ is a surjection), $\mathbb{R}$ the set the set of all real numbers, $\mathbb{Q}$ the set of all rational numbers and let $\mathbb{N}$ be the set of all natural numbers. Let X be a normed space with norm $\|\cdot\|_X$ and Y be a Banach space with norm $\|\cdot\|_Y$. Denote by $M(G,X) =$ the set of all mappings from G into X, let $L^\infty(G,X) = \{f : G \to X | \ \|f\|_\infty := \sup_{x\in G} \|f(x)\|_X < \infty\}$ and $\mathbb{R}^+ = \{r \in \mathbb{R} : r \geqslant 0\}$. The sets $M(G,Y)$, $M(G^r,X)$ and $M(G^r,\mathbb{R}^+))$ can be defined similarly.

REMARK. If G is an n-divisible abelian group and n is even, then G must be a 2-divisible abelian group. For a positive integer $r > 1$, let G^r be the r copies of G, i.e.,

$$G^r = \{(x_1,x_2,\ldots,x_r) : x_j \in G\}.$$

Next, we give the definitions of the stability of functional inequalities and functional equations.

DEFINITION 1.1. Given mappings $E : M(G,X) \to M(G^r,\mathbb{R}^+)$, $\varphi : G^r \to \mathbb{R}^+$ and $\psi : G \to \mathbb{R}^+$, if

$$E(f)(x_1,x_2,\ldots,x_r) \leqslant \varphi(x_1,x_2,\ldots,x_r), \qquad \forall x_1,x_2,\ldots,x_r \in G,$$

implies that there exists $g \in M(G,X)$ such that $E(g) \leqslant 0$ and $\|f(x) - g(x)\|_\infty \leqslant \psi(x)$ $(\forall x \in G)$, then we say that the inequality $E(f) \leqslant 0$ is (φ,ψ)-stable in $M(G,X)$. In this case, we also say that the solutions of the inequality $E(f) \leqslant 0$ is (φ,ψ)-stable in $M(G,X)$.

DEFINITION 1.2. Given mappings $E : M(G,X) \to M(G^r,X)$, $\varphi : G^r \to \mathbb{R}^+$ and $\psi : G \to \mathbb{R}^+$, if

$$\|E(f)(x_1,x_2,\ldots,x_r)\|_\infty \leqslant \varphi(x_1,x_2,\ldots,x_r), \qquad \forall x_1,x_2,\ldots,x_r \in G,$$

implies that there exists $g \in M(G,X)$ such that $E(g) = 0$ and $\|f(x) - g(x)\|_\infty \leqslant \psi(x)$ $(\forall x \in G)$, then we say that the equation $E(f) = 0$ is (φ,ψ)-stable in $M(G,X)$. In this case, we also say that the solutions of the equation $E(f) = 0$ is (φ,ψ)-stable in $M(G,X)$

One of the most famous functional equations is the additive functional equation (1.0). In 1821, it was first solved by A. L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of A. L. Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solution of the additive functional equation (1.0) is called an additive function.

It is well known that if an additive function $f : \mathbb{R} \to \mathbb{R}$ satisfies one of the following conditions:

(a) f is continuous at a point;
(b) f is monotonic on an interval of positive length;
(c) f is bounded on an interval of positive length;

(d) f is integrable;

(e) f is measurable,

then f is of the form $f(x) = cx$ with a real constant c. That is to say f has the linearity. That is, if a solution of the additive equation (1.0) satisfies one of the very weak conditions (a) to (e), then it does have the linearity. But every additive functional which is not linear displays a very strange behavior. More precisely, the graph of every additive functional $f:\mathbb{R}\to\mathbb{R}$ which is not of the form $f(x) = cx$ is dense in $\mathbb{R}^2$.

There are a number of variations of the additive functional equations.

