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INEQUALITIES AND MONOTONICITY PROPERTIES FOR SOME SPECIAL FUNCTIONS

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Abstract. The monotonicity, convexity, log-convexity and complete monotonicity properties for some special functions are proved, and some inequalities are established.

1. Introduction

A function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geqslant 0 \quad \text{for all} \quad x \in I \quad \text{and} \quad n = 0, 1, 2, \ldots. \tag{1}$$

If the inequality (1) is strict, then f is said to be strictly completely monotonic on I. It is known (Bernstein's Theorem) that f is completely monotonic on $(0,\infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\, \mathrm{d}\mu(t),$$

where μ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x > 0$, see [13, p.161]. A detailed collection of the most important properties of completely monotonic functions can be found in [13, Chapter IV].

A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is called strictly convex (concave), if

$$a_{n+2} - 2a_{n+1} + a_n > (<)\, 0 \quad \text{for all} \quad n \geqslant 1.$$

A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is called strictly log-convex (log-concave), if it is positive and

$$a_{n+1}^2 < (>)\, a_n a_{n+2} \quad \text{for all integers} \quad n \geqslant 1.$$

By the arithmetic-geometric mean inequality, the log-convexity implies the convexity, and the concavity implies the log-concavity.

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The classical gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, \mathrm{d}t \quad (x > 0)$$

is one of the most important functions in analysis and its applications. The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed [1, pp. 259-260] as

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\, \mathrm{d}t, \tag{2}$$

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt}\, \mathrm{d}t \tag{3}$$

for $x > 0$ and $n \in \mathbb{N}$, where $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty} D_n, \quad \text{where} \quad D_n = \sum_{k=1}^{n} \frac{1}{k} - \log n. \tag{4}$$

It is also known as the Euler-Mascheroni constant. According to Glaisher [8], the use of the symbol γ is probably due to the geometer Lorenzo Mascheroni (1750-1800) who used it in 1790 while Euler used the letter C. Euler's constant plays an important role in Analysis (Gamma function, Bessel functions, exponential-integral, ...) and occurs frequently in Number Theory (order of magnitude of arithmetical functions for instance [9]).

Direct use of formula (4) to compute the Euler's constant is of poor interest since the convergence is very slow. The Euler-Maclaurin summation can be used to have a complete asymptotic expansion of the harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$,

$$H_n - \log n = \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \frac{1}{n^{2k}},$$

where the B_{2k} are the Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{5}$$

First four Bernoulli numbers with even indices are

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \tag{6}$$

and then

$$\gamma = H_n - \log n - \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{1}{240n^8} + \cdots.$$

In 1991, R. M. Young [14] presented an elegant geometrical proof for the double inequality

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n}, \quad n = 1,2,\dots. \tag{7}$$

In [3, 4, 6, 12], other bounds for $D_n - \gamma$ were established. Since

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}, \tag{8}$$

the inequality (7) can be written as

$$\frac{1}{2(n+1)} < \psi(n+1) - \log n < \frac{1}{2n}, \quad n = 1,2,\dots. \tag{9}$$

Motivated by the inequality (9), we establish the following results.

THEOREM 1. *The function* $f(x) = (x+1)[\psi(x+1) - \log x]$ *is strictly completely monotonic on* $(0,\infty)$. *The function* $g(x) = x[\psi(x+1) - \log x]$ *is a so-called Bernstein function on* $(0,\infty)$*, that is,* $g > 0$ *and* g' *is strictly completely monotonic on* $(0,\infty)$.

REMARK 1. From the representations [1, pp. 258-259]

$$\psi(x) = \log x - \frac{1}{2x} + O(x^{-2}),$$
$$\psi(x+1) = \psi(x) + \frac{1}{x},$$

we conclude

$$\lim_{x\to\infty} f(x) = \frac{1}{2} \quad \text{and} \quad \lim_{x\to\infty} g(x) = \frac{1}{2},$$

the inequalities (7) are immediate consequences of the fact that f is strictly decreasing on $(0,\infty)$ and g is strictly increasing on $(0,\infty)$.

