SEVERAL $q$–INTEGRAL INEQUALITIES

YU MIAO AND FENG QI

(communicated by Sh. Abramovich)


1. Introduction

1.1. The $q$-derivative and $q$-integral

For $0 < q < 1$, the $q$-analog of the derivative, denoted by $D_q$ in what follows, may be defined [26] by

$$ D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0. \quad (1) $$

If $f'(0)$ exists, then $D_q f(0) = f'(0)$. As $q$ tends to $1^-$, the $q$-derivative reduces to the usual derivative.

The $q$-analog of integration may be given [27] by

$$ \int_0^1 f(x) \, dq x = (1-q) \sum_{i=0}^{\infty} f(q^i) q^i, \quad (2) $$

which reduces to $\int_0^1 f(x) \, dx$ in the case $q \to 1^-$. More generally, the $q$-Jackson integral from 0 to $a \in \mathbb{R}$ can be defined [10, 11] by

$$ \int_0^a f(x) \, dq x = a(1-q) \sum_{k=0}^{\infty} f(aq^k) q^k \quad (3) $$

provided the sum converges absolutely. The $q$-Jackson integral on a general interval $[a, b]$ may be defined [10, 11] by

$$ \int_a^b f(x) \, dq x = \int_0^b f(x) \, dq x - \int_0^a f(x) \, dq x. \quad (4) $$


Keywords and phrases: $q$-derivative, $q$-integral, Qi type integral inequality, $q$-integral inequalities.

The second author was partially supported by the China Scholarship Council.
The $q$-Jackson integral and $q$-derivative are related by the “fundamental theorem of quantum calculus” which can be restated [11, p. 73] as follows: If $F$ is an anti $q$-derivative of the function $f$, namely $D_q F = f$, continuous at $x = a$, then

$$\int_a^b f(x) \, d_q x = F(b) - F(a); \quad (5)$$

For any function $f$ one has

$$D_q \left( \int_a^x f(t) \, d_q t \right) = f(x). \quad (6)$$

It is easy to check the $q$-analog of Leibniz’s rule

$$D_q [f(x)g(x)] = f(x)D_q [g(x)] + g(x)D_q [f(x)]. \quad (7)$$

For $b > 0$ and $a = bq^n$ with $n \in \mathbb{N}$, denote

$$[a, b)_q = \{ bq^k : 0 \leq k \leq n \} \quad \text{and} \quad (a, b]_q = [aq^{-1}, b]_q. \quad (8)$$


1.2. Two related open problems

In [20], the following problem was posed: Under what conditions does the inequality

$$\int_a^b [f(x)]^t \, d x \geq \left[ \int_a^b f(x) \, d x \right]^{t-1} \quad (9)$$

hold for $t > 1$?

In [19], the above open problem was extended as follows: Under what conditions does the inequality

$$\int_a^b f^\alpha (x) \, d x \geq \left[ \int_a^b f(x) \, d x \right]^\beta \quad (10)$$

hold for positive real numbers $\alpha$ and $\beta$?

There have been a lot of literature, for instances, [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 24, 25, 28, 29, 30, 31, 32] and related references therein, about investigations of the above open problems. In these investigations, different and various tools, ideas, methods and techniques, such as Jensen’s inequality [14, 17], the convexity method [8, 22, 31], functional inequalities in abstract spaces [2, 3, 14], probability measures viewpoint [8, 15], Hölder inequality and its reversed variants [3, 19], analytical methods [18, 25] and Cauchy’s mean value theorem [7, 21], have been created.

1.3. Main results

Motivated by (9) and (10), it is much natural to consider the following replanted problem: Under what conditions does the inequality

$$\int_a^b f^\alpha (x) d_q x \geq \left[ \int_a^b f(x) \, d_q x \right]^\beta \quad (11)$$
hold for positive real numbers \( \alpha \) and \( \beta \) ? In other words, whether can all the inequalities obtained in \([1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 28, 29, 30, 31, 32]\) and related references therein, the so-called Qi type integral inequalities for the usual integrals, be replanted into the \( q \)-integral cases or other cases?

In [6], given are some sufficient conditions such that the inequality (11) validates, which can be recited as the following three propositions.

