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SEVERAL q–INTEGRAL INEQUALITIES

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Abstract. In the present paper, several q-integral inequalities are shown, which stem from [F. Qi, Several integral inequalities, *J. Inequal. Pure Appl. Math.*, **1**(2)(2000), Art. 19; Available online at `http://jipam.vu.edu.au/article.php?sid=113`].

1. Introduction

1.1. The q-derivative and q-integral

For $0 < q < 1$, the q-analog of the derivative, denoted by D_q in what follows, may be defined [26] by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0. \tag{1}$$

If $f'(0)$ exists, then $D_q f(0) = f'(0)$. As q tends to 1^-, the q-derivative reduces to the usual derivative.

The q-analog of integration may be given [27] by

$$\int_0^1 f(x)\,\mathrm{d}_q x = (1-q)\sum_{i=0}^{\infty} f(q^i)q^i, \tag{2}$$

which reduces to $\int_0^1 f(x)\,\mathrm{d}\,x$ in the case $q \to 1^-$.

More generally, the q-Jackson integral from 0 to $a \in \mathbb{R}$ can be defined [10, 11] by

$$\int_0^a f(x)\,\mathrm{d}_q x = a(1-q)\sum_{k=0}^{\infty} f(aq^k)q^k \tag{3}$$

provided the sum converges absolutely. The q-Jackson integral on a general interval $[a,b]$ may be defined [10, 11] by

$$\int_a^b f(x)\,\mathrm{d}_q x = \int_0^b f(x)\,\mathrm{d}_q x - \int_0^a f(x)\,\mathrm{d}_q x. \tag{4}$$

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The q-Jackson integral and q-derivative are related by the "fundamental theorem of quantum calculus" which can be restated [11, p. 73] as follows: If F is an anti q-derivative of the function f, namely $D_qF = f$, continuous at $x = a$, then

$$\int_a^b f(x)\,\mathrm{d}_q x = F(b) - F(a); \tag{5}$$

For any function f one has

$$D_q\left(\int_a^x f(t)\,\mathrm{d}_q t\right) = f(x). \tag{6}$$

It is easy to check the q-analog of Leibniz's rule

$$D_q[f(x)g(x)] = f(x)D_q[g(x)] + g(x)D_q[f(x)]. \tag{7}$$

For $b > 0$ and $a = bq^n$ with $n \in \mathbb{N}$, denote

$$[a,b]_q = \left\{bq^k : 0 \leqslant k \leqslant n\right\} \quad \text{and} \quad (a,b]_q = \left[aq^{-1},b\right]_q. \tag{8}$$

1.2. Two related open problems

In [20], the following problem was posed: Under what conditions does the inequality

$$\int_a^b [f(x)]^t\,\mathrm{d}\,x \geqslant \left[\int_a^b f(x)\,\mathrm{d}\,x\right]^{t-1} \tag{9}$$

hold for $t > 1$?

In [19], the above open problem was extended as follows: Under what conditions does the inequality

$$\int_a^b f^{\alpha}(x)\,\mathrm{d}\,x \geqslant \left[\int_a^b f(x)\,\mathrm{d}\,x\right]^{\beta} \tag{10}$$

hold for positive real numbers α and β?

There have been a lot of literature, for instances, [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 24, 25, 28, 29, 30, 31, 32] and related references therein, about investigations of the above open problems. In these investigations, different and various tools, ideas, methods and techniques, such as Jensen's inequality [14, 17], the convexity method [8, 22, 31], functional inequalities in abstract spaces [2, 3, 14], probability measures viewpoint [8, 15], Hölder inequality and its reversed variants [3, 19], analytical methods [18, 25] and Cauchy's mean value theorem [7, 21], have been created.

1.3. Main results

Motivated by (9) and (10), it is much natural to consider the following replanted problem: Under what conditions does the inequality

$$\int_a^b f^{\alpha}(x)\,\mathrm{d}_q x \geqslant \left[\int_a^b f(x)\,\mathrm{d}_q x\right]^{\beta} \tag{11}$$

hold for positive real numbers α and β? In other words, whether can all the inequalities obtained in [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 28, 29, 30, 31, 32] and related references therein, the so-called Qi type integral inequalities for the usual integrals, be replanted into the q-integral cases or other cases?

In [6], given are some sufficient conditions such that the inequality (11) validates, which can be recited as the following three propositions.