REMARK 2. From the integral representations

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) \mathrm{d}t, \quad x > 0, \tag{10}$$

$$\ln x = \int_0^{+\infty} \frac{e^{-t} - e^{-xt}}{t} \, \mathrm{d}t, \quad x > 0, \tag{11}$$

we conclude that

$$D_n - \gamma = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) e^{-nt} \, \mathrm{d}t. \tag{12}$$

By using the following generalization of the Schwarz inequality

$$\int_0^\infty g(t)[f(t)]^m \, \mathrm{d}t \int_0^\infty g(t)[f(t)]^n \, \mathrm{d}t \geqslant \left[\int_0^\infty g(t)[f(t)]^{(m+n)/2} \, \mathrm{d}t\right]^2, \tag{13}$$

where f and g are two nonnegative functions of a real variable and m and n belong to a set S of real numbers, such that the integrals in (13) exist, A. Laforgia and P. Natalini [11] proved that the inequality

$$(D_{n+1}-\gamma)^2 \leqslant (D_n-\gamma)(D_{n+2}-\gamma) \tag{14}$$

holds for all integers $n \geqslant 1$. The inequality (14) is called in the literature the Turán-type inequality.

It is well-known that the complete monotonicity implies the log-convexity [7]. By Theorem 1, the function $f(x) = (x+1)[\psi(x+1)-\log x]$ is strictly log-convex on $(0,\infty)$, and $g(x) = x[\psi(x+1)-\log x]$ is strictly concave on $(0,\infty)$. From log-convexity of the sequence $\{f(n)\}_{n=1}^{\infty}$ and log-concavity of the sequence $\{g(n)\}_{n=1}^{\infty}$ (Note that the concavity of the sequence $\{g(n)\}_{n=1}^{\infty}$ implies its log-concavity), we obtain that for all integers $n \geqslant 1$, then

$$\frac{(n+2)^2}{(n+1)(n+3)}(D_{n+1}-\gamma)^2 < (D_n-\gamma)(D_{n+2}-\gamma) < \frac{(n+1)^2}{n(n+2)}(D_{n+1}-\gamma)^2. \tag{15}$$

Obviously, the left inequality of (15) is an improvement of the inequality (14).

The convergence of the sequence D_n to γ is very slow. In 1993, D. W. DeTemple [6] studied a modified sequence which converges faster and proved

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}, \quad n \geqslant 1, \tag{16}$$

where

$$R_n = \sum_{k=1}^{n} \frac{1}{k} - \log\left(n+\frac{1}{2}\right).$$

Now let

$$H(n) = n^2(R_n-\gamma), \quad n \geqslant 1.$$

By (8), we see that

$$H(n) = (R_n-\gamma)n^2 = \left[\psi(n+1) - \log\left(n+\frac{1}{2}\right)\right]n^2. \tag{17}$$

Some computer experiments led M. Vuorinen to conjecture that $H(n)$ increases on the interval $[1,\infty)$ from $H(1) = -\gamma+1-\log(3/2) = 0.0173\ldots$ to $1/24 = 0.0416\ldots$. E. A. Karatsuba [10] proved that for all integers $n \geqslant 1$, $H(n) < H(n+1)$, by clever use of Stirling formula and Fourier series. Some computer experiments also seem to indicate that $[(n+1)/n]^2 H(n)$ is a decreasing convex function [5].

The following Theorem 2 shows the monotonicity and convexity properties of $H(n)$, $[(n+1/2)/n]^2 H(n)$ and $[(n+1)/n]^2 H(n)$.

THEOREM 2. *Let* $H(n)$ $(n = 1,2,\ldots)$ *be defined by* (17). *Then for all integers* $n \geqslant 1$, $H(n)$ *and* $[(n+1/2)/n]^2 H(n)$ *are both strictly increasing concave sequences, while* $[(n+1)/n]^2 H(n)$ *is strictly decreasing convex sequence.*

REMARK 3. By the asymptotic formula [2, p. 550]

$$\psi(x) = \log\left(x - \frac{1}{2}\right) + \frac{1}{24(x-1/2)^2} + O(x^{-4}) \quad \text{as} \quad x \to \infty,$$

we conclude that

$$\lim_{n\to\infty} H(n) = \frac{1}{24} \quad \text{and} \quad \lim_{n\to\infty} [(n+1)/n]^2 H(n) = \frac{1}{24}. \tag{18}$$

From the monotonicity of $H(n)$, $[(n+1)/n]^2 H(n)$ and (18), we obtain the inequality (16).