**Proposition 1.** ([6, Proposition 3.2]) If \( t \geq 3 \) and the function \( f(x) \) satisfies
\[
f(a) \geq 0 \quad \text{and} \quad D_q f(x) \geq (t - 2)(x - a)^{t-3}
\] for \( x \in (a, b]_q \), then
\[
\int_a^b [f(x)]^t \, d_q x \geq \left[ \int_a^b f(qx) \, d_q x \right]^{t-1}.
\] (13)

**Proposition 2.** ([6, Proposition 3.5]) If \( p \geq 1 \) and the function \( f(x) \) satisfies \( f(a) \geq 0 \) and \( D_q f(x) \geq p \) for \( x \in (a, b]_q \), then
\[
\int_a^b [f(x)]^{p+2} \, d_q x \geq \frac{1}{(b - a)^{p-1}} \left[ \int_a^b f(qx) \, d_q x \right]^{p+1}.
\] (14)

**Proposition 3.** ([6, Proposition 3.7]) If \( f(x) \) satisfies \( f(a) \geq 0 \) and \( D_q f(x) \geq 1 + q \) for \( (a, b]_q \), then
\[
\int_a^b [f(x)]^{2p+1} \, d_q x > \left\{ \int_a^b [f(x)]^p \, d_q x \right\}^2
\] (15)
is valid for all \( p > 0 \).

The main aim of this paper is to provide more sufficient conditions such that the inequality (11) is valid, and so several \( q \)-integral inequalities, our main results, are presented as the following three theorems.

**Theorem 1.** If \( f(x) \) is a non-negative and increasing function on \([a, b]_q \) and satisfies
\[
(\alpha - 1)f^{\alpha-2}(qx)D_q f(x) \geq \beta(\beta - 1)f^{\beta-1}(x-a)^{\beta-2}
\] (16)
for \( \alpha \geq 1 \) and \( \beta \geq 1 \), then
\[
\int_a^b f^{\alpha}(x) \, d_q x \geq \left[ \int_a^b f(x) \, d_q x \right]^\beta.
\] (17)

**Theorem 2.** If \( f(x) \) is a non-negative and increasing function on \([bq^{n+m}, b]_q \) for \( m, n \in \mathbb{N} \) and satisfies
\[
(\alpha - 1)D_q f(x) \geq \beta(\beta - 1)f^{\beta-\alpha+1}(q^m x - a)^{\beta-2}
\] (18)
on \([a, b]_q\) and for \(\alpha, \beta \geq 1\), then

\[
\int_a^b f^\alpha(x) \, dq \, x \geq \left[ \int_a^b f(q^\alpha x) \, dq \, x \right]^\beta.
\] (19)

**Theorem 3.** If \(f(x)\) is a non-negative function on \([0, b]_q\) and satisfies

\[
\int_x^b f^\beta(t) \, dq \, t \geq \int_x^b t^\beta \, dq \, t
\] (20)

for \(x \in [0, b]_q\) and \(\beta > 0\), then the inequality

\[
\int_0^b f^{\beta+\alpha}(t) \, dq \, t \geq \int_0^b t^\alpha f^\beta(t) \, dq \, t
\] (21)

holds for all positive numbers \(\alpha\) and \(\beta\).

2. A lemma

In order to prove our main results, the following lemma is necessary.

**Lemma 1.** ([6, Lemma 3.1]) Let \(p \geq 1\) and \(g(x)\) be a non-negative and monotonic function on \([a, b]_q\). Then

\[
p g^{p-1}(qx) D_q g(x) \leq D_q [g^p(x)] \leq p g^{p-1}(x) D_q g(x), \quad x \in (a, b]_q.
\] (22)

3. Proofs of main results

Now we are in a position to demonstrate our main results, the three theorems stated in Section 1.3.

**Proof of Theorem 1.** For \(x \in [a, b]_q\), let

\[
F(x) = \int_a^x [f(t)]^\alpha \, dq \, t - \left[ \int_a^x f(t) \, dq \, t \right]^\beta
\]

and

\[
h(x) = \int_a^x f(t) \, dq \, t.
\]

In virtue of Lemma 1, it follows that

\[
D_q F(x) = f^\alpha(x) - D_q [h^\beta(x)] \geq f^\alpha(x) - \beta h^{\beta-1}(x) f(x) = f(x) [f^{\alpha-1}(x) - \beta h^{\beta-1}(x)] \leq f(x) g(x).
\]

Since \(f(x)\) is a non-negative and increasing function, then

\[
h(x) = \int_a^x f(t) \, dq \, t \leq f(x)(x - a).
\]
By virtue of Lemma 1 again, it follows that

\[
D_qg(x) = D_q\left[f^{\alpha-1}(x)\right] - \beta D_q\left[h^{\beta-1}(x)\right]
\]

\[
\geq (\alpha - 1)f^{\alpha-2}(qx)D_qf(x) - \beta(\beta - 1)h^{\beta-2}(x)f(x)
\]

\[
\geq (\alpha - 1)f^{\alpha-2}(qx)D_qf(x) - \beta(\beta - 1)(x-a)^{\beta-2}f^{\beta-1}(x),
\]

which means \(D_qg(x) \geq 0\) by (16), and \(g(x) \geq 0\) and \(D_qF(x) \geq 0\), and so \(F(x) \geq 0\). The proof of Theorem 1 is complete. □

