

A GENERALIZATION OF MULTIPLE HARDY–HILBERT’S INTEGRAL INEQUALITY

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Abstract. In this paper, we give a new generalization of multiple Hardy-Hilbert’s inequality with the best constant factor. As applications, the equivalent form and some particular results are derived.

1. Introduction

Let $1/p + 1/q = 1$ ($p > 1$), $f, g \geq 0$,

$$0 < \int_0^\infty f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^q(x) dx < \infty.$$

The well known Hardy-Hilbert’s integral inequality (see [1]) is given by

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x) dx \right)^{1/p} \left(\int_0^\infty g^q(x) dx \right)^{1/q}; \quad (1.1)$$

and an equivalent form is given by

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \int_0^\infty f^p(x) dx; \quad (1.2)$$

where the constant factors $\pi/\sin(\pi/p)$ and $[\pi/\sin(\pi/p)]^p$ are the best possible. These inequalities play an important role in analysis and operator theory (see [2]). Recently, many generalizations and refinements of these inequalities were also obtained.

At present, because of the requirement of higher-dimensional analysis and operator theory, multiple Hardy-Hilbert’s inequalities have been studied. In 2003, Yang [6] obtain the following multiple extension of (1.1) as:

If $\alpha \in \mathbb{R}$, $n \in \mathbb{N} \setminus \{1\}$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\lambda > n - \min_{1 \leq i \leq n} \{p_i\}$ and $f_i \geq 0$ satisfy

$$0 < \int_\alpha^\infty (x - \alpha)^{n-1-\lambda} f_i^{p_i}(x) dx < \infty \quad (i = 1, 2, \dots, n),$$

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then

$$\int_{\alpha}^{\infty} \dots \int_{\alpha}^{\infty} \frac{1}{\left(\sum_{j=1}^n x_j - n\alpha\right)^{\lambda}} \prod_{i=1}^n f_i(x_i) dx_1 \dots dx_n$$

$$< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{p_i + \lambda - n}{p_i}\right) \left\{ \int_{\alpha}^{\infty} (x - \alpha)^{n-1-\lambda} f_i^{p_i}(x) dx \right\}^{\frac{1}{p_i}},$$
(1.3)

where the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{p_i + \lambda - n}{p_i}\right)$ is the best possible. In particular, for $\alpha = 0$, the following multiple extension of inequality (1.1) holds.

$$\int_0^{\infty} \dots \int_0^{\infty} \frac{1}{\left(\sum_{j=1}^n x_j\right)^{\lambda}} \prod_{i=1}^n f_i(x_i) dx_1 \dots dx_n$$

$$< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{p_i + \lambda - n}{p_i}\right) \left\{ \int_0^{\infty} x^{n-1-\lambda} f_i^{p_i}(x) dx \right\}^{\frac{1}{p_i}},$$
(1.4)

where the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{p_i + \lambda - n}{p_i}\right)$ is the best possible.

In 2005, Yang [7] obtain another multiple extension of (1.1) as follows:

If $n \in \mathbb{N} \setminus \{1\}$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\lambda > 0$ and $f_i \geq 0$ satisfy

$$0 < \int_0^{\infty} x^{p_i-1-\lambda} f_i^{p_i}(x) dx < \infty \quad (i = 1, 2, \dots, n),$$

then

$$\int_0^{\infty} \dots \int_0^{\infty} \frac{1}{\left(\sum_{j=1}^n x_j\right)^{\lambda}} \prod_{i=1}^n f_i(x_i) dx_1 \dots dx_n$$

$$< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right) \left\{ \int_0^{\infty} x^{p_i-1-\lambda} f_i^{p_i}(x) dx \right\}^{\frac{1}{p_i}},$$
(1.5)

where the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$ is the best possible.