REMARK 4. From the monotonicity of $[(n+1/2)/n]^2 H(n)$ and the limit relation $\lim_{n\to\infty}[(n+1/2)/n]^2 H(n) = \frac{1}{24}$, we obtain

$$R_n - \gamma < \frac{1}{24(n+\frac{1}{2})^2}, \quad n \geqslant 1. \tag{19}$$

Obviously, the upper in (19) is sharper than one in (16).

The inequality (19) can be written as

$$\frac{1}{\sqrt{24[\psi(n+1) - \log(n+1/2)]}} - n > \frac{1}{2}. \tag{20}$$

From the asymptotic formula [2, p. 550]

$$\psi(x) = \log\left(x - \frac{1}{2}\right) + \frac{1}{24(x-\frac{1}{2})^2} - \frac{7}{960(x-\frac{1}{2})^4} + O(x^{-6}) \quad \text{as} \quad x \to \infty,$$

we obtain

$$\begin{aligned}
\frac{1}{\sqrt{24[\psi(x+1) - \log(x+1/2)]}} - x &= \frac{1 - x\sqrt{24[\psi(x+1) - \log(x+1/2)]}}{\sqrt{24[\psi(x+1) - \log(x+1/2)]}} \\
&= \frac{1 - x\sqrt{\frac{1}{(x+1/2)^2} - \frac{7}{40(x+1/2)^4} + O(x^{-6})}}{\sqrt{\frac{1}{(x+1/2)^2} - \frac{7}{40(x+1/2)^4} + O(x^{-6})}} \\
&= \frac{x + \frac{1}{2} - x\sqrt{1 - \frac{7}{40(x+1/2)^2} + O(x^{-4})}}{\sqrt{1 - \frac{7}{40(x+1/2)^2} + O(x^{-4})}} \\
&= \frac{\frac{1}{2} + \frac{7x}{80(x+1/2)^2} + O(x^{-3})}{1 - \frac{7}{80(x+1/2)^2} + O(x^{-4})} \to \frac{1}{2} \quad \text{as} \quad x \to \infty,
\end{aligned}$$

and then,

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{24[\psi(n+1) - \log(n+1/2)]}} - n\right] = \frac{1}{2}. \tag{21}$$

Hence, the constant $\frac{1}{2}$ in the upper bound of (19) is the best possible.

By Theorem 2, the sequences $[(n+1)/n]^2 H(n)$ is convex. The following Theorem 3 further considers its log-convexity.

THEOREM 3. *Let* $H(n)$ $(n=1,2,\ldots)$ *be defined by* (17)*. Then for all integers* $n \geqslant 1$*, the sequences* $[(n+1)/n]^2 H(n)$ *is strictly log-convex.*

REMARK 5. The concavity of the sequence $[(n+1/2)/n]^2 H(n)$ implies its log-concavity. From log-convexity of the sequence $[(n+1)/n]^2 H(n)$ and log-concavity of the sequence $[(n+1/2)/n]^2 H(n)$ we obtain the Turán-type inequality

$$\begin{aligned} \frac{(n+2)^4}{(n+1)^2(n+3)^2}(R_{n+1}-\gamma)^2 &< (R_n-\gamma)(R_{n+2}-\gamma) \\ &< \frac{(n+\frac{3}{2})^4}{(n+\frac{1}{2})^2(n+\frac{5}{2})^2}(R_{n+1}-\gamma)^2 \end{aligned} \tag{22}$$

for all integers $n \geqslant 1$.

2. Proofs of theorems

Proof of Theorem 1. Using the representations [1, p. 259]

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right) dt \tag{23}$$

and

$$\log x = \int_0^\infty \frac{e^{-t}-e^{-xt}}{t} dt, \tag{24}$$

we imply

$$f(x) = (x+1)\int_0^\infty \delta(t) e^{-(x+1)t} dt,$$

where

$$\delta(t) = \frac{1}{t} - \frac{1}{e^t-1}, \ t>0.$$

Easy computations reveal that the function δ is strictly decreasing on $(0,\infty)$ with $\lim_{x\to 0}\delta(t) = \frac{1}{2}$ and $\lim_{x\to 0}\delta(t) = 0$.

