Journal of
Mathematical
Inequalities
Volume 3, Number 1 (2009), 99–106

SOME BOUNDS ON SAMPLE PARAMETERS WITH REFINEMENTS OF SAMUELSON AND BRUNK INEQUALITIES

R. SHARMA, A. KAURA, M. GUPTA AND S. RAM

(*communicated by J. Matkowski*)

Abstract. We derive bounds on the extreme deviation of a finite universe in terms of its range and standard deviation. The bounds for the range and ratio of the extreme values are obtained in terms of the standard means.

1. Introduction

Let $x_1, x_2, \ldots, x_n$ denote n real numbers with arithmetic mean

$$A = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{1.1}$$

and standard deviation

$$S = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - A)^2}\,. \tag{1.2}$$

The Samuelson inequality asserts that for a sample of size n no observation x_i $(i = 1, 2, \ldots, n)$ can lie more than $\sqrt{n-1}$ standard deviation away from the mean [1]

$$d \leqslant \sqrt{n-1}\, S, \tag{1.3}$$

where

$$d = \max\{|x_i - A|, i = 1, 2, \ldots, n\} \tag{1.4}$$

is maximum deviation from the mean. The complementary Brunk inequalities says that [2]

$$S \leqslant \sqrt{n-1}\, d_1 \quad and \quad S \leqslant \sqrt{n-1}\, d_2\,, \tag{1.5}$$

where $d_1 = M - A$, $d_2 = A - m$ and $m \leqslant x_i \leqslant M$, $(i = 1, 2, \ldots, n)$. Such inequalities as the above, their further refinements and extensions have been studied by several authors. In particular, Bhatia and Davis have proved that [3]

$$S^2 \leqslant (M - A)(A - m)\,. \tag{1.6}$$

Mathematics subject classification (2000): 60E15.
Keywords and phrases: Karl Pearson coefficient of dispersion, standard deviation, harmonic mean and Samuelson inequality.

Paper JMI-03-09

What remains unnoticed in (1.6) is an interesting observation that the upper bound can be expressed in terms of maximum deviation d, and range r defined by

$$r = M - m. \tag{1.7}$$

This provides refinements of the Brunk and Samuelson inequalities, and we get an upper bound for the maximum deviation in terms of the standard deviation and range of the sample (Theorem-2.1, below). We obtain one more refinement of the Samuelson inequality (Theorem-2.2, below).

The Karl Pearson coefficient of dispersion V, defined by

$$V = \frac{S}{A}, \tag{1.8}$$

is a widely used measure of dispersion. We obtain bounds for V in terms of m and M (Theorem 2.3, below).

Let H, G and m_2 be the harmonic mean, geometric mean and root mean square, respectively, defined by

$$H = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1}, \tag{1.9}$$

$$G = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \tag{1.10}$$

and

$$m_2 = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}. \tag{1.11}$$

We obtain a lower bound for r in terms of A and H and a complementary upper bound in terms of G and m_2 (Theorem-2.4, below). Some bounds for the ratio $\frac{M}{m}$ are also proved (Theorem-2.5, below). The bounds for the difference $A - H$ are derived (Theorem-2.6, below) which affect further improvements on the corresponding bounds proved in [4],

$$\frac{(M-m)\,S^2}{M\,(M-m) - S^2} \leqslant A - H \leqslant \frac{(M-m)\,S^2}{m\,(M-m) + S^2}. \tag{1.12}$$

2. Main Results

THEOREM 2.1. *For $m \leqslant x_i \leqslant M$, $(i = 1, 2, \ldots, n)$ and under the above notations*

$$\frac{r}{2} - \sqrt{\left(\frac{r}{2}\right)^2 - S^2} \leqslant \{d_1, d_2, d\} \leqslant \frac{r}{2} + \sqrt{\left(\frac{r}{2}\right)^2 - S^2}\,. \tag{2.1}$$

The upper bound (2.1) *gives a refinement of the Samuelson inequality for $nS \geqslant \sqrt{n-1}\, r$. An equivalent formulation of* (2.1) *is:*

$$S^2 \leqslant d\,(r-d) = d_i\,(r-d_i)\ ,\ \ i = 1, 2. \tag{2.2}$$

This provides refinements of the Brunk inequalities.

