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SHARP NORM INEQUALITIES FOR THE TRUNCATED HILBERT TRANSFORM

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Abstract. We split the classical Hilbert Transform H into the sum of two convolution integrals $H^{(\delta)}+R^{(\delta)}$, where the kernel of $H^{(\delta)}$ is supported away from the origin in $\{|t| \geqslant \delta\}$, while the kernel of $R^{(\delta)}$ is supported near the origin in $\{|t| \leqslant \delta\}$. We prove that the L^p-norm of $H^{(\delta)}$, known in the literature as the *Truncated Hilbert Transform*, is equal to the norm of H. Namely $||H^{(\delta)}||_{p,p} = \cot(\pi/2p)$ for $2 \leqslant p < +\infty$, and $||H^{(\delta)}||_{p,p} = \tan(\pi/2p)$ for $1 < p \leqslant 2$. We then prove that the L^p-norm of $R^{(\delta)}$ is stricly larger. In particular $||R^{(\delta)}||_{2,2} = 1.17897\ldots$, a constant related to the Gibbs phenomenon.

1. Introduction and statement of the two main theorems

Let us define, for any fixed $\delta > 0$, the following two convolution integrals

$$(H^{(\delta)}f)(x) = p.v.\frac{1}{\pi}\int_{\{|t|\geqslant\delta\}} \frac{f(x-t)}{t}\,dt \tag{1.1}$$

$$(R^{(\delta)}f)(x) = p.v.\frac{1}{\pi}\int_{\{|t|\leqslant\delta\}} \frac{f(x-t)}{t}\,dt. \tag{1.2}$$

The operator $H^{(\delta)}$ is usually called the *Truncated Hilbert Transform* in the literature. Both $H^{(\delta)}$ and $R^{(\delta)}$ map boundedly $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ for $1 < p < \infty$. Clearly $H = H^{(\delta)} + R^{(\delta)}$, where H is the Hilbert Transform. We will denote by n_p the the best constant in the inequality of M. Riesz $\|Hf\|_p \leqslant n_p\,\|f\|_p$. This constant coincides with the operator norm $||H||_{p,p}$ of the Hilbert Transform H seen as a map from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$. It is given explicitly by the formula

$$n_p = ||H||_{p,p} = \begin{cases} \tan\left(\frac{\pi}{2p}\right) & \text{if } 1 < p \leqslant 2 \\ \cot\left(\frac{\pi}{2p}\right) & \text{if } 2 \leqslant p < \infty. \end{cases} \tag{1.3}$$

See Pichorides [6] or Grafakos [3] for a proof of this fact. It is easy to see and well known that both $||H^{(\delta)}||_{p,p}$ and $||R^{(\delta)}||_{p,p}$ do not depend on the truncation constant $\delta > 0$ (Lemma 1). Furthermore, both norms $||H^{(\delta)}||_{p,p}$ and $||R^{(\delta)}||_{p,p}$ are bounded below by n_p (Lemma 2). Our two main results are

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THEOREM 1. *Let $H^{(\delta)}$ be the Truncated Hilbert Transform* (1.1). *Then* $||H^{(\delta)}||_{p,p} = n_p$ *for* $1 < p < +\infty$, *where* $n_p = ||H||_{p,p}$ *is the expression* (1.3).

THEOREM 2. *Let $R^{(\delta)}$ be the operator* (1.2) *and let $n_p = ||H||_{p,p}$ be the expression* (1.3). *Then $||R^{(\delta)}||_{p,p} > n_p$ for an open interval of exponents p containing $p = 2$. In particular $||R^{(\delta)}||_{2,2} = \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}\, dt = 1.17897974447...$ a constant related to the Gibbs phenomenon. The operator $R^{(\delta)}$ satisfies the upper norm-estimate $||R^{(\delta)}||_{p,p} \leqslant 2n_p$.*

2. Two lemmas and proof of the two theorems

LEMMA 1. *The norms $||H^{(\delta)}||_{p,p}$ and $||R^{(\delta)}||_{p,p}$ do not depend on the truncation constant $\delta > 0$ for all $1 < p < +\infty$.*

Proof. This is easily seen by horizontal and vertical rescaling.

LEMMA 2. *The lower norm-estimates $||H^{(\delta)}||_{p,p} \geqslant n_p$ and $||R^{(\delta)}||_{p,p} \geqslant n_p$ both hold for all $1 < p < +\infty$.*

Proof. By lemma 1 we have that $||H^{(\delta)}||_{p,p}$ is constant as $\delta \to 0^+$, on the other hand $\lim\limits_{\delta\to 0^+} H^{(\delta)}f = Hf$ for any fixed $f \in L^p(\mathbb{R})$, therefore Fatou's Lemma implies that $||H^{(\delta)}||_{p,p} \geqslant n_p$. In a similar fashion, having observed that $\lim\limits_{\delta\to+\infty} R^{(\delta)}f = Hf$, Fatou's Lemma implies that $||R^{(\delta)}||_{p,p} \geqslant n_p$. Note that $\lim\limits_{\delta\to+\infty} H^{(\delta)}f = 0$ and $\lim\limits_{\delta\to 0^+} R^{(\delta)}f = 0$.