For $x>0$, $n=0,1,2,\ldots$, we have

$$\begin{aligned} (-1)^n f^{(n)}(x) &= (-1)^n \sum_{k=0}^{n} \binom{n}{k} (x+1)^{(n-k)} \left(\int_0^\infty \delta(t) e^{-(x+1)t} dt \right)^{(k)} \\ &= (x+1)\int_0^\infty \delta(t) e^{-(x+1)t} t^n dt - n\int_0^\infty \delta(t) e^{-(x+1)t} t^{n-1} dt \\ &= \int_0^{n/(x+1)} \delta(t) e^{-(x+1)t} t^{n-1} [(x+1)t-n] dt \\ &\quad + \int_{n/(x+1)}^\infty \delta(t) e^{-(x+1)t} t^{n-1} [(x+1)t-n] dt \end{aligned} \tag{25}$$

$$
\begin{aligned}
&> \delta(n/(x+1)) \int_0^{n/(x+1)} e^{-(x+1)t} t^{n-1} [(x+1)t - n] dt \\
&\quad + \delta(n/(x+1)) \int_{n/(x+1)}^{\infty} e^{-(x+1)t} t^{n-1} [(x+1)t - n] dt \\
&= \delta(n/(x+1)) \int_0^{\infty} e^{-(x+1)t} t^{n-1} [(x+1)t - n] dt.
\end{aligned}
$$

Since

$$
\frac{m!}{(x+s)^{m+1}} = \int_0^{\infty} t^m e^{-(x+s)t}\, \mathrm{d}t \quad (x > 0;\ s \geqslant 0,\ m = 0, 1, 2, \ldots), \tag{26}
$$

we conclude

$$
\int_0^{\infty} e^{-(x+1)t} t^{n-1} [(x+1)t - n]\, \mathrm{d}t = 0,
$$

so that (25) implies

$$
(-1)^n f^{(n)}(x) > 0 \quad (x > 0,\ n = 0, 1, 2, \ldots).
$$

Hence, the function f is strictly completely monotonic on $(0, \infty)$.

Using the representations (23) and (24), we imply

$$
g(x) = x \int_0^{\infty} \omega(t) e^{-xt} dt, \tag{27}
$$

where

$$
\omega(t) = \frac{e^t}{t} - \frac{1}{e^t - 1} - 1, \quad t > 0. \tag{28}
$$

Easy computations reveal that

$$
\frac{t^2(e^t-1)^2}{e^t} \omega'(t) = \sum_{n=4}^{\infty} \left[(n-2)2^{n-1} - 2(n-1) \right] \frac{t^n}{n!} > 0, \quad t > 0, \tag{29}
$$

hence, the function ω is strictly increasing on $(0, \infty)$, and $\omega(t) > \lim_{t \to 0} \omega(t) = \frac{1}{2}$, and then $g(x) > 0$. For $x > 0$, $n = 1, 2, \ldots$, we have

$$
\begin{aligned}
(-1)^n g^{(n)}(x) &= (-1)^n \sum_{k=0}^{n} \binom{n}{k} x^{(n-k)} \left(\int_0^{\infty} \omega(t) e^{-xt} dt \right)^{(k)} \\
&= x \int_0^{\infty} \omega(t) e^{-xt} t^n dt - n \int_0^{\infty} \omega(t) e^{-xt} t^{n-1} dt \\
&= \int_0^{n/x} \omega(t) e^{-xt} t^{n-1} (xt - n) dt \\
&\qquad + \int_{n/x}^{\infty} \omega(t) e^{-xt} t^{n-1} (xt - n) dt \qquad (30) \\
&< \omega(n/x) \int_0^{n/x} e^{-xt} t^{n-1} (xt - n) dt \\
&\qquad + \omega(n/x) \int_{n/x}^{\infty} e^{-xt} t^{n-1} (xt - n) dt \\
&= \omega(n/x) \int_0^{\infty} e^{-xt} t^{n-1} (xt - n) dt = 0.
\end{aligned}
$$

Hence, g is a Bernstein function on $(0,\infty)$. The proof of Theorem 1 is complete. □