PROPOSITION 1. ([6, Proposition 3.2]) *If $t \geqslant 3$ and the function $f(x)$ satisfies $f(a) \geqslant 0$ and*

$$D_q f(x) \geqslant (t-2)(x-a)^{t-3} \tag{12}$$

for $x \in (a,b]_q$, then

$$\int_a^b [f(x)]^t \,\mathrm{d}_q x \geqslant \left[\int_a^b f(qx)\,\mathrm{d}_q x\right]^{t-1}. \tag{13}$$

PROPOSITION 2. ([6, Proposition 3.5]) *If $p \geqslant 1$ and the function $f(x)$ satisfies $f(a) \geqslant 0$ and $D_q f(x) \geqslant p$ for $x \in (a,b]_q$, then*

$$\int_a^b [f(x)]^{p+2} \,\mathrm{d}_q x \geqslant \frac{1}{(b-a)^{p-1}}\left[\int_a^b f(qx)\,\mathrm{d}_q x\right]^{p+1}. \tag{14}$$

PROPOSITION 3. ([6, Proposition 3.7]) *If $f(x)$ satisfies $f(a) \geqslant 0$ and $D_q f(x) \geqslant 1+q$ for $(a,b]_q$, then*

$$\int_a^b [f(x)]^{2p+1} \,\mathrm{d}_q x > \left\{\int_a^b [f(x)]^p \,\mathrm{d}_q x\right\}^2 \tag{15}$$

is valid for all $p > 0$.

The main aim of this paper is to provide more sufficient conditions such that the inequality (11) is valid, and so several q-integral inequalities, our main results, are presented as the following three theorems.

THEOREM 1. *If $f(x)$ is a non-negative and increasing function on $[a,b]_q$ and satisfies*

$$(\alpha-1)f^{\alpha-2}(qx)D_q f(x) \geqslant \beta(\beta-1)f^{\beta-1}(x)(x-a)^{\beta-2} \tag{16}$$

for $\alpha \geqslant 1$ and $\beta \geqslant 1$, then

$$\int_a^b f^{\alpha}(x)\,\mathrm{d}_q x \geqslant \left[\int_a^b f(x)\,\mathrm{d}_q x\right]^{\beta}. \tag{17}$$

THEOREM 2. *If $f(x)$ is a non-negative and increasing function on $[bq^{n+m},b]_q$ for $m,n \in \mathbb{N}$ and satisfies*

$$(\alpha-1)D_q f(x) \geqslant \beta(\beta-1)f^{\beta-\alpha+1}(q^m x)(x-a)^{\beta-2} \tag{18}$$

on $[a,b]_q$ *and for* $\alpha,\beta \geqslant 1$, *then*

$$\int_a^b f^{\alpha}(x)\,\mathrm{d}_q x \geqslant \left[\int_a^b f(q^m x)\,\mathrm{d}_q x\right]^{\beta}. \tag{19}$$

THEOREM 3. *If* $f(x)$ *is a non-negative function on* $[0,b]_q$ *and satisfies*

$$\int_x^b f^{\beta}(t)\,\mathrm{d}_q t \geqslant \int_x^b t^{\beta}\,\mathrm{d}_q t \tag{20}$$

for $x \in [0,b]_q$ *and* $\beta > 0$, *then the inequality*

$$\int_0^b f^{\beta+\alpha}(t)\,\mathrm{d}_q t \geqslant \int_0^b t^{\alpha} f^{\beta}(t)\,\mathrm{d}_q t \tag{21}$$

holds for all positive numbers α *and* β.

2. A lemma

In order to prove our main results, the following lemma is necessary.

LEMMA 1. ([6, Lemma 3.1]) *Let* $p \geqslant 1$ *and* $g(x)$ *be a non-negative and monotonic function on* $[a,b]_q$. *Then*

$$pg^{p-1}(qx)D_q g(x) \leqslant D_q[g^p(x)] \leqslant pg^{p-1}(x)D_q g(x), \quad x \in (a,b]_q. \tag{22}$$

3. Proofs of main results

Now we are in a position to demonstrate our main results, the three theorems stated in Section 1.3.

Proof of Theorem 1. For $x \in [a,b]_q$, let

$$F(x) = \int_a^x [f(t)]^{\alpha}\,\mathrm{d}_q t - \left[\int_a^x f(t)\,\mathrm{d}_q t\right]^{\beta}$$

and

$$h(x) = \int_a^x f(t)\,\mathrm{d}_q t.$$

In virtue of Lemma 1, it follows that

$$\begin{aligned} D_q F(x) &= f^{\alpha}(x) - D_q\left[h^{\beta}(x)\right] \geqslant f^{\alpha}(x) - \beta h^{\beta-1}(x) f(x) \\ &= f(x)\left[f^{\alpha-1}(x) - \beta h^{\beta-1}(x)\right] \triangleq f(x)g(x). \end{aligned}$$