EXAMPLE 1.1. The following equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$$

is called a Jensen's functional equation. Every solution of a Jensen's functional equation is called a Jensen . It is well known that a f between real vector spaces with $f(0) = 0$ is a Jensen if and only if it is an additive (see [26] or [27]). We may refer to the paper [28] of H. Haruki and Th. M. Rassias for the entire solutions of a generalized Jensen's functional equation.

DEFINITION 1.3. For a mapping $f : G \to X$, the equation

$$f(x) + f(y) + nf(z) = nf\left(\frac{x+y}{n} + z\right), \qquad \forall x,y,z \in G, n \in \mathbb{N}\setminus\{0\} \tag{1.2}$$

is said to be a generalized Cauchy-Jensen equation (GCJE, shortly). Specially, when $n = 2$, it is called a Cauchy-Jensen equation (CJE, shortly).

2. Functional inequalities associated with GCJE

In this section, let G be an n-divisible abelian group for some positive integer n.

PROPOSITION 2.1. *A mapping $f : G \to X$ is additive if and only if it satisfies*

$$\|f(x) + f(y) + nf(z)\|_X \leqslant \left\|nf\left(\frac{x+y}{n} + z\right)\right\|_X, \qquad \forall x,y,z \in G. \tag{2.1}$$

Proof. Sufficiency. Suppose that the condition (2.1) is satisfied. Letting $x = y = z = 0$ in (2.1) implies that $\|(n+2)f(0)\|_X \leqslant \|nf(0)\|_X$. So $f(0) = 0$. Replacing x by $-nz$ and letting $y = 0$ in (2.1) yield that

$$\|f(-nz) + nf(z)\|_X \leqslant \|f(0)\|_X = 0, \qquad \forall z \in G.$$

Thus, $f(-nz) = -nf(z)$, $\forall z \in G$. By letting $z = -\frac{x+y}{n}$ in (2.1), we get

$$\begin{aligned}\left\|f(x) + f(y) + nf\left(-\frac{x+y}{n}\right)\right\|_X &= \|f(x) + f(y) - f(x+y)\|_X \\ &\leqslant \left\|nf\left(\frac{x+y}{n} - \frac{x+y}{n}\right)\right\|_X \\ &= \|f(0)\|_X.\end{aligned}$$

Hence

$$f(x+y) = f(x) + f(y), \qquad \forall x,y \in G.$$

Necessity. Let f be additive. Then

$$f(x+y) = f(x) + f(y), \qquad \forall x,y \in G,$$

and so

$$f(rx) = rf(x), \qquad \forall r \in \mathbb{Q},\ x \in G.$$

Thus

$$f(x)+f(y)+nf(z) = nf\left(\frac{x+y}{n}\right) + nf(z) = nf\left(\frac{x+y}{n}+z\right), \qquad \forall x,y,z \in G.$$

Hence

$$\|f(x)+f(y)+nf(z)\|_X \leqslant \left\|nf,\left(\frac{x+y}{n}+z\right)\right\|_X, \qquad \forall x,y,z \in G.$$

This completes the proof. □

According to the proof of Proposition 2.1, we can get the following corollary.

COROLLARY 2.1. *For a mapping $f : G \to X$, the following statements are equivalent.*

(a) f is additive.

(b) $f(x)+f(y)+nf(z) = nf(\frac{x+y}{n}+z)$, $\qquad \forall x,y,z \in G$.

(c) $\|f(x)+f(y)+nf(z)\|_X \leqslant \|nf(\frac{x+y}{n}+z)\|_X$, $\qquad \forall x,y,z \in G$.

Clearly, a vector space is an n-divisible abelian group, so Corollary 2.1 is right when G is a vector space.

3. GHUR-stability of functional inequalities and functional equations associated with CJE

In this section, let G be a 2-divisible abelian group and $f : G \to Y$.