Proof. Inequality (1.6) can be solved to find the following bounds for mean:

$$\frac{m+M}{2} - \sqrt{\left(\frac{r}{2}\right)^2 - S^2} \leqslant A \leqslant \frac{m+M}{2} + \sqrt{\left(\frac{r}{2}\right)^2 - S^2} \ . \tag{2.3}$$

It is evident that

$$d = M - A \text{ or } A - m \, . \tag{2.4}$$

In either case, (2.3) implies (2.1), and $r \leqslant nd$. The remaining assertions of the theorem are now immediate. □

THEOREM 2.2. *For $m \leqslant x_i \leqslant M$, $(i = 1, 2, \ldots, n)$ and $n \geqslant 3$,*

$$\frac{r}{2} - \sqrt{\frac{n-2}{2}\left(S^2 - \frac{r^2}{2n}\right)} \leqslant \{d_1, d_2, d\} \leqslant \frac{r}{2} + \sqrt{\frac{n-2}{2}\left(S^2 - \frac{r^2}{2n}\right)} \, , \tag{2.5}$$

or equivalently

$$S^2 \geqslant \frac{r^2}{2n} + \frac{2}{n-2}\left(d - \frac{r}{2}\right)^2 \geqslant \frac{d^2}{n-1}. \tag{2.6}$$

This provides a refinement of the Samuelson inequality.

Proof. There is no loss of generality in assuming that $x_1 = m$ and $x_n = M$. We can write (1.1) in the form

$$A = \frac{m+M}{n} + \frac{n-2}{n} \frac{x_2 + \ldots + x_{n-1}}{n-2} \ . \tag{2.7}$$

On using Cauchy inequality we get that

$$\left(A - \frac{m+M}{n}\right)^2 \leqslant \frac{n-2}{n^2}\left(nS^2 + nA^2 - m^2 - M^2\right) \ . \tag{2.8}$$

Inequality (2.8) yields the following bounds for mean:

$$\frac{m+M}{2} - \sqrt{\frac{n-2}{2}\left(S^2 - \frac{r^2}{2n}\right)} \leqslant A \leqslant \frac{m+M}{2} + \sqrt{\frac{n-2}{2}\left(S^2 - \frac{r^2}{2n}\right)}. \tag{2.9}$$

By (2.4), $d = M - A$ or $A - m$, in both cases, (2.9) implies (2.5). Further it is easily seen that

$$\frac{r^2}{2n} + \frac{2}{n-2}\left(d - \frac{r}{2}\right)^2 \geqslant \frac{d^2}{n-1} \tag{2.10}$$

if and only if

$$(nd - (n-1)r)^2 \geqslant 0. \tag{2.11}$$

Hence (2.6) provides a refinement of the Samuelson inequality. □

THEOREM 2.3. *For* $0 < m \leqslant x_i \leqslant M,\ (i = 1, 2, \ldots, n)$

$$V \leqslant \frac{M-m}{2\sqrt{Mm}} \tag{2.12}$$

and, for $n \geqslant 3$

$$V \geqslant \frac{M-m}{\sqrt{(n-1)\left(m^2+M^2\right)+2mM}}\ , \tag{2.13}$$

where V *is defined by* (1.8).

Proof. It follows from (1.6) that

$$V \leqslant \frac{\sqrt{(M-A)(A-m)}}{A}, \qquad A > 0. \tag{2.14}$$

On using derivatives we find that the right hand side expression of (2.14) has maximum at

$$A = \frac{2mM}{m+M}. \tag{2.15}$$

Substituting (2.15) into (2.14) we obtain (2.12) . Similarly it follows from (2.6) that

$$\frac{S^2}{A^2} \geqslant \frac{r^2}{2nA^2} + \frac{2}{(n-2)A^2}\left(d - \frac{r}{2}\right)^2. \tag{2.16}$$

The right hand side expression of (2.16) has minimum at

$$A = \frac{(n-1)\left(m^2+M^2\right)+2mM}{n(m+M)}\ . \tag{2.17}$$

Substituting (2.17) into (2.16) we obtain (2.13). □

THEOREM 2.4. *For* $0 < m \leqslant x_i \leqslant M,\ (i = 1, 2, \ldots, n)$

$$r \geqslant 2\sqrt{H(A-H)} \tag{2.18}$$

and

$$r \leqslant \sqrt{n\left(m_2^2 - G^2\right)}, \tag{2.19}$$

where H, G *and* m_2 *are respectively defined by* (1.9), (1.10) *and* (1.11).