Proof of Theorem 1. Because of Lemma 1 we can work with $\delta = 1$. We have

$$H^{(1)}f(x) = \text{p.v.}\frac{1}{\pi}\int_{\{|t|\geqslant 1\}} \frac{f(x-t)}{t}\, dt = \int_{\mathbb{R}} m(\xi)\hat{f}(\xi)e^{2\pi i x\xi} d\xi \tag{2.1}$$

where we have associated to $H^{(1)}$ the multiplier function $m(\xi)$ which is equal to the Fourier Transform of $\frac{1}{\pi y}\chi_{\{|y|\geqslant 1\}}(y)$. We now observe that our truncated Hilbert transform can be written as the composition of H with another operator K as follows

$$H^{(1)}f(x) = \int_{\mathbb{R}} [-i\ \text{sgn}(\xi)]\ [i\ \text{sgn}(\xi)m(\xi)]\ \hat{f}(\xi)e^{2\pi i x\xi} d\xi = HKf(x).$$

In fact, the multiplier function $-i\ \text{sgn}(\xi)$ defines the Hilbert Transform H, while the other multiplier function $i\ \text{sgn}(\xi)m(\xi)$ defines another operator K as a Fourier multiplier. Clearly, K is also given by a convolution with the kernel function

$$k(x) = -H\left(\frac{1}{\pi y}\chi_{\{|y|\geqslant 1\}}(y)\right)(x)$$

which we now compute explicitly

$$\begin{aligned}
k(x) &= \text{p.v.} - \frac{1}{\pi^2} \int_{\{|y|\geqslant 1\}} \frac{1}{(x-y)y}\, dy = \text{p.v.} - \frac{1}{\pi^2\, x} \int_{\{|y|\geqslant 1\}} \left(\frac{1}{y} + \frac{1}{x-y} \right) dy \\
&= -\frac{1}{\pi^2\, x} \lim_{b\to+\infty} \left(\log \frac{b}{|x-b|} - \log \frac{1}{|x-1|} + \log \frac{1}{|x+1|} - \log \frac{b}{|x+b|} \right) \\
&= \frac{1}{\pi^2\, x}\ \log \left| \frac{x+1}{x-1} \right|.
\end{aligned}$$

We claim that $||K||_{p,p} = 1$ for all $1 < p < +\infty$. The theorem follows, because $||H^{(1)}||_{p,p} = ||HK||_{p,p} \leqslant ||H||_{p,p}\ ||K||_{p,p} = n_p$ and this inequality, together with the reverse inequality of Lemma 2, proves that $||H^{(1)}||_{p,p} = n_p$.

To prove our claim let us observe that $k(x) = \dfrac{1}{\pi^2\, x}\ \log \left| \dfrac{x+1}{x-1} \right|$ is an integrable function over the line $\mathbb{R}$. In fact, k has two integrable logarithmic spikes at $x = -1$ and $x = 1$, a removable singularity at $x = 0$, and it is $O(1/x^2)$ as $x \to \pm\infty$. It's also easy to check that k is even and non-negative. If we show that

$$\int_{\mathbb{R}} \frac{1}{\pi^2\, x}\ \log \left| \frac{x+1}{x-1} \right|\, dx = 1 \tag{2.2}$$

then our claim will be a consequence of Minkowski's inequality, because convolution operators with non-negative and integrable kernels have (p,p) norm equal to the integral of their kernel. It is possible, but not completely straightforward, to compute this integral directly. However, we can simply observe that our integral (2.2) is also equal to $\hat{k}(0)$ where, remembering formula (2.1) and the definition of k, we have:

$$\begin{aligned}
\hat{k}(\xi) &= i\, \text{sgn}(\xi) \int_{\mathbb{R}} \frac{1}{\pi y} \chi_{\{|y|\geqslant 1\}}(y) e^{-2\pi i y \xi}\, dy \\
&= 2i\, \text{sgn}(\xi) \int_1^{+\infty} \frac{1}{\pi y} (-i\, \sin(2\pi y \xi))\, dy \\
&= \frac{2}{\pi}\ \text{sgn}(\xi)\ \int_1^{+\infty} \frac{\sin(2\pi y\xi)}{y}\, dy = 1 - \frac{2}{\pi}\ \text{sgn}(\xi) S(2\pi\xi).
\end{aligned}$$

In the last equality we have used the special function $S(x) = \int_0^x \frac{\sin t}{t}\, dt$ and the well-known principal-value integral $\displaystyle\int_{\mathbb{R}} \frac{\sin t}{t}\, dt = \pi$. Note that $\hat{k}(\xi)$ is an even function, continuous at $\xi = 0$ (although its derivative does not exist there) with $\hat{k}(0) = 1$. The theorem is proven. $\square$