THEOREM 3.1. *Let $\varphi : G^3 \to \mathbb{R}^+$ satisfy $\lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) = 0$, $\forall x,y,z \in G$ and*

$$\check{\varphi}(x,z) := \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \varphi(2^{n+1}x, 0, -2^n z) < \infty, \qquad \forall x,z \in G.$$

Suppose that $f : G \to Y$ is a mapping such that $f(-x) = -f(x)$ for all $x \in G$ and for all $x,y,z \in G$,

$$\|f(x)+f(y)+2f(z)\|_Y \leqslant \left\|2f\left(\frac{x+y}{2}+z\right)\right\|_Y + \varphi(x,y,z), \tag{3.1}$$

then there exists a unique additive mapping $h : G \to Y$ such that

$$\|f(x)-h(x)\|_Y \leqslant \check{\varphi}(x,x), \qquad \forall x \in G. \tag{3.2}$$

Proof. Replacing x by $2x$ and letting $y=0$ and $z=-x$ in (3.1), we get

$$\|f(2x)-2f(x)\|_Y \leqslant \varphi(2x,0,-x), \qquad \forall x \in G. \tag{3.3}$$

That is,

$$\|f(x)-\frac{1}{2}f(2x)\|_Y \leqslant \frac{1}{2}\varphi(2x,0,-x), \qquad \forall x \in G. \tag{3.4}$$

It follows from (3.4) that

$$\begin{aligned}
\left\|\frac{1}{2^l}f(2^l x)-\frac{1}{2^m}f(2^m x)\right\|_Y &\leqslant \sum_{k=l}^{m-1}\left\|\frac{1}{2^k}f(2^k x)-\frac{1}{2^{k+1}}f(2^{k+1}x)\right\|_Y \\
&= \sum_{k=l}^{m-1}\frac{1}{2^k}\left\|f(2^k x)-\frac{1}{2}f(2^{k+1}x)\right\|_Y \\
&\leqslant \sum_{k=l}^{m-1}\frac{1}{2^{k+1}}\varphi(2^{k+1}x,0,-2^k x)
\end{aligned} \tag{3.5}$$

for all nonnegative integers m and n with $m>l$ and $x \in G$. Since for all $x \in G$, the series

$$\sum_{k=0}^{\infty}\frac{1}{2^{k+1}}\varphi(2^{k+1}x,0,-2^k x)$$

converges, (3.5) implies that $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in G$ and therefore convergent since Y is a Banach space. Put

$$h(x)=\lim_{n\to\infty}\frac{1}{2^n}f(2^n x), \qquad \forall x \in G.$$

Moveover, letting $l=0$ and $m \to \infty$ in (3.5) yield (3.2). Moreover, we see from (3.1) that

$$\begin{aligned}
\|h(x)+h(y)+2h(z)\|_Y &= \lim_{n\to\infty}\left\|\frac{1}{2^n}f(2^n x)+\frac{1}{2^n}f(2^n y)+2\cdot\frac{1}{2^n}f(2^n z)\right\|_Y \\
&= \lim_{n\to\infty}\frac{1}{2^n}\|f(2^n x)+f(2^n y)+2f(2^n z)\|_Y \\
&\leqslant \lim_{n\to\infty}\frac{1}{2^n}\left(2\left\|f\left(2^n\left(\frac{x+y}{2}+z\right)\right)\right\|_Y+\varphi(2^n x,2^n y,2^n z)\right) \\
&= \left\|2h(\frac{x+y}{2}+z)\right\|_Y.
\end{aligned}$$

Thus

$$\|h(x)+h(y)+2h(z)\|_Y \leqslant \left\|2h\left(\frac{x+y}{2}+z\right)\right\|_Y, \qquad \forall x,y,z \in G.$$

It follows from Proposition 2.1 that h is additive.