In order prove our Theorem 2 and Theorem 3 we need to the following results [2]: For $x>\frac{1}{2}$, $N=0,1,2,\ldots$,

$$\log\left(x-\frac{1}{2}\right)-\sum_{k=1}^{2N}\frac{B_{2k}(1/2)}{2k(x-\frac{1}{2})^{2k}}<\psi(x) \\ <\log\left(x-\frac{1}{2}\right)-\sum_{k=1}^{2N+1}\frac{B_{2k}(1/2)}{2k(x-\frac{1}{2})^{2k}} \tag{31}$$

and

$$\frac{(n-1)!}{(x-\frac{1}{2})^{n}}+\sum_{k=1}^{2N+1}\frac{B_{2k}(1/2)}{(2k)!}\frac{(n+2k-1)!}{(x-\frac{1}{2})^{n+2k}}<(-1)^{n+1}\psi^{(n)}(x) \\ <\frac{(n-1)!}{(x-\frac{1}{2})^{n}}+\sum_{k=1}^{2N}\frac{B_{2k}(1/2)}{(2k)!}\frac{(n+2k-1)!}{(x-\frac{1}{2})^{n+2k}},\quad n=1,2,\ldots, \tag{32}$$

where

$$B_k(1/2)=-\left(1-\frac{1}{2^{k-1}}\right)B_k,\quad k=0,1,2,\ldots,$$

B_k are Bernoulli numbers defined by (5). By (6) we get

$$B_2(1/2)=-\frac{1}{12},\quad B_4(1/2)=\frac{7}{240},\quad B_6(1/2)=-\frac{31}{1344},\quad B_8(1/2)=\frac{127}{3840}.$$

From (31), we obtain for $x>\frac{1}{2}$,

$$\frac{1}{24(x-\frac{1}{2})^2}-\frac{7}{960(x-\frac{1}{2})^4}<\psi(x)-\log\left(x-\frac{1}{2}\right)<\frac{1}{24(x-\frac{1}{2})^2}. \tag{33}$$

$$\frac{1}{24(x-\frac{1}{2})^2}-\frac{7}{960(x-\frac{1}{2})^4}<\psi(x)-\log\left(x-\frac{1}{2}\right) \\ <\frac{1}{24(x-\frac{1}{2})^2}-\frac{7}{960(x-\frac{1}{2})^4}+\frac{31}{8064(x-\frac{1}{2})^6}. \tag{34}$$

From (32), we obtain for $x>\frac{1}{2}$,

$$\frac{1}{12(x-\frac{1}{2})^3}-\frac{7}{240(x-\frac{1}{2})^5}<\frac{1}{x-\frac{1}{2}}-\psi'(x)<\frac{1}{12(x-\frac{1}{2})^3}, \tag{35}$$

$$\frac{1}{12(x-\frac{1}{2})^3}-\frac{7}{240(x-\frac{1}{2})^5}+\frac{31}{1344(x-\frac{1}{2})^7}-\frac{127}{3840(x-\frac{1}{2})^9} \\ <\frac{1}{x-\frac{1}{2}}-\psi'(x)<\frac{1}{12(x-\frac{1}{2})^3}-\frac{7}{240(x-\frac{1}{2})^5}+\frac{31}{1344(x-\frac{1}{2})^7}, \tag{36}$$

$$0 < \psi''(x) + \frac{1}{(x-\frac{1}{2})^2} < \frac{1}{4(x-\frac{1}{2})^4}, \tag{37}$$

$$\begin{aligned} \frac{1}{4(x-\frac{1}{2})^4} - \frac{7}{48(x-\frac{1}{2})^6} &< \psi''(x) + \frac{1}{(x-\frac{1}{2})^2} \\ &< \frac{1}{4(x-\frac{1}{2})^4} - \frac{7}{48(x-\frac{1}{2})^6} + \frac{31}{920(x-\frac{1}{2})^8}. \end{aligned} \tag{38}$$

Now we are in position to prove our Theorem 2 and Theorem 3.

Proof of Theorem 2. Let $a \geqslant 0$ be a real number and $f_a(x)$ be defined by

$$f_a(x) = (x+a)^2 \left[\psi(x+1) - \log\left(x+\frac{1}{2}\right)\right], \quad x > -\frac{1}{2}. \tag{39}$$

Differentiation yields

$$f_a'(x) = 2(x+a)\left[\psi(x+1) - \log\left(x+\frac{1}{2}\right)\right] - (x+a)^2\left[\frac{1}{x+\frac{1}{2}} - \psi'(x+1)\right].$$

$$\begin{aligned} f_a''(x) = 2\left[\psi(x+1) - \log\left(x+\frac{1}{2}\right)\right] - 4(x+a)\left[\frac{1}{x+\frac{1}{2}} - \psi'(x+1)\right] \\ + (x+a)^2\left[\psi''(x+1) + \frac{1}{(x+\frac{1}{2})^2}\right]. \end{aligned}$$