In 2005, Yang et al. [8] obtain another multiple extension of (1.1) as follows:

If $n \in \mathbb{N} \setminus \{1\}$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\lambda > 0$, $\sum_{i=1}^n \tilde{A}_i = \lambda - n$ and $f_i \geq 0$ satisfy

$$0 < \int_0^{\infty} x^{-1-p_i \tilde{A}_i} f_i^{p_i}(x) dx < \infty \quad (i = 1, 2, \dots, n),$$

then

$$\int_0^{\infty} \dots \int_0^{\infty} \frac{1}{\left(\sum_{j=1}^n x_j\right)^{\lambda}} \prod_{i=1}^n f_i(x_i) dx_1 \dots dx_n$$

$$< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\tilde{A}_i + 1\right) \left\{ \int_0^{\infty} x^{-1-p_i \tilde{A}_i} f_i^{p_i}(x) dx \right\}^{\frac{1}{p_i}},$$
(1.6)

where the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma(\tilde{A}_i + 1)$ is the best possible.

The main objective of this paper is to build a new generalization of multiple Hardy-Hilbert's inequality, using Gamma function, n -functions $u_i, i = 1, 2, \dots, n$ and $(n + 1)$ -parameters $\lambda, \phi_i, i = 1, 2, \dots, n$, with a best constant factor, which is more generalized inequality and from which all of the above inequalities are obtained by specialising the parameters. As application, we give the equivalent inequality and certain particular results.

The layout of this paper is as follows: In section 2, we prove three lemmas describing the interplays between multiple beta integral and the Gamma function. In section 3, we establish the inequality that is a generalization of multiple Hardy-Hilberts integral inequalities described in [6], [7], [8] and obtain the equivalent form of this inequality. In the last section, we discussed about some of the particular cases of these inequalities involving power function, logarithm function, exponential function, trigonometric function and inverse trigonometric function, as it enhances further the applicability of these integral inequalities.

2. Some lemmas

In this section we shall prove lemmas, which play crucial roles in proving our main results.

We set the following notations: Let $n \in \mathbb{N} \setminus \{1\}, p_i > 1 (i = 1, 2, \dots, n), \sum_{i=1}^n \frac{1}{p_i} = 1, \lambda > 0, \phi_i > 0 (i = 1, 2, \dots, n)$ with $\sum_{i=1}^n \phi_i = \lambda$. Suppose for every $i = 1, 2, \dots, n; u_i : (a_i, b_i) \rightarrow (0, \infty)$, is a strictly increasing differentiable function such that $u_i(a_i) = 0$ and $u_i(b_i) = \infty$.

We need the following formula on the γ -function (cf. Wang et al. [3]),

$$\int_0^\infty \frac{1}{(1+t)^{p+q}} t^{p-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{2.1}$$

where $\Gamma(p)$ is the gamma function.

LEMMA 2.1. *If $k \in \mathbb{N}, r_i > 0 (i = 1, 2, \dots, k + 1)$ and $\sum_{i=1}^{k+1} r_i = \lambda$, then*

$$\int_0^\infty \dots \int_0^\infty \frac{1}{(1 + \sum_{i=1}^k t_i)^\lambda} \prod_{i=1}^k t_i^{r_i-1} dt_1 \dots dt_k = \frac{1}{\Gamma(\lambda)} \prod_{j=1}^{k+1} \Gamma(r_j). \tag{2.2}$$

Proof. Setting $v = \frac{t_1}{1 + \sum_{i=2}^k t_i}$ and using (2.1), we have

$$\begin{aligned} & \int_0^\infty \dots \int_0^\infty \frac{1}{(1 + \sum_{i=1}^k t_i)^\lambda} \prod_{i=1}^k t_i^{r_i-1} dt_1 \dots dt_k \\ &= \int_0^\infty \dots \int_0^\infty \frac{\prod_{i=2}^k t_i^{r_i-1}}{(1 + \sum_{i=2}^k t_i)^{\lambda-r_1}} dt_2 \dots dt_k \int_0^\infty \frac{v^{r_1-1}}{(1+v)^\lambda} dv \\ &= \frac{\Gamma(r_1)\Gamma(\lambda - r_1)}{\Gamma(\lambda)} \int_0^\infty \dots \int_0^\infty \frac{\prod_{i=2}^k t_i^{r_i-1}}{(1 + \sum_{i=2}^k t_i)^{\lambda-r_1}} dt_2 \dots dt_k. \end{aligned}$$

Repeating the above process, we obtain (2.2). This completes the lemma.