Next, let $g : G \to Y$ be another additive mapping satisfying

$$\|f(x)-g(x)\|_Y \leqslant \check{\varphi}(x,x), \qquad \forall x \in G.$$

Then for every $x \in G$, we have

$$\begin{aligned}\|h(x)-g(x)\|_Y &= \left\|\frac{1}{2^n}h(2^n x)-\frac{1}{2^n}g(2^n x)\right\|_Y \\ &\leqslant \left\|\frac{1}{2^n}f(2^n x)-\frac{1}{2^n}h(2^n x)\right\|_Y + \left\|\frac{1}{2^n}f(2^n x)-\frac{1}{2^n}g(2^n x)\right\|_Y \\ &\leqslant 2\frac{1}{2^n}\check{\varphi}(2^n x, 2^n x) \\ &= 2\sum_{m=n}^{\infty}\frac{1}{2^{m+1}}\varphi(2^{m+1}x, 0, -2^m x) \\ &\to 0 \ (n \to \infty).\end{aligned}$$

This implies that $g = h$. Thus, the mapping h is unique. The proof is completed. □

Suppose that G is a normed space with norm $\|\cdot\|$. If we put $\varphi(x,y,z) = \theta\|x\|^p\|y\|^q\|z\|^t$ and $\varphi(x,y,z) = \theta(\|x\|^p+\|y\|^q+\|z\|^t)$ in Theorem 3.1, respectively, then we get the Corollaries 3.1 and 3.2.

COROLLARY 3.1. *Let $p,t \neq 0$, $q > 0$, $\theta > 0$ and let $f : G \to Y$ be an odd mapping. If*

$$\|f(x)+f(y)+2f(z)\|_Y \leqslant \|2f(\frac{x+y}{2}+z)\|_Y + \theta\|x\|^p\|y\|^q\|z\|^t, \qquad \forall x,y,z \in G,$$

then f is an additive mapping.

COROLLARY 3.2. *Let $p,t > 0$, $p,t < 1$, $q \neq 0$, $\theta > 0$ and let $f : G \to Y$ be an odd mapping. If*

$$\|f(x)+f(y)+2f(z)\|_Y \leqslant \|2f(\frac{x+y}{2}+z)\|_Y + \theta(\|x\|^p+\|y\|^q+\|z\|^t), \qquad \forall x,y,z \in G,$$

then there exists a unique additive mapping $h : G \to Y$ such that

$$\|f(x)-h(x)\|_Y \leqslant \theta\left(\frac{2^p}{2-2^p}\|x\|^p+\frac{1}{2-2^t}\|x\|^t\right), \qquad \forall x \in G.$$

THEOREM 3.2. *Let $\varphi : G^3 \to \mathbb{R}^+$ satisfy*

$$\check{\varphi}(x,z) := \sum_{n=0}^{\infty}2^n\varphi\left(\frac{1}{2^n}x, 0, -\frac{1}{2^{n+1}}z\right) < \infty, \qquad \forall x,z \in G$$

and $\lim_{n\to\infty}2^n\varphi(\frac{1}{2^n}x,\frac{1}{2^n}y,\frac{1}{2^n}z) = 0$ *for all $x,y,z \in G$. If $f : G \to Y$ is an odd mapping such that $\forall x,y,z \in G$,*

$$\|f(x)+f(y)+2f(z)\|_Y \leqslant \left\|2f\left(\frac{x+y}{2}+z\right)\right\|_Y + \varphi(x,y,z), \tag{3.6}$$

then there exists a unique additive mapping $h : G \to Y$ *such that*

$$\|f(x) - h(x)\|_Y \leqslant \check{\varphi}(x,x), \qquad \forall x \in G. \tag{3.7}$$

Proof. Replacing x by $2x$ and letting $y = 0$ and $z = -x$ in (3.6), we get

$$\|f(2x) - 2f(x)\|_Y \leqslant \varphi(2x, 0, -x), \qquad \forall x \in G. \tag{3.8}$$

Replacing x by $\frac{x}{2}$ in (3.8), we get

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_Y \leqslant \varphi\left(x, 0, -\frac{x}{2}\right), \qquad \forall x \in G. \tag{3.9}$$

