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GENERALIZATIONS OF SOME PROPERTIES OF CONVEX FUNCTIONS

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Abstract. In this paper, generalizations of some properties of convex functions proved in [4, 5] are given.

1. Introduction

In 1965, T. Popoviciu [10] gave the following theorem that characterizes convex functions:

THEOREM A. *If n is a nonnegative integer number $\geqslant 3$ and k a nonnegative integer number which verify the inequalities $2 \leqslant k \leqslant n-1$, the real continuous function f, defined on the nonempty interval I, is convex on I iff the inequality*

$$k \cdot \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} f\left(\frac{x_{i_1}+x_{i_2}+\ldots+x_{i_k}}{k}\right) \leqslant \binom{n-2}{k-2} \cdot \left[\frac{n-k}{k-1} \cdot \sum_{i=1}^{n} f(x_i) + n \cdot f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\right] \tag{1}$$

holds for all $x_1, x_2, \ldots, x_n \in I$.

For $n=3$, $k=2$, we find the well known Popoviciu's inequality

$$2 \cdot \left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right] \leqslant f(x)+f(y)+f(z)+3 \cdot f\left(\frac{x+y+z}{3}\right) \tag{2}$$

for all $x, y, z \in I$.

In 1974, J. C. Burkill [3], showing the importance of the inequality (2), stated the following result:

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THEOREM B. *If the function* $f:[a,b]\to\mathbb{R}$ $(a,b\in\mathbb{R}, a<b)$ *is convex on* $[a,b]$ *and twice differentiable on* (a,b), *then the inequality*

$$(p+q)\cdot f\left(\frac{px+qy}{p+q}\right)+(q+r)\cdot f\left(\frac{qy+rz}{q+r}\right)+(r+p)\cdot f\left(\frac{rz+px}{r+p}\right) \leqslant p\cdot f(x)+q\cdot f(y)+r\cdot f(z)+(p+q+r)\cdot f\left(\frac{px+qy+rz}{p+q+r}\right) \tag{3}$$

is valid for all $x,y,z\in[a,b]$ *and for all* $p,q,r\in(0,+\infty)$.

Later, P. M. Vasić and Lj. R. Stanković [12] and V. J. Baston [2] in 1976, then A. Lupaş [7] in 1982, proved the inequality (3) removing the differentiability condition on f.

P. M. Vasić and Lj. R. Stanković [12] also showed that the following generalization of the inequality (1):

$$\sum_{1\leqslant i_1<i_2<\ldots<i_k\leqslant n}\left(\sum_{j=1}^{k}p_{i_j}\right)\cdot f\left(\frac{\sum_{j=1}^{k}p_{i_j}x_{i_j}}{\sum_{j=1}^{k}p_{i_j}}\right) \leqslant \binom{n-2}{k-2}\cdot\left(\frac{n-k}{k-1}\cdot\sum_{i=1}^{n}p_i f(x_i)+\left(\sum_{i=1}^{n}p_i\right)\cdot f\left(\frac{\sum_{i=1}^{n}p_ix_i}{\sum_{i=1}^{n}p_i}\right)\right), \tag{4}$$

where $k,n\in\mathbb{N}$, $n\geqslant 3$, $2\leqslant k\leqslant n-1$, $p_i>0$, $x_i\in[a,b]$, $i=1,2,\ldots,n$, is equivalent to the inequality (3).

In [4], we proved the following result:

THEOREM C. *A real continuous function* f, *defined on a nonempty interval* I, *is convex on* I *iff the inequality*

$$3\cdot\left[f\left(\frac{2x+y}{3}\right)+f\left(\frac{x+2y}{3}\right)+f\left(\frac{2y+z}{3}\right)+f\left(\frac{y+2z}{3}\right)+f\left(\frac{2z+x}{3}\right)+f\left(\frac{z+2x}{3}\right)\right] \leqslant 4\cdot[f(x)+f(y)+f(z)]+6\cdot f\left(\frac{x+y+z}{3}\right) \tag{5}$$

holds for all $x,y,z\in I$.

In [5], we gave the following result:

THEOREM D. *Let* $f:I\to\mathbb{R}$ *be a real continuous convex function, where* I *is a nonempty interval. Then, we have:*

$$2\cdot\left[f\left(\frac{x+y+z}{3}\right)+f\left(\frac{y+z+t}{3}\right)+f\left(\frac{z+t+x}{3}\right)+f\left(\frac{t+x+y}{3}\right)\right] \tag{6}$$

$$\leqslant f(x)+f(y)+f(z)+f(t)+4\cdot f\left(\frac{x+y+z+t}{4}\right)$$

for all $x,y,z,t\in I$.