Since $f(x)$ is a non-negative and increasing function, then

$$h(x) = \int_a^x f(t)\,\mathrm{d}_q t \leqslant f(x)(x-a).$$

By virtue of Lemma 1 again, it follows that

$$\begin{aligned} D_q g(x) &= D_q\left[f^{\alpha-1}(x)\right] - \beta D_q\left[h^{\beta-1}(x)\right] \\ &\geqslant (\alpha-1)f^{\alpha-2}(qx)D_q f(x) - \beta(\beta-1)h^{\beta-2}(x)f(x) \\ &\geqslant (\alpha-1)f^{\alpha-2}(qx)D_q f(x) - \beta(\beta-1)(x-a)^{\beta-2}f^{\beta-1}(x), \end{aligned}$$

which means $D_q g(x) \geqslant 0$ by (16), and $g(x) \geqslant 0$ and $D_q F(x) \geqslant 0$, and so $F(x) \geqslant 0$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Let

$$F(x) = \int_a^x [f(t)]^\alpha \,\mathrm{d}_q t - \left[\int_a^x f(q^m t)\,\mathrm{d}_q t\right]^\beta$$

and

$$h(x) = \int_a^x f(q^m t)\,\mathrm{d}_q t.$$

Utilizing Lemma 1 gives

$$\begin{aligned} D_q F(x) &= f^\alpha(t) - D_q\left[h^\beta(x)\right] \geqslant f^\alpha(t) - \beta h^{\beta-1}(x) f(q^m x) \\ &\geqslant f(x)\left[f^{\alpha-1}(x) - \beta h^{\beta-1}(x)\right] \triangleq f(x)g(x). \end{aligned}$$

Since $f(x)$ is a non-negative and increasing function, then

$$h(x) = \int_a^x f(q^m t)\,\mathrm{d}_q t \leqslant f(q^m x)(x-a).$$

Using Lemma 1 once again leads to

$$\begin{aligned} D_q g(x) &= D_q\left[f^{\alpha-1}(x)\right] - \beta D_q\left[h^{\beta-1}(x)\right] \\ &\geqslant (\alpha-1)f^{\alpha-2}(qx)D_q f(x) - \beta(\beta-1)h^{\beta-2}(x)f(q^m x) \\ &\geqslant (\alpha-1)f^{\alpha-2}(qx)D_q f(x) - \beta(\beta-1)(x-a)^{\beta-2}f^{\beta-1}(q^m x) \\ &= f^{\alpha-2}(qx)\left[(\alpha-1)D_q f(x) - \beta(\beta-1)(x-a)^{\beta-2}f^{\beta-\alpha+1}(q^m x)\right], \end{aligned}$$

which means $D_q g(x) \geqslant 0$ by (18), and $g(x) \geqslant 0$ and $D_q F(x) \geqslant 0$, and so $F(x) \geqslant 0$. The proof of Theorem 2 is complete. □

Proof of Theorem 3. By a fact in [11, p 106-107] that $D_q x^t = [t]_q x^{t-1}$ for $t \in \mathbb{R}$, where $[t]_q = \frac{1-q^t}{1-q}$, it is easy to see that

$$\begin{aligned} \int_0^b x^\beta f^\alpha(x)\,\mathrm{d}_q x &= \frac{1}{[\beta]_q}\int_0^b f^\alpha(x)\left(\int_0^x u^{\beta-1}\,\mathrm{d}_q u\right)\mathrm{d}_q x \\ &= \frac{1}{[\beta]_q}\int_0^b u^{\beta-1}\left[\int_u^b f^\alpha(x)\,\mathrm{d}_q x\right]\mathrm{d}_q u \\ &\geqslant \frac{1}{[\beta]_q}\int_0^b u^{\beta-1}\left(\int_u^b x^\alpha\,\mathrm{d}_q x\right)\mathrm{d}_q u \\ &= \frac{1}{[\beta]_q}\int_0^b x^\alpha\left(\int_0^x u^{\beta-1}\,\mathrm{d}_q u\right)\mathrm{d}_q x = \int_0^b x^{\alpha+\beta}\,\mathrm{d}_q x. \end{aligned} \tag{23}$$

Using the arithmetic-geometric inequality yields

$$\frac{\beta}{\alpha+\beta} f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha+\beta} x^{\alpha+\beta} \geqslant x^{\alpha} f^{\beta}(x). \tag{24}$$

Integrating on both sides of (24) and combining with (23) reveal

$$\frac{\beta}{\alpha+\beta} \int_0^b f^{\alpha+\beta}(x)\,\mathrm{d}_q x + \frac{\alpha}{\alpha+\beta} \int_0^b x^{\alpha+\beta}\,\mathrm{d}_q x \geqslant \int_0^b x^{\alpha} f^{\beta}(x)\,\mathrm{d}_q x \geqslant \int_0^b x^{\beta+\alpha}\,\mathrm{d}_q x. \tag{25}$$

The proof of Theorem 3 is complete. □

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