Hence, it follows from (3.9) that

$$\begin{aligned}
\left\|2^l f\left(\frac{1}{2^l}x\right) - 2^m f\left(\frac{1}{2^m}x\right)\right\|_Y &\leqslant \sum_{k=l}^{m-1} \left\|2^k f\left(\frac{1}{2^k}x\right) - 2^{k+1} f\left(\frac{1}{2^{k+1}}x\right)\right\|_Y \\
&= \sum_{k=l}^{m-1} 2^k \left\|f\left(\frac{1}{2^k}x\right) - f\left(\frac{1}{2^{k+1}}x\right)\right\|_Y \\
&\leqslant \sum_{k=l}^{m-1} 2^k \varphi\left(\frac{1}{2^k}x, 0, -\frac{1}{2^{k+1}}x\right).
\end{aligned}$$

for all nonnegative integers m and n with $m > l$ and $x \in G$. Since for all $x, y, z \in G$, the series

$$\sum_{k=0}^{\infty} 2^k \varphi\left(\frac{1}{2^k}x, 0, -\frac{1}{2^{k+1}}x\right)$$

converges, we see that $\{2^n f(\frac{1}{2^n}x)\}$ is a Cauchy sequence for all $x \in G$ and then converges since Y is a Banach space. Put

$$h(x) = \lim_{n\to\infty} 2^n f\left(\frac{1}{2^n}x\right), \qquad \forall x \in G.$$

The remainder is similar to the proof of Theorem 3.1. This completes the proof. □

Suppose that G is a normed space with norm $\|\cdot\|$. If we put $\varphi(x,y,z) = \theta\|x\|^p\|y\|^q\|z\|^t$ and $\varphi(x,y,z) = \theta(\|x\|^p + \|y\|^q + \|z\|^t)$ in Theorem 3.2, respectively, then we get Corollaries 3.3 and 3.4.

COROLLARY 3.3. *Let* $p, t \neq 0, q > 0$, $\theta > 0$ *and* $f : G \to Y$ *be an odd mapping. If*

$$\|f(x) + f(y) + 2f(z)\|_Y \leqslant \left\|2f\left(\frac{x+y}{2} + z\right)\right\|_Y + \theta\|x\|^p\|y\|^q\|z\|^t, \qquad \forall x, y, z \in G,$$

then f *is an additive mapping.*

COROLLARY 3.4. *Let $p,t>0$, $p,t<1$, $q\neq 0$, $\theta>0$ and $f:G\to Y$ be an odd mapping. If*

$$\|f(x)+f(y)+2f(z)\|_Y \leqslant \left\|2f\left(\frac{x+y}{2}+z\right)\right\|_Y + \theta(\|x\|^p+\|y\|^q+\|z\|^t), \qquad \forall x,y,z\in G,$$

then there exists a unique additive mapping $h:G\to Y$ such that

$$\|f(x)-h(x)\|_Y \leqslant \theta\left(\frac{2^p}{2^p-2}\|x\|^p+\frac{1}{2^t-2}\|x\|^t\right), \qquad \forall x\in G.$$

By the definitions of the stability of inequality, if we define $E:M(G,Y)\to M(G^r,\mathbb{R}^*)$ as

$$(Ef)(x,y) := \|f(x)+f(y)+2f(z)\|_Y - \left\|2f\left(\frac{x+y}{2}+z\right)\right\|_Y,$$

then the inequality $Ef\leqslant 0$ is $(\varphi,\check{\varphi})$-stable in $M(G,Y)$ where $(\varphi,\check{\varphi})$ is as in Theorem 3.1 and Theorem 3.2, respectively.