LEMMA 2.2. For $j = 1, 2, \dots, n$; define the weight function $\omega_j(x_j)$ as:

$$\omega_j(x_j) = \int_{a_1}^{b_1} \dots \int_{a_{j-1}}^{b_{j-1}} \int_{a_{j+1}}^{b_{j+1}} \dots \int_{a_n}^{b_n} \frac{1}{(\sum_{k=1}^n u_k(x_k))^\lambda} \times \prod_{1 \leq i \neq j \leq n} (u_i(x_i))^{\phi_i - 1} u'_i(x_i) dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n. \tag{2.3}$$

Then

$$\omega_j(x_j) = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma(\phi_i) (u_j(x_j))^{-\phi_j}, \quad (j = 1, 2, \dots, n). \tag{2.4}$$

Proof. Setting

$$t_i = \begin{cases} \frac{u_i(x_i)}{u_j(x_j)}, & \text{for } i = 1, 2, 3, \dots, j-1, \\ \frac{u_{i+1}(x_{i+1})}{u_j(x_j)}, & \text{for } i = j, j+1, \dots, n-1; \end{cases}$$

$$r_i = \begin{cases} \phi_i, & \text{for } i = 1, 2, 3, \dots, j-1, \\ \phi_{i+1}, & \text{for } i = j, j+1, \dots, n-1; \end{cases}$$

and using (2.2), we have

$$\begin{aligned} \omega_j(x_j) &= \int_0^\infty \dots \int_0^\infty \frac{1}{(u_j(x_j))^\lambda (1 + \sum_{k=1}^{n-1} t_k)^\lambda} \\ &\quad \times \prod_{i=1}^{n-1} t_i^{r_i - 1} (u_j(x_j))^{\lambda - \phi_j - n + 1} (u_j(x_j))^{n-1} dt_1 \dots dt_{n-1} \\ &= (u_j(x_j))^{-\phi_j} \int_0^\infty \dots \int_0^\infty \frac{1}{(1 + \sum_{k=1}^{n-1} t_k)^\lambda} \prod_{i=1}^{n-1} t_i^{r_i - 1} dt_1 \dots dt_{n-1} \\ &= \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma(\phi_i) (u_j(x_j))^{-\phi_j}. \end{aligned}$$

This completes the lemma.

LEMMA 2.3. Take $\tilde{a}_j = u^{-1}(1)$ ($j = 1, 2, \dots, n$). For sufficiently small $\varepsilon > 0$,

$$\begin{aligned} I &:= \int_{\tilde{a}_1}^{b_1} \dots \int_{\tilde{a}_n}^{b_n} \frac{1}{(\sum_{i=1}^n u_i(x_i))^\lambda} \prod_{j=1}^n (u_j(x_j))^{\phi_j - \frac{\varepsilon}{p_j} - 1} u'_j(x_j) dx_1 \dots dx_n \\ &\geq \frac{1}{\varepsilon} \left\{ \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) + o(1) \right\} - \sum_{j=1}^{n-1} \left(\phi_j - \frac{\varepsilon}{p_j} \right)^{-2} O_j(1). \end{aligned} \tag{2.5}$$

Proof. Setting $v_j = u_j(x_j)$ and $t_i = \frac{v_i}{v_n}, i = 1, 2, \dots, n - 1$, we have

$$\begin{aligned}
 I &:= \int_1^\infty \dots \int_1^\infty \frac{1}{(\sum_{i=1}^n v_i)^\lambda} \prod_{j=1}^n v_j^{\phi_j - \frac{\varepsilon}{p_j} - 1} dv_1 \dots dv_n \\
 &= \int_1^\infty \frac{1}{v_n^{\varepsilon+1}} \left[\int_{\frac{1}{v_n}}^\infty \dots \int_{\frac{1}{v_n}}^\infty \frac{1}{(1 + \sum_{i=1}^{n-1} t_i)^\lambda} \prod_{j=1}^{n-1} t_j^{\phi_j - \frac{\varepsilon}{p_j} - 1} dt_1 \dots dt_{n-1} \right] dv_n \\
 &\geq \int_1^\infty \frac{1}{v_n^{\varepsilon+1}} dv_n \int_0^\infty \dots \int_0^\infty \frac{1}{(1 + \sum_{i=1}^{n-1} t_i)^\lambda} \prod_{j=1}^{n-1} t_j^{\phi_j - \frac{\varepsilon}{p_j} - 1} dt_1 \dots dt_{n-1} \\
 &\quad - \int_1^\infty \frac{1}{v_n} \sum_{j=1}^{n-1} A_j(v_n) dv_n \\
 &= I_1 - I_2 \text{ (say),}
 \end{aligned} \tag{2.6}$$