In the following, we shall give some generalizations of the inequalities (5) and (6). Their proofs are based on a result due to T. Popoviciu [8, 9] and to K.Toda [11], contained in:

THEOREM E. *(a) Every function from the sequence*

$$f_n(t)=\alpha t+\beta+\sum_{k=0}^{n}p_k\cdot|t-t_k|,\quad n=1,2,\ldots \tag{7}$$

where $t\in[a,b]$, $a,b\in\mathbb{R}$, $a<b$; $\alpha,\beta\in\mathbb{R}$; $p_k\geqslant 0$, $t_k\in[a,b]$ $(k=1,2,\ldots)$, *is convex on* $[a,b]$.

(b) Every convex function f *on* $[a,b]$ *is the uniform limit of a sequence* f_n *of the form* (7).

2. Main results

A first generalization of the inequality (5) is given by:

THEOREM 2.1. *Let* $f:[a,b]\to\mathbb{R}$ *be a continuous convex function, where* $a,b\in\mathbb{R}$, $a<b$. *Then, the inequality*

$$3\cdot\sum_{1\leqslant i<j\leqslant n}\left[f\left(\frac{2x_i+x_j}{3}\right)+f\left(\frac{x_i+2x_j}{3}\right)\right]\leqslant(3n-5)\cdot\sum_{i=1}^{n}f(x_i)+2n\cdot f\left(\frac{\sum_{i=1}^{n}x_i}{n}\right) \tag{8}$$

holds for all $x_1,x_2,\ldots,x_n\in[a,b]$ *and for all* $n\in\mathbb{N}$, $n\geqslant 3$.

Proof. We first state the following inequality:

$$\sum_{1\leqslant i<j\leqslant n}(|2a_i+a_j|+|a_i+2a_j|)\leqslant(3n-5)\cdot\sum_{i=1}^{n}|a_i|+2\cdot\left|\sum_{i=1}^{n}a_i\right| \tag{9}$$

for all $n\in\mathbb{N}$, $n\geqslant 3$ and for all $a_1,a_2,\ldots,a_n\in\mathbb{R}$.

D. Adamović [1] proved the relation:

$$\sum_{1\leqslant i<j\leqslant n}|a_i+a_j|\leqslant(n-2)\cdot\sum_{i=1}^{n}|a_i|+\left|\sum_{i=1}^{n}a_i\right| \tag{10}$$

for all $n\in\mathbb{N}$, $n\geqslant 3$ and for all $a_1,a_2,\ldots,a_n\in\mathbb{R}$.

Note that

$$|2a_i+a_j|+|a_i+2a_j| \leqslant |a_i|+|a_j|+2\cdot|a_i+a_j|, \tag{11}$$

where $i,j\in\mathbb{N}$, $1\leqslant i<j\leqslant n$, $n\in\mathbb{N}$, $n\geqslant 3$.

From (11), we deduce the inequality

$$\sum_{1\leqslant i<j\leqslant n}(|2a_i+a_j|+|a_i+2a_j|) \leqslant (n-1)\cdot\sum_{i=1}^{n}|a_i|+2\cdot\sum_{1\leqslant i<j\leqslant n}|a_i+a_j|. \tag{12}$$

Multiplying (10) by 2 and adding the obtained relation to (12), we find (9).

In the following, we apply Theorem E.

In order to verify that the inequality (8) is valid for all continuous and convex functions on the interval $[a,b]$, it suffices to show that it holds for the functions

$$f_1(t)=\alpha t+\beta,\ a\leqslant t\leqslant b \quad\text{and}\quad f_2(t)=|t-\lambda|,\ a\leqslant t\leqslant b,$$

where $\lambda\in[a,b]$ is arbitrary but fixed. Obviously, f_1 satisfies (8). Thus, we only need to show that f_2 also satisfies (8). For real numbers $x_1,x_2,\dots,x_n,\lambda\in[a,b]$, the inequality (8) becomes:

$$\sum_{1\leqslant i<j\leqslant n}(|2x_i+x_j-3\lambda|+|x_i+2x_j-3\lambda|) \leqslant (3n-5)\cdot\sum_{i=1}^{n}|x_i-\lambda|+2\cdot\left|\sum_{i=1}^{n}x_i-n\lambda\right|. \tag{13}$$

Considering $a_1=x_1-\lambda$, $a_2=x_2-\lambda$, $\dots$, $a_n=x_n-\lambda$ in (13), we obtain (9). Thus, the inequality (8) is verified by every function of the form (7). Passing at limit with $n\to\infty$ in the inequality obtained from (8) by interchanging f with f_n, we deduce the inequality that we wanted to prove.