THEOREM 3.3. *Let $\varphi: G^3\to\mathbb{R}^+$ satisfy*

$$\check{\varphi}(x,z) := \sum_{n=0}^{\infty}\frac{1}{2^{n+2}}\left(\varphi(2^{n+1}x,0,-2^n z)+\varphi(-2^{n+1}x,0,2^n z)\right) < \infty$$

for all $x,z\in G$ and $\lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n x,2^n y,2^n z)=0$, for all $x,y,z\in G$. If $f:G\to Y$ is a mapping such that $f(0)=0$ and for all $x,y,z\in G$,

$$\left\|f(x)+f(y)+2f(z)-2f\left(\frac{x+y}{2}+z\right)\right\|_Y \leqslant \varphi(x,y,z), \tag{3.10}$$

then there exists a unique additive mapping $h:G\to Y$ such that

$$\|f(x)-h(x)\|_Y \leqslant \check{\varphi}(x,x), \qquad \forall x\in G. \tag{3.11}$$

Proof. Replacing x by $2x$ and letting $y=0$ and $z=-x$ in (3.10), we get

$$\|f(2x)+2f(-x)\|_Y \leqslant \varphi(2x,0,-x), \qquad \forall x\in G. \tag{3.12}$$

Replacing x by $-x$ in (3.12), we get

$$\|f(-2x)+2f(x)\|_Y \leqslant \varphi(-2x,0,x), \qquad \forall x\in G. \tag{3.13}$$

Put $g(x)=\frac{f(x)-f(-x)}{2}$. Using (3.12) and (3.13) yield that

$$\|2g(x)-g(2x)\|_Y \leqslant \frac{1}{2}(\varphi(2x,0,-x)+\varphi(-2x,0,x)), \qquad \forall x\in G.$$

That is,

$$\|g(x)-\frac{1}{2}g(2x)\|_Y \leqslant \frac{1}{4}(\varphi(2x,0,-x)+\varphi(-2x,0,x)), \qquad \forall x \in G. \tag{3.14}$$

It follows from (3.14) that

$$\begin{aligned}\left\|\frac{1}{2^l}g(2^lx)-\frac{1}{2^m}g(2^mx)\right\|_Y &\leqslant \sum_{k=l}^{m-1}\left\|\frac{1}{2^k}g(2^kx)-\frac{1}{2^{k+1}}g(2^{k+1}x)\right\|_Y \\ &= \sum_{k=l}^{m-1}\frac{1}{2^k}\left\|g(2^kx)-\frac{1}{2}g(2^{k+1}x)\right\|_Y \\ &\leqslant \sum_{k=l}^{m-1}\frac{1}{2^{k+2}}(\varphi(2^{k+1}x,0,-2^kx)+\varphi(-2^{k+1}x,0,2^kx))\end{aligned} \tag{3.15}$$

for all nonnegative integers m and n with $m>l$ and $x \in G$. Since for all $x \in G$, the series

$$\sum_{n=0}^{\infty}\frac{1}{2^{n+2}}\left(\varphi(2^{n+1}x,0,-2^nx)+\varphi(-2^{n+1}x,0,2^nx)\right)$$

converges, (3.15) implies that $\{\frac{1}{2^n}g(2^nx)\}$ is a Cauchy sequence for all $x \in G$ and therefore converges since Y is complete. Put

$$h(x)=\lim_{n\to\infty}\frac{1}{2^n}g(2^nx), \qquad \forall x \in G.$$

Moveover, by letting $l=0$ and $m\to\infty$ in (3.15), (3.11) follows. It follows from (3.10) that

$$\begin{aligned}&\left\|h(x)+h(y)+2h(z)-2h\left(\frac{x+y}{2}+z\right)\right\|_Y \\ &= \lim_{n\to\infty}\left\|\frac{1}{2^n}f(2^nx)+\frac{1}{2^n}f(2^ny)+2\cdot\frac{1}{2^n}f(2^nz)-2\frac{1}{2^n}f\left(\frac{2^nx+2^ny}{2}+2^nz\right)\right\|_Y \\ &= \lim_{n\to\infty}\frac{1}{2^n}\left\|f(2^nx)+f(2^ny)+2f(2^nz)-2f\left(\frac{2^nx+2^ny}{2}+2^nz\right)\right\|_Y \\ &\leqslant \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^nx,2^ny,2^nz) \\ &= 0.\end{aligned}$$

This implies that

$$h(x)+h(y)+2h(z)=h(\frac{x+y}{2}+z), \qquad \forall x,y,z \in G.$$

Now, Corollary 2.1 yields that that h is additive.