From (31) and (32) we obtain the asymptotic formulas

$$\psi(x) = \log\left(x - \frac{1}{2}\right) + \frac{1}{24(x-\frac{1}{2})^2} - \frac{7}{960(x-\frac{1}{2})^4} + O(x^{-6}), \tag{40}$$

$$\psi'(x) = \frac{1}{x-\frac{1}{2}} - \frac{1}{12(x-\frac{1}{2})^3} + \frac{7}{240(x-\frac{1}{2})^5} + O(x^{-7}), \tag{41}$$

which concludes that

$$\lim_{x\to\infty} f_0'(x) = \lim_{x\to\infty} f_{1/2}'(x) = \lim_{x\to\infty} f_1'(x) = 0. \tag{42}$$

By (33), (35) and (37), we obtain

$$\begin{aligned} f_0''(x) &< 2\left[\frac{1}{24(x+\frac{1}{2})^2}\right] - 4x\left[\frac{1}{12(x+\frac{1}{2})^3} - \frac{7}{240(x+\frac{1}{2})^5}\right] + x^2\left[\frac{1}{4(x+\frac{1}{2})^4}\right] \\ &= -\frac{5x^2 - \frac{23}{4}x - \frac{5}{8}}{60(x+\frac{1}{2})^5} < 0 \quad \text{for} \quad x \geqslant 2, \end{aligned}$$

and then, $f_0'(x) > \lim_{x\to\infty} f_0'(x) = 0$ for $x \geqslant 2$. Hence, the function f_0 is strictly increasing concave for $x \geqslant 2$.

A dierct calculation produces

$$f_0(1) = \left(1-\gamma-\log\frac{3}{2}\right) = 0.01731922699\ldots,$$
$$f_0(2) = 4\left(1+\frac{1}{2}-\gamma-\log\frac{5}{2}\right) = 0.025957441\ldots,$$
$$f_0(3) = 9\left(1+\frac{1}{2}+\frac{1}{3}-\gamma-\log\frac{7}{2}\right) = 0.03019229938\ldots,$$
$$f_0(1)-2f_0(2)+f_0(3) = -0.00440335563\ldots < 0.$$

Thus, the sequence $H(n)$ is strictly increasing concave for all integers $n \geqslant 1$.

By (34), (35) and (38), we obtain

$$\begin{aligned} f''_{1/2}(x) &< 2\left[\frac{1}{24(x+\frac{1}{2})^2}-\frac{7}{960(x+\frac{1}{2})^4}+\frac{31}{8064(x+\frac{1}{2})^6}\right] \\ &\quad -4\left(x+\frac{1}{2}\right)\left[\frac{1}{12(x+\frac{1}{2})^3}-\frac{7}{240(x+\frac{1}{2})^5}\right] \\ &\quad +\left(x+\frac{1}{2}\right)^2\left[\frac{1}{4(x+\frac{1}{2})^4}-\frac{7}{48(x+\frac{1}{2})^6}+\frac{31}{192(x+\frac{1}{2})^6}\right] \\ &= -\frac{\frac{21}{480}(x+\frac{1}{2})^2-\frac{341}{4032}}{(x+\frac{1}{2})^6} < 0 \quad \text{for} \quad x \geqslant 1, \end{aligned}$$

and then, $f'_{1/2}(x) > \lim_{x\to\infty} f'_{1/2}(x) = 0$ for $x \geqslant 1$. Hence, the function $f_{1/2}$ is strictly increasing concave for $x \geqslant 1$, and the sequence $[(n+1/2)/n]^2 H(n)$ is strictly increasing concave for all integers $n \geqslant 1$.