REMARK 2.1. Considering $n=3$ in the inequality (8), we find (5).

Further, we shall give a common generalization of the inequalities (2) and (5).

THEOREM 2.2. *Let* $f:[a,b]\to\mathbb{R}$ *be a continuous and convex function, where* $a,b\in\mathbb{R}$, $a<b$. *Then, the inequality*

$$(q+r)\cdot\sum_{1\leqslant i<j\leqslant n}\left[f\left(\frac{qx_i+rx_j}{q+r}\right)+f\left(\frac{rx_i+qx_j}{q+r}\right)\right] \leqslant [(n-1)q+(n-3)r]\cdot\sum_{i=1}^{n}f(x_i)+2rn\cdot f\left(\frac{\sum_{i=1}^{n}x_i}{n}\right) \tag{14}$$

holds $(\forall)n\in\mathbb{N}$, $n\geqslant 3$, $(\forall)q,r\in(0,+\infty)$, $q\geqslant r$, *and* $(\forall)x_1,x_2,\dots,x_n\in[a,b]$.

Proof. We first state the following inequality

$$\sum_{1\leqslant i<j\leqslant n}(|qa_i+ra_j|+|ra_i+qa_j|) \leqslant [(n-1)q+(n-3)r]\cdot\sum_{i=1}^{n}|a_i|+2r\cdot\left|\sum_{i=1}^{n}a_i\right|, \tag{15}$$

where $n \in \mathbb{N},\ n \geqslant 3;\ a_1, a_2, \ldots, a_n \in \mathbb{R};\ q, r \in (0, +\infty),\ q \geqslant r$.

Noting that

$$|qa_i + ra_j| + |ra_i + qa_j| \leqslant (q-r)(|a_i| + |a_j|) + 2r \cdot |a_i + a_j|,$$

where $i, j \in \mathbb{N},\ 1 \leqslant i < j \leqslant n,\ n \in \mathbb{N},\ n \geqslant 3$ and $q, r \in (0, +\infty),\ q \geqslant r$, we find

$$\sum_{1 \leqslant i < j \leqslant n} (|qa_i + ra_j| + |ra_i + qa_j|) \leqslant (n-1) \cdot (q-r) \cdot \sum_{i=1}^{n} |a_i| + 2r \cdot \sum_{1 \leqslant i < j \leqslant n} |a_i + a_j|. \quad (16)$$

From (10) and (16), we deduce (15).

The rest of the proof is similar to the corresponding part from the proof of Theorem 2.1.

REMARK 2.2. Considering $n = 3,\ q = 2,\ r = 1$ in (14), we find (5) and considering $n = 3,\ q = r = 1$ in (14), we obtain (2).

Another generalization of the inequality (5) is given by:

THEOREM 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a continuous and convex function, where $a, b \in \mathbb{R},\ a < b$. Then, the inequality*

$$\sum_{1 \leqslant i < j \leqslant n} \left[(qp_i + rp_j) \cdot f\left(\frac{qp_i x_i + rp_j x_j}{qp_i + rp_j}\right) + (rp_i + qp_j) \cdot f\left(\frac{rp_i x_i + qp_j x_j}{rp_i + qp_j}\right) \right] \quad (17)$$

$$\leqslant [(n-1)q + (n-3)r] \cdot \sum_{i=1}^{n} p_i \cdot f(x_i) + 2r \left(\sum_{i=1}^{n} p_i\right) \cdot f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\right)$$

holds $(\forall)\, n \in \mathbb{N},\ n \geqslant 3,\ (\forall)\, p_1, p_2, \ldots, p_n, q, r \in (0, +\infty),\ q \geqslant r$, and $(\forall)\, x_1, x_2, \ldots, x_n \in [a, b]$.

Proof. The proof is similar to that of Theorem 2.1 and Theorem 2.2.

In the following, we shall give two generalizations of the inequality (6). The first result is contained in:

THEOREM 2.4. *If $f : [a, b] \to \mathbb{R}\ (a, b \in \mathbb{R}, a < b)$ is a continuous convex function, then the inequality*

$$6 \cdot \sum_{1 \leqslant i_1 < i_2 < i_3 \leqslant n} f\left(\frac{x_{i_1} + x_{i_2} + x_{i_3}}{3}\right) \leqslant (n-1) \cdot \left[(n-3) \cdot \sum_{i=1}^{n} f(x_i) + n \cdot f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\right] \quad (18)$$

is valid for all $x_1, x_2, \ldots, x_n \in [a, b]$ and for all $n \in \mathbb{N},\ n \geqslant 4$.