Proof of Theorem 2. We have

$$R^{(1)} f(x) = \text{p.v.} \frac{1}{\pi} \int_{\{|t|\leqslant 1\}} \frac{f(x-t)}{t}\, dt = \int_{\mathbb{R}} r(\xi) \hat{f}(\xi) e^{2\pi i x \xi} d\xi \tag{2.3}$$

where $r(\xi)$ is a suitable multiplier function. Also, our operator $R^{(1)}$ can be written as the composition of H with another operator G as follows

$$R^{(1)}f(x) = \int_{\mathbb{R}} [-i\,\mathrm{sgn}(\xi)]\ [i\,\mathrm{sgn}(\xi) r(\xi)]\ \hat{f}(\xi) e^{2\pi i x\xi} d\xi = HGf(x). \tag{2.4}$$

The multiplier function $-i\,\mathrm{sgn}(\xi)$ defines the Hilbert Transform H, while $i\,\mathrm{sgn}(\xi)$ $r(\xi)$ defines another operator G as a Fourier multiplier. We could compute the convolution kernel $g(x)$ of G with a procedure similar to the one we used in the proof of theorem 1, but it is easier to observe that from the equality $H = H^{(1)} + R^{(1)}$ it follows

$$g(x) = \delta(x) - \frac{1}{\pi^2\, x}\ \log\left|\frac{x+1}{x-1}\right|. \tag{2.5}$$

We have denoted by $\delta(x)$ the Dirac unit mass concentrated at the origin. It is easy to check that $(HG + HK)f(x) = Hf(x)$ as it must be. From $H = H^{(1)} + R^{(1)}$ it also follows that the multiplier function r appearing in (2.3) and in (2.4) must be given by

$$r(\xi) = -\frac{2i}{\pi}\, S(2\pi\xi) \tag{2.6}$$

where $S(x) = \int_0^x \frac{\sin t}{t}\, dt$ as before. The L^p-norm asymmetry of $H^{(1)}$ and $R^{(1)}$ is due to the different form of the two convolution kernels $k(x)$ and $g(x)$. From (2.5) we see that g is not given by an integrable function and it is not positive (Minkowski's inequality is no longer directly useful to obtain sharp constants). From (2.6) we see that $||R^{(1)}||_{2,2}$ which is equal to the L^∞ norm of the corresponding multiplier function $r(\xi)$, is given by

$$\sup_{\xi\in\mathbb{R}} \frac{2}{\pi}\ |S(2\pi\xi)| = \frac{2}{\pi}\, S(\pi) = \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t}\, dt = 1.17897974447...$$

a "Gibbs" constant that is strictly larger than $1 = n_2$. The continuity of operator norms as functions of p implies that, at least for an open interval of exponents containing $p = 2$, we must have $||R^{(\delta)}||_{p,p} > n_p$. Finally, taking (p, p) operator-norms on both sides of the equality $R^{(1)} = H - H^{(1)}$ and applying Theorem 1 we obtain the upper estimate $||R^{(\delta)}||_{p,p} \leqslant 2n_p$. $\square$

3. Related problems and final remarks

The (outer) Truncated Hilbert Transform (1.1) has a discrete analogue D which maps boundedly the space of bilateral sequence $\{b_n\} \in l^p(\mathbb{Z})$ into itself for $1 < p < \infty$. It is defined by

$$(Db)_n = \text{p.v.} \frac{1}{\pi} \sum_{k\neq 0} \frac{b_{n-k}}{k}. \tag{3.1}$$

where k runs over all the non-zero integers in $\mathbb{Z}$ and the bilateral sum is taken in the principal value sense (limit as $N \to \infty$ of the balanced partial sums from $-N$ to N, with cancellations playing a role in convergence). A challenging conjecture that dates back to the 1920s is that the (p,p) norms of the discrete operator D coincide with the norms of the classical Hilbert Transform, i.e., with the expression n_p in (1.3). See Laeng [4] for some history, some partial results, and some information about other kinds of discrete Hilbert Transform. Theorem 1 is conceivably relevant for this problem, because there is a stricter analogy between the operators (1.1) and (3.1) than there is between the classical Hilbert Transform and (3.1). More precisely, it can be proven (theorem 4.5 in [4]) that when $p=2^n$ for $n=1,2,\ldots$ and when $p=2^n/(2^n-1)$ then in fact $||D||_{p,p}=n_p$. It can also be proven (theorem 4.6 in [4]) that the difference between the norms $||D||_{p,p}$ and the norms $||H^{(\delta)}||_{p,p}$ of the Truncated Hilbert Transform cannot exceed, for $1<p<\infty$, a small absolute constant independent of p. Putting together Theorem 1 here and the two results quoted above, we see that the conjecture is true on a discrete infinite subset of exponents and if there is a norm discrepancy for other exponents this discrepancy is much smaller (especially for p close to 1 or ∞) than the discrepancy given by complex Riesz interpolation between any two exponents where the conjecture holds.

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