By (34), (36) and (38), we obtain

$$\begin{aligned} f''_1(x) &> 2\left[\frac{1}{24(x+\frac{1}{2})^2}-\frac{7}{960(x+\frac{1}{2})^4}\right] \\ &\quad -4(x+1)\left[\frac{1}{12(x+\frac{1}{2})^3}-\frac{7}{240(x+\frac{1}{2})^5}+\frac{31}{1344(x+\frac{1}{2})^7}\right] \\ &\quad +(x+1)^2\left[\frac{1}{4(x+\frac{1}{2})^4}-\frac{7}{48(x+\frac{1}{2})^6}\right] \\ &= \frac{280(x+\frac{1}{2})^4+63(x+\frac{1}{2})^3-290(x+\frac{1}{2})^2-432.5(x+\frac{1}{2})-155}{3360(x+\frac{1}{2})^7} \\ &> 0 \quad \text{for} \quad x \geqslant \frac{3}{2}. \end{aligned}$$

and then, $f'_1(x) < \lim_{x\to\infty} f'_1(x) = 0$ for $x \geqslant \frac{3}{2}$. Hence, the function f_1 is strictly decreasing convex for $x \geqslant \frac{3}{2}$.

A dierct calculation produces

$$\begin{aligned}
f_1(1) &= 4\left(1-\gamma-\log\frac{3}{2}\right) = 0.06927690796\ldots,\\
f_1(2) &= 9\left(1+\frac{1}{2}-\gamma-\log\frac{5}{2}\right) = 0.0584042429\ldots,\\
f_1(3) &= 16\left(1+\frac{1}{2}+\frac{1}{3}-\gamma-\log\frac{7}{2}\right) = 0.0536751989\ldots,\\
f_1(1) &- 2f_1(2)+f_1(3) = 0.00614362106\ldots > 0.
\end{aligned}$$

Hence, the sequence $[(n+1)/n]^2H(n)$ is strictly decreasing convex for all integers $n \geqslant 1$. The proof of Theorem 2 is complete. □

Proof of Theorem 3. Let $f_1(x)$ be defined by

$$f_1(x) = (x+1)^2\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right], \quad x>0.$$

Direct calculation produces

$$\begin{aligned}
&\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right][\log f_1(x)]''\\
&= \left[\psi''(x+1)+\frac{1}{(x+\frac{1}{2})^2}\right]\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right]\\
&\qquad -\left[\frac{1}{x+\frac{1}{2}}-\psi'(x+1)\right]^2-\frac{2}{(x+1)^2}\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right]^2.
\end{aligned}$$

By (33), (35) and (38), we obtain

$$\begin{aligned}
&\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right]^2[\log f_1(x)]''\\
&> \left[\frac{1}{4(x+\frac{1}{2})^4}-\frac{7}{48(x+\frac{1}{2})^6}\right]\left[\frac{1}{24(x+\frac{1}{2})^2}-\frac{7}{960(x+\frac{1}{2})^4}\right]\\
&\qquad -\left[\frac{1}{12(x+\frac{1}{2})^3}\right]^2-\frac{2}{(x+1)^2}\left[\frac{1}{24(x+\frac{1}{2})^2}\right]^2\\
&= \frac{40x^3-21x^2-142x-\frac{169}{2}}{11520(x+\frac{1}{2})^8(x+1)^2}+\frac{49}{46080(x+\frac{1}{2})^{10}} > 0 \quad \text{for} \quad x \geqslant 3.
\end{aligned}$$

A dierct calculation produces

$$\begin{aligned}
f_1(1)f_1(3) &= 0.003718451 > 0.003411055 = f_1^2(2),\\
f_1(2)f_1(4) &= 0.002979014\ldots > 0.002881026 = f_1^2(3) > 0.
\end{aligned}$$

Hence, the sequences $[(n+1)/n]^2 H(n)$ $(n = 1,2,\ldots)$ is strictly log-convex. The proof of Theorem 3 is complete. □

By Theorem 2, we pose the following conjecture.

CONJECTURE 1. *Let*

$$H(x) = \left[\psi(x+1) - \log\left(x+\frac{1}{2}\right)\right] x^2, \quad x > 0. \tag{43}$$

Then,

(i) The functions $H(x)$ *and* $[(x+1/2)/x]^2 H(x)$ *are both so-called Bernstein function on* $(0,\infty)$. *That is,*

$$H(x) > 0, \quad (-1)^n (H(x))^{(n+1)} > 0 \tag{44}$$

for $x > 0, n = 0,1,2,\ldots$, *and*

$$[(x+1/2)/x]^2 H(x) > 0, \quad (-1)^n ([(x+1/2)/x]^2 H(x))^{(n+1)} > 0 \tag{45}$$

for $x > 0, n = 0,1,2,\ldots$.

(ii) The function $[(x+1)/x]^2 H(x)$ *is strictly completely monotonic on* $(0,\infty)$.

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