where, for $j = 1, 2, \dots, n - 1$, $A_j(v_n)$ is defined by

$$A_j(v_n) = \int \dots \int_{D_j} \frac{1}{(1 + \sum_{i=1}^{n-1} t_i)^\lambda} \prod_{j=1}^{n-1} t_j^{\phi_j - \frac{\varepsilon}{p_j} - 1} dt_1 \dots dt_{n-1},$$

where $D_j = \{(t_1, \dots, t_{n-1}) \mid 0 < t_j \leq \frac{1}{v_n}, 0 < t_k < \infty (k \neq j)\}$.

By (2.2), as $\varepsilon \rightarrow 0^+$, we have

$$\begin{aligned}
 &\int_0^\infty \dots \int_0^\infty \frac{1}{(1 + \sum_{i=1}^{n-1} t_i)^\lambda} \prod_{j=1}^{n-1} t_j^{\phi_j - \frac{\varepsilon}{p_j} - 1} dt_1 \dots dt_{n-1} \\
 &\rightarrow \int_0^\infty \dots \int_0^\infty \frac{1}{(1 + \sum_{i=1}^{n-1} t_i)^\lambda} \prod_{j=1}^{n-1} t_j^{\phi_j - 1} dt_1 \dots dt_{n-1} = \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j).
 \end{aligned}$$

Hence

$$I_1 = \frac{1}{\varepsilon} \left\{ \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) + o(1) \right\}$$

Now we estimate the integrals $A_j(v_n), j = 1, 2, \dots, n - 1$.

(a) For $n = 2$, there is only one $A_j(v_n)$, that is $A_1(v_n)$. We have

$$\begin{aligned}
 A_1(v_n) &= \int_0^{\frac{1}{v_n}} \frac{1}{(1 + t_1)^\lambda} t_1^{\phi_1 - \frac{\varepsilon}{p_1} - 1} dt_1 \\
 &\leq \int_0^{\frac{1}{v_n}} t_1^{\phi_1 - \frac{\varepsilon}{p_1} - 1} dt_1 \\
 &= \left(\phi_1 - \frac{\varepsilon}{p_1} \right)^{-1} v_n^{\frac{\varepsilon}{p_1} - \phi_1}.
 \end{aligned}$$

(b) For $n \in \mathbb{N} \setminus \{1, 2\}$, without loss of generality, we estimate $A_1(v_n)$. By (2.2), we have

$$\begin{aligned} A_1(v_n) &\leq \int_0^\infty \dots \int_0^\infty \frac{\prod_{j=2}^{n-1} t_j^{\phi_j - \frac{\varepsilon}{p_j} - 1}}{(1 + \sum_{i=2}^{n-1} t_i)^\lambda} dt_2 \dots dt_{n-1} \int_0^{\frac{1}{v_n}} t_1^{\phi_1 - \frac{\varepsilon}{p_1} - 1} dt_1 \\ &\leq \frac{v_n^{\frac{\varepsilon}{p_1} - \phi_1}}{\phi_1 - \frac{\varepsilon}{p_1}} \int_0^\infty \dots \int_0^\infty \frac{\prod_{j=2}^{n-1} t_j^{\phi_j - \frac{\varepsilon}{p_j} - 1}}{(1 + \sum_{i=2}^{n-1} t_i)^{\lambda - \phi_1 + \varepsilon \left(1 - \frac{1}{p_1}\right)}} dt_2 \dots dt_{n-1} \\ &= \frac{\prod_{j=2}^n \Gamma\left(\phi_j - \frac{\varepsilon}{p_j}\right)}{\Gamma\left(\lambda - \phi_1 + \varepsilon \left(1 - \frac{1}{p_1}\right)\right)} \frac{v_n^{\frac{\varepsilon}{p_1} - \phi_1}}{\phi_1 - \frac{\varepsilon}{p_1}}. \end{aligned}$$