Proof. We shall show that

$$2\cdot\sum_{1\leqslant i_1<i_2<i_3\leqslant n}|a_{i_1}+a_{i_2}+a_{i_3}|\leqslant(n-1)\cdot\left[(n-3)\cdot\sum_{i=1}^{n}|a_i|+\left|\sum_{i=1}^{n}a_i\right|\right] \tag{19}$$

for all $n\in\mathbb{N}$, $n\geqslant 4$ and for all $a_1,a_2,\ldots,a_n\in\mathbb{R}$.

In 1963, D.Z. Doković [6] proved the following inequality

$$\sum_{1\leqslant i_1<i_2<\ldots<i_k\leqslant n}|a_{i_1}+a_{i_2}+\ldots+a_{i_k}|\leqslant\binom{n-2}{k-2}\cdot\left(\frac{n-k}{k-1}\cdot\sum_{i=1}^{n}|a_i|+\left|\sum_{i=1}^{n}a_i\right|\right) \tag{20}$$

for all $n\in\mathbb{N}$, $n\geqslant 3$, for all $k\in\{2,3,\ldots,n-1\}$ and for all $a_1,a_2,\ldots,a_n\in\mathbb{R}$.

Considering $k=3$ in (20), we find

$$\sum_{1\leqslant i_1<i_2<i_3\leqslant n}|a_{i_1}+a_{i_2}+a_{i_3}|\leqslant\frac{(n-2)(n-3)}{2}\cdot\sum_{i=1}^{n}|a_i|+(n-2)\cdot\left|\sum_{i=1}^{n}a_i\right|. \tag{21}$$

Note that

$$\frac{(n-1)(n-2)}{2}\cdot\left|\sum_{i=1}^{n}a_i\right|\leqslant\sum_{1\leqslant i_1<i_2<i_3\leqslant n}|a_{i_1}+a_{i_2}+a_{i_3}| \tag{22}$$

for all $n\in\mathbb{N}$, $n\geqslant 3$ and for all $a_1,a_2,\ldots,a_n\in\mathbb{R}$.

Multiplying the relation (21) with $n-1$ and adding the obtained result to the inequality (22), we deduce the following inequality, equivalent to (19):

$$\begin{aligned}(n-2)\cdot&\sum_{1\leqslant i_1<i_2<i_3\leqslant n}|a_{i_1}+a_{i_2}+a_{i_3}|\\&\leqslant\frac{(n-1)(n-2)(n-3)}{2}\cdot\sum_{i=1}^{n}|a_i|+\frac{(n-1)(n-2)}{2}\cdot\left|\sum_{i=1}^{n}a_i\right|.\end{aligned}$$

The rest of the proof is based on Theorem E and is similar to the corresponding part from the proof of Theorem 2.1.

REMARK 2.3. Considering $n=4$ in (18), we find (6).

The second generalization of the inequality (6) is given by:

THEOREM 2.5. *If $f:[a,b]\to\mathbb{R}$ $(a,b\in\mathbb{R},a<b)$ is a continuous and convex function, then the following inequality holds:*

$$\begin{aligned}2\cdot&\sum_{1\leqslant i_1<i_2<i_3\leqslant n}(p_{i_1}+p_{i_2}+p_{i_3})\cdot f\left(\frac{p_{i_1}x_{i_1}+p_{i_2}x_{i_2}+p_{i_3}x_{i_3}}{p_{i_1}+p_{i_2}+p_{i_3}}\right)\\&\leqslant(n-1)\cdot\left[(n-3)\cdot\sum_{i=1}^{n}p_i\cdot f(x_i)+\left(\sum_{i=1}^{n}p_i\right)\cdot f\left(\frac{\sum_{i=1}^{n}p_ix_i}{\sum_{i=1}^{n}p_i}\right)\right],\end{aligned} \tag{23}$$

$(\forall)\,n\in\mathbb{N}$, $n\geqslant 4$, $(\forall)\,p_1,p_2,\ldots,p_n\in(0,+\infty)$, *and* $(\forall)\,x_1,x_2,\ldots,x_n\in[a,b]$.

Proof. The proof, being similar to that of Theorem 2.4, is omitted.

REMARK 2.4. Considering $n = 4$ and $p_1 = p_2 = p_3 = p_4 = 1$, we find (6).

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