Proof. From inequality [3]:

$$A \leqslant m + M - \frac{mM}{H} \tag{2.20}$$

it follows that

$$M - m \geqslant \frac{H(A-m)}{H-m} - m. \tag{2.21}$$

The right hand side expression of (2.21) has minimum at

$$m = H - \sqrt{H(A-H)}. \tag{2.22}$$

Substituting (2.22) into (2.21) we get (2.18).

To prove (2.19) we write

$$m_2^2 = \frac{m^2+M^2}{n} + \frac{n-2}{n}\frac{x_2^2+\ldots+x_{n-1}^2}{n-2} \tag{2.23}$$

and on using inequality

$$\frac{x_2^2+\ldots+x_{n-1}^2}{n-2} \geqslant (x_2x_3\ldots x_{n-1})^{\frac{2}{n-2}} \tag{2.24}$$

we find that

$$m_2^2 - G^2 \geqslant \frac{m^2+M^2}{n} + \frac{n-2}{n}\left[\frac{G}{(mM)^{\frac{1}{n}}}\right]^{\frac{2n}{n-2}} - G^2. \tag{2.25}$$

Minimising the right hand side expression of (2.25) we get

$$m_2^2 - G^2 \geqslant \frac{(M-m)^2}{n}. \tag{2.26}$$

Inequality (2.19) now follows immediately from (2.26). □

THEOREM 2.5. *For* $0 < m \leqslant x_i \leqslant M,\ (i = 1,2,\ldots,n)$

$$\frac{M}{m} \geqslant \left[\frac{S}{A} + \sqrt{1+\frac{S^2}{A^2}}\right]^2, \tag{2.27}$$

$$\frac{M}{m} \geqslant \left[\sqrt{\frac{A}{H}} + \sqrt{\frac{A}{H}-1}\right]^2, \tag{2.28}$$

$$\frac{M}{m} \leqslant \left[\left(\frac{A}{G}\right)^{\frac{n}{2}} + \sqrt{\left(\frac{A}{G}\right)^{n}-1}\right]^2, \tag{2.29}$$

$$\frac{M}{m} \leqslant \left[\left(\frac{G}{H}\right)^{\frac{n}{2}} + \sqrt{\left(\frac{G}{H}\right)^{n}-1}\right]^2, \tag{2.30}$$

$$\frac{M}{m} \leqslant \left(\frac{m_2}{G}\right)^{n} + \sqrt{\left(\frac{m_2}{G}\right)^{2n}-1} \tag{2.31}$$

and

$$\frac{M}{m} \leqslant \left[1 + \frac{\sqrt{\alpha}}{2}\left(\sqrt{\alpha+4}+\sqrt{\alpha}\right)\right]^2, \tag{2.32}$$

where

$$\alpha = n\left(\sqrt{\frac{A}{H}} - 1\right). \tag{2.33}$$

Proof. For $m > 0,$ we find from inequality (1.6) that

$$\frac{M}{m} \geqslant \frac{S^2 + A^2 - mA}{m(A-m)}. \tag{2.34}$$

On using derivatives we get that

$$\frac{S^2 + A^2 - mA}{m(A-m)} \geqslant \left[\frac{A}{\sqrt{S^2+A^2} - S}\right]^2. \tag{2.35}$$

Combining (2.34) and (2.35) we get on simplification (2.27).

It follows from inequality (2.20) that

$$\frac{M}{m} \geqslant \frac{H(A-m)}{m(H-m)}. \tag{2.36}$$

Also

$$\frac{H(A-m)}{m(H-m)} \geqslant \left[\sqrt{\frac{A}{H}} + \sqrt{\frac{A}{H} - 1}\right]^2. \tag{2.37}$$

Combining (2.36) and (2.37) we obtain (2.28).