By virtue of the results of (a) and (b), for $j = 1, 2, \dots, n - 1$, we have

$$A_j(v_n) \leq \left(\phi_j - \frac{\varepsilon}{p_j}\right)^{-1} v_n^{\frac{\varepsilon}{p_j} - \phi_j} O_j(1).$$

Hence

$$\begin{aligned} I_2 &\leq \sum_{j=1}^{n-1} \left(\phi_j - \frac{\varepsilon}{p_j}\right)^{-1} O_j(1) \int_1^\infty v_n^{\frac{\varepsilon}{p_j} - \phi_j - 1} dv_n \\ &= \sum_{j=1}^{n-1} \left(\phi_j - \frac{\varepsilon}{p_j}\right)^{-2} O_j(1). \end{aligned}$$

Now from (2.6), we get (2.5). The lemma is proved.

3. Main results

In this section, we shall establish the integral inequality that generalizes the results proved in [6], [7], [8] and the constant factor obtained is the best possible. The equivalent form of the integral inequality is also obtained.

THEOREM 3.1. *If $f_j \geq 0$ ($j = 1, 2, \dots, n$), satisfy*

$$0 < \int_{a_j}^{b_j} (u_j(x))^{p_j(1-\phi_j)-1} (u'_j(x))^{1-p_j} f_j^{p_j}(x) dx < \infty,$$

then

$$\begin{aligned} &\int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \frac{1}{(\sum_{i=1}^n u_i(x_i))^\lambda} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n \\ &< \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \prod_{j=1}^n \left\{ \int_{a_j}^{b_j} (u_j(x_j))^{p_j(1-\phi_j)-1} (u'_j(x_j))^{1-p_j} f_j^{p_j}(x_j) dx_j \right\}^{\frac{1}{p_j}} \end{aligned} \tag{3.1}$$

where the constant factors $\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j)$ is the best possible.

Proof. By Hölder's inequality and (2.3), we have

$$\begin{aligned}
 I &:= \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \frac{1}{(\sum_{i=1}^n u_i(x_i))^\lambda} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n \\
 &= \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \prod_{j=1}^n \left\{ \frac{f_j(x_j)}{(\sum_{i=1}^n u_i(x_i))^{\frac{\lambda}{p_j}}} \frac{(u_j(x_j))^{(\phi_j-1)(\frac{1}{p_j}-1)}}{(u'_j(x_j))^{1-\frac{1}{p_j}}} \right. \\
 &\quad \left. \times \prod_{1 \leq i \neq j \leq n} (u_i(x_i))^{\frac{(\phi_i-1)}{p_j}} (u'_i(x_i))^{\frac{1}{p_j}} \right\} dx_1 \dots dx_n \\
 &\leq \prod_{j=1}^n \left\{ \int_{a_j}^{b_j} \omega_j(x_j) (u_j(x_j))^{(\phi_j-1)(1-p_j)} (u'_j(x_j))^{1-p_j} f_j^{p_j}(x_j) dx_j \right\}^{\frac{1}{p_j}}
 \end{aligned} \tag{3.2}$$

If there is an equality in (3.2), then there exists constants C_1, C_2, \dots, C_n not all zero, such that for any $i \neq k \in \{1, 2, \dots, n\}$ (see [4]),

$$\begin{aligned}
 C_i &\frac{f_i^{p_i}(x_i)}{(\sum_{j=1}^n u_j(x_j))^\lambda} \frac{(u_i(x_i))^{(\phi_i-1)(1-p_i)}}{(u'_i(x_i))^{p_i-1}} \prod_{1 \leq j \neq i \leq n} (u_j(x_j))^{\phi_j-1} u'_j(x_j) \\
 &= C_k \frac{f_k^{p_k}(x_k)}{(\sum_{j=1}^n u_j(x_j))^\lambda} \frac{(u_k(x_k))^{(\phi_k-1)(1-p_k)}}{(u'_k(x_k))^{p_k-1}} \prod_{1 \leq j \neq k \leq n} (u_j(x_j))^{\phi_j-1} u'_j(x_j) \\
 &\text{a.e. in } (a_1, b_1) \times \dots \times (a_n, b_n).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 &C_1 f_i^{p_i}(x_i) (u_i(x_i))^{p_i(1-\phi_i)} (u'_i(x_i))^{-p_i} \\
 &= C_k f_k^{p_k}(x_k) (u_k(x_k))^{p_k(1-\phi_k)} (u'_k(x_k))^{-p_k} \\
 &= C \text{ a.e. in } (a_1, b_1) \times \dots \times (a_n, b_n),
 \end{aligned}$$