Next, let $g : G \to Y$ be another additive mapping satisfying

$$\|f(x)-g(x)\|_Y \leqslant \check{\varphi}(x,x), \qquad \forall x \in G.$$

Then for every $x \in G$, we have

$$\begin{aligned}\|h(x)-g(x)\|_Y &= \left\|\frac{1}{2^n}h(2^n x)-\frac{1}{2^n}g(2^n x)\right\|_Y \\ &\leqslant \left\|\frac{1}{2^n}f(2^n x)-\frac{1}{2^n}h(2^n x)\right\|_Y + \left\|\frac{1}{2^n}f(2^n x)-\frac{1}{2^n}g(2^n x)\right\|_Y \\ &\leqslant 2\frac{1}{2^n}\check{\varphi}(2^n x, 2^n x) \\ &= 2\sum_{m=n}^{\infty}\frac{1}{2^{m+2}}\left(\varphi(2^{m+1}x,0,-2^m x)+\varphi(-2^{m+1}x,0,2^m x)\right) \\ &\to 0 \ (n\to\infty).\end{aligned}$$

Consequently, $h(x)=g(x)$, for all $x \in G$. This shows that the mapping h is unique and completes the proof. □

Suppose that G is a normed space with a norm $\|\cdot\|$. If we put $\varphi(x,y,z) = \theta\|x\|^p\|y\|^q\|z\|^t$ and $\varphi(x,y,z) = \theta(\|x\|^p+\|y\|^q+\|z\|^t)$ in Theorem 3.3, respectively, then we get Corollaries 3.5 and 3.6.

COROLLARY 3.5. *Let $q>0$, $p,t \neq 0$, $\theta > 0$ and let $f : G \to Y$ be a mapping with $f(0)=0$. If*

$$\left\|f(x)+f(y)+2f(z)-2f\left(\frac{x+y}{2}+z\right)\right\|_Y \leqslant \theta\|x\|^p\|y\|^q\|z\|^t, \qquad \forall x,y,z \in G,$$

then f is an additive mapping.

COROLLARY 3.6. *Let $p,t>0$, $p,t<1$, $q \neq 0$, $\theta > 0$ and let $f : G \to Y$ be a mapping with $f(0)=0$. If*

$$\left\|f(x)+f(y)+2f(z)-2f\left(\frac{x+y}{2}+z\right)\right\|_Y \leqslant \theta(\|x\|^p+\|y\|^q+\|z\|^t), \qquad \forall x,y,z \in G,$$

then there exists a unique additive mapping $h : G \to Y$ such that

$$\|f(x)-h(x)\|_Y \leqslant \theta\left(\frac{2^p}{2-2^p}\|x\|^p+\frac{1}{2-2^t}\|x\|^t\right), \qquad \forall x \in G.$$

THEOREM 3.4. *Let $\varphi : G^3 \to \mathbb{R}^+$ satisfy*

$$\check{\varphi}(x,z) := \sum_{n=0}^{\infty}2^{n-1}\left(\varphi(2^{-n}x,0,-2^{-(n+1)}z)+\varphi(-2^{-n}x,0,2^{-(n+1)}z)\right) < \infty$$

for all $x,z \in G$ *and* $\lim_{n\to\infty} 2^n \varphi(\frac{1}{2^n}x, \frac{1}{2^n}y, \frac{1}{2^n}z) = 0$, $\forall x,y,z \in G$. *If* $f : G \to Y$ *is a mapping such that* $f(0) = 0$ *and for all* $x,y,z \in G$,

$$\left\| f(x) + f(y) + 2f(z) - 2f\left(\frac{x+y}{2} + z\right) \right\|_Y \leqslant \varphi(x,y,z), \tag{3.16}$$

then there exists a unique additive mapping $h : G \to Y$ *such that*

$$\|f(x) - h(x)\|_Y \leqslant \check{\varphi}(x,x), \qquad \forall x \in G. \tag{3.17}$$