\textit{Proof of Theorem 2.} Let

\[
F(x) = \int_a^x \left[f(t)\right]^\alpha d_q t - \left[\int_a^x f(q^m t) d_q t\right]^\beta
\]

and

\[
h(x) = \int_a^x f(q^m t) d_q t.
\]

Utilizing Lemma 1 gives

\[
D_qF(x) = f^\alpha(t) - D_q\left[h^\beta(x)\right] \geq f^\alpha(t) - \beta h^{\beta-1}(x)f(q^m x)
\]

\[
\geq f(x)\left[f^{\alpha-1}(x) - \beta h^{\beta-1}(x)\right] \triangleq f(x)g(x).
\]

Since \(f(x)\) is a non-negative and increasing function, then

\[
h(x) = \int_a^x f(q^m t) d_q t \leq f(q^m x)(x-a).
\]

Using Lemma 1 once again leads to

\[
D_qg(x) = D_q\left[f^{\alpha-1}(x)\right] - \beta D_q\left[h^{\beta-1}(x)\right]
\]

\[
\geq (\alpha - 1)f^{\alpha-2}(qx)D_qf(x) - \beta(\beta - 1)h^{\beta-2}(x)f(q^m x)
\]

\[
\geq (\alpha - 1)f^{\alpha-2}(qx)D_qf(x) - \beta(\beta - 1)(x-a)^{\beta-2}f^{\beta-1}(q^m x)
\]

\[
= f^{\alpha-2}(qx)\left[(\alpha - 1)D_qf(x) - \beta(\beta - 1)(x-a)^{\beta-2}f^{\beta-1}(q^m x)\right],
\]

which means \(D_qg(x) \geq 0\) by (18), and \(g(x) \geq 0\) and \(D_qF(x) \geq 0\), and so \(F(x) \geq 0\). The proof of Theorem 2 is complete. □

\textit{Proof of Theorem 3.} By a fact in [11, p 106-107] that \(D_qx^\ell = [\ell]_q x^{\ell - 1}\) for \(t \in \mathbb{R}\), where \([\ell]_q = \frac{1-q}{1-q^\ell}\), it is easy to see that

\[
\int_0^b x^\beta f^\alpha(x) d_q x = \frac{1}{[\beta]_q} \int_0^b \left(\int_0^x u^{\beta-1} d_q u\right) d_q x
\]

\[
= \frac{1}{[\beta]_q} \int_0^b u^{\beta-1} \left[\int_u^b f^\alpha(x) d_q x\right] d_q u
\]

\[
\geq \frac{1}{[\beta]_q} \int_0^b u^{\beta-1} \left(\int_u^b x^\alpha d_q x\right) d_q u
\]

\[
= \frac{1}{[\beta]_q} \int_0^b x^\alpha \left(\int_0^x u^{\beta-1} d_q u\right) d_q x = \int_0^b x^{\alpha+\beta} d_q x.
\]

(23)
Using the arithmetic-geometric inequality yields
\[ \frac{\beta}{\alpha + \beta} f^{\alpha + \beta}(x) + \frac{\alpha}{\alpha + \beta} x^{\alpha + \beta} \geq x^{\alpha} f^{\beta}(x). \] (24)

Integrating on both sides of (24) and combining with (23) reveal
\[ \frac{\beta}{\alpha + \beta} \int_{\alpha}^{\beta} f^{\alpha + \beta}(x) \, d_q x + \frac{\alpha}{\alpha + \beta} \int_{\alpha}^{\beta} x^{\alpha + \beta} \, d_q x = \int_{\alpha}^{\beta} x^{\alpha} f^{\beta}(x) \, d_q x \geq \int_{\alpha}^{\beta} x^{\beta + \alpha} \, d_q x. \] (25)

The proof of Theorem 3 is complete. □

Acknowledgements. The authors would like to express many thanks to the anonymous referees for their corrections and comments on this manuscript.

REFERENCES


(Received October 5, 2008)

Yu Miao  
*College of Mathematics and Information Science*  
*Henan Normal University*  
*Xinxiang City*  
*Henan Province*  
*453007, China*  
e-mail: yumiao728@yahoo.com.cn, yumiao728@gmail.com

Feng Qi  
*Research Institute of Mathematical Inequality Theory*  
*Henan Polytechnic University*  
*Jiaozuo City*  
*Henan Province*  
*454010, China*  
e-mail: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

URL: http://qifeng618.spaces.live.com