To prove (2.29) we observe that

$$A = \frac{m+M}{n} + \frac{x_2 + \ldots + x_{n-1}}{n} \tag{2.38}$$

and

$$\frac{x_2 + \ldots + x_{n-1}}{n-2} \geqslant (x_2 \ldots x_{n-1})^{\frac{1}{n-2}}. \tag{2.39}$$

Therefore

$$\frac{A}{G} \geqslant \frac{m+M}{nG} + \frac{n-2}{nG}\left(\frac{G^n}{mM}\right)^{\frac{1}{n-2}}. \tag{2.40}$$

The right hand side expression of (2.40) has minimum at

$$G = (mM)^{\frac{1}{n}}\left(\frac{m+M}{2}\right)^{\frac{n-2}{n}}. \tag{2.41}$$

Combining (2.40) and (2.41) we get on simplification

$$\frac{M}{m} - 2\left(\frac{A}{G}\right)^{\frac{n}{2}}\sqrt{\frac{M}{m}} + 1 \leqslant 0. \tag{2.42}$$

From (2.42) we easily get (2.29).

In the same manner we can see that

$$\frac{M}{m}-2\left(\frac{G}{H}\right)^{\frac{n}{2}}\sqrt{\frac{M}{m}}+1\leqslant 0 \tag{2.43}$$

and

$$\left(\frac{M}{m}\right)^{2}-2\left(\frac{m_2}{G}\right)^{n}\frac{M}{m}+1\leqslant 0. \tag{2.44}$$

From(2.43) and (2.44) we respectively get (2.30) and (2.31).

On using Cauchy's inequality it is easily seen that

$$\frac{A}{H}\geqslant\frac{1}{n^2}\left[n-2+\frac{m+M}{\sqrt{mM}}\right]^2. \tag{2.45}$$

From (2.45) we get that

$$\frac{M}{m}-(2+\alpha)\sqrt{\frac{M}{m}}+1\leqslant 0. \tag{2.46}$$

Inequality (2.32) now follows immediately from (2.46). □

THEOREM 2.6. *For* $0<m\leqslant x_i\leqslant M,\ (i=1,2,\ldots,n)$

$$A-H\geqslant\frac{2S^2}{m+M+\sqrt{(M-m)^2-4S^2}} \tag{2.47}$$

and

$$A-H\leqslant\frac{2S^2}{m+M-\sqrt{(M-m)^2-4S^2}}. \tag{2.48}$$

Proof. For $m<H<M$, we have [4]

$$S^2\leqslant\frac{M(A-H)(M-A)}{M-H} \tag{2.49}$$

and

$$S^2\geqslant\frac{m(A-m)(A-H)}{H-m}. \tag{2.50}$$

From(2.49) and (2.50) we respectively find that

$$A-H\geqslant\frac{(M-A)S^2}{M(M-A)-S^2} \tag{2.51}$$

and

$$A-H\leqslant\frac{(A-m)S^2}{m(A-m)+S^2}. \tag{2.52}$$

The right hand side expression of (2.51) is an increasing function of A and assumes its minimum at

$$A = \frac{m+M}{2} - \sqrt{\left(\frac{M-m}{2}\right)^2 - S^2}\,. \tag{2.53}$$

Substituting (2.53) in (2.51) we get (2.47). Similarly, the right hand side expression of (2.52) assumes its maximum at

$$A = \frac{m+M}{2} + \sqrt{\left(\frac{M-m}{2}\right)^2 - S^2}\,. \tag{2.54}$$

Substituting (2.54) in (2.52) we get (2.48). □

Acknowledgement. The first author thanks Prof. Rajendra Bhatia for his helpful guidance and suggestions, and Indian Statistical Institute for sponsoring his visit to New Delhi in January 2007, when this work was begun.

REFERENCES

[1] P. A. SAMUELSON, *How deviant can you be?*, Journal of the American Statistical Association, 63, 1522–1525, (1968).
[2] H. D. BRUNK, *Note on two papers of K. R. Nair,* Journal of the Indian Society of Agricultural Statistics, 11, 186–189, (1959).
[3] R. BHATIA AND C. DAVIS, *A better bound on the variance,* The American Mathematical Monthly, 107, 353–357, (2000).
[4] R. SHARMA, *Some more inequalities for arithmetic mean, harmonic mean and variance*, Journal of Mathematical Inequalities, 2, 109-114, (2008).

(Received July 18, 2008)

R. Sharma, A. Kaura, M. Gupta and S. Ram
Department of Mathematics
Himachal Pradesh University
Shimla-5,
India – 171 005
e-mail: rajesh_hpu_math@yahoo.co.in