Proof. Replacing x by $2x$ and letting $y = 0$ and $z = -x$ in (3.16), we get

$$\|f(2x) + 2f(-x)\|_Y \leqslant \varphi(2x,0,-x), \qquad \forall x \in G. \tag{3.18}$$

Replacing x by $-x$ in (3.18), we get

$$\|f(-2x) + 2f(x)\|_Y \leqslant \varphi(-2x,0,x), \qquad \forall x \in G. \tag{3.19}$$

Put $g(x) = \frac{f(x)-f(-x)}{2}$. By (3.18) and (3.19), we get

$$\|2g(x) - g(2x)\|_Y \leqslant \frac{1}{2}\left(\varphi(2x,0,-x) + \varphi(-2x,0,x)\right), \qquad \forall x \in G. \tag{3.20}$$

Replacing x by $\frac{x}{2}$ in (3.20), we get

$$\left\| g(x) - 2g\left(\frac{x}{2}\right) \right\|_Y \leqslant \frac{1}{2}\left(\varphi\left(x,0,-\frac{x}{2}\right) + \varphi\left(-x,0,\frac{x}{2}\right)\right), \qquad \forall x \in G. \tag{3.21}$$

The remainder is similar to the proof of Theorem 3.3. This completes the proof. $\square$

Suppose that G is a normed space with the norm $\|\cdot\|$. If we put $\varphi(x,y,z) = \theta\|x\|^p\|y\|^q\|z\|^t$ and $\varphi(x,y,z) = \theta(\|x\|^p + \|y\|^q + \|z\|^t)$ in Theorem 3.4, respectively, then we get Corollaries 3.7 and 3.8.

COROLLARY 3.7. *Let* $p,t \neq 0$, $q > 0$, $\theta > 0$ *and let* $f : G \to Y$ *be a mapping with* $f(0) = 0$. *If*

$$\left\| f(x) + f(y) + 2f(z) - 2f\left(\frac{x+y}{2} + z\right) \right\|_Y \leqslant \theta\|x\|^p\|y\|^q\|z\|^t, \quad \forall x,y,z \in G,$$

then f *is an additive mapping.*

COROLLARY 3.8. *Let* $p,t > 1, q \neq 0$, $\theta > 0$ *and let* $f : G \to X$ *be a mapping with* $f(0)=0$. *If*

$$\left\| f(x) + f(y) + 2f(z) - 2f\left(\frac{x+y}{2} + z\right) \right\|_Y \leqslant \theta(\|x\|^p + \|y\|^q + \|z\|^t), \quad \forall x,y,z \in G,$$

then there exists a unique additive mapping $h : G \to Y$ *such that*

$$\|f(x) - h(x)\|_Y \leqslant \theta\left(\frac{2^p}{2-2^p}\|x\|^p + \frac{1}{2-2^t}\|x\|^t\right), \qquad \forall x \in G.$$

By the definition of the stability of equation, if we define $E : M(G,Y) \to M(G^r,Y)$ as

$$(Ef)(x,y) := f(x) + f(y) + 2f(z) - 2f\left(\frac{x+y}{2} + z\right),$$

then the equation $Ef = 0$ is $(\varphi, \check{\varphi})$-stable in $M(G,Y)$ where $(\varphi, \check{\varphi})$ is as in Theorem 3.3 and Theorem 3.4, respectively.

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