where C is a constant. Without loss of generality, suppose that $C_i \neq 0$. Then we have

$$\begin{aligned}
 &\int_{a_i}^{b_i} (u_i(x))^{p_i(1-\phi_i)-1} (u'_i(x))^{1-p_i} f_i^{p_i}(x) dx \\
 &= \frac{C}{C_i} \int_{a_i}^{b_i} \frac{u'_i(x)}{u_i(x)} dx = \frac{C}{C_i} \int_0^\infty \frac{1}{t} dt = \infty
 \end{aligned}$$

which contradicts to

$$0 < \int_{a_i}^{b_i} (u_i(x))^{p_i(1-\phi_i)-1} (u'_i(x))^{1-p_i} f_i^{p_i}(x) dx < \infty.$$

Hence by (2.4) and (3.2), we get (3.1).

For sufficiently small $\varepsilon > 0$, setting

$$\tilde{f}_j(x) = \begin{cases} 0 & \text{if } x \in (a_j, \tilde{a}_j) \ (\tilde{a}_j = u_j^{-1}(1)), \\ (u_j(x))^{\phi_j - \frac{\varepsilon}{p_j} - 1} u'_j(x) & \text{if } x \in [\tilde{a}_j, b_j]. \end{cases}$$

we have

$$\prod_{j=1}^n \left\{ \int_{a_j}^{b_j} (u_j(x_j))^{p_j(1-\phi_j)-1} (u'_j(x_j))^{1-p_j} \tilde{f}_j^{p_j}(x_j) dx_j \right\}^{\frac{1}{p_j}} = \frac{1}{\varepsilon} \tag{3.3}$$

If the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j)$ in (3.1) is not the best possible, then there exists a positive constant $K < \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j)$, such that (3.1) is still valid if we replace $\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j)$ by K . In particular, by (3.3) and (2.5), we have

$$\begin{aligned} & \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) + o(1) - \varepsilon \sum_{j=1}^{n-1} \left(\phi_j - \frac{\varepsilon}{p_j} \right)^{-2} O_j(1) \\ & < \varepsilon \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \frac{1}{(\sum_{i=1}^n u_i(x_i))^\lambda} \prod_{j=1}^n \tilde{f}_j(x_j) dx_1 \dots dx_n \\ & < \varepsilon K \prod_{j=1}^n \left\{ \int_{a_j}^{b_j} (u_j(x_j))^{p_j(1-\phi_j)-1} (u'_j(x_j))^{1-p_j} \tilde{f}_j^{p_j}(x_j) dx_j \right\}^{\frac{1}{p_j}} \\ & = K, \end{aligned}$$

and then $\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \leq K (\varepsilon \rightarrow 0^+)$. This contradiction leads to the conclusion that the constant factor in (3.1) is the best possible. The theorem is proved.

We shall now obtain an equivalent form of the integral inequality (3.1).

THEOREM 3.2. *If $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$ and $f_j \geq 0$ ($j = 1, 2, \dots, n - 1$), satisfy*

$$0 < \int_{a_j}^{b_j} (u_j(x))^{p_j(1-\phi_j)-1} (u'_j(x))^{1-p_j} f_j^{p_j}(x) dx < \infty,$$

then we obtain an equivalent inequality of (3.1) as:

$$\begin{aligned} & \int_{a_n}^{b_n} \frac{u'_n(x_n)}{(u_n(x_n))^{1-q\phi_n}} \left[\int_{a_1}^{b_1} \dots \int_{a_{n-1}}^{b_{n-1}} \frac{1}{(\sum_{i=1}^n u_i(x_i))^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^q dx_n \\ & < \left[\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \right]^{q-1} \prod_{j=1}^{n-1} \left\{ \int_{a_j}^{b_j} (u_j(x_j))^{p_j(1-\phi_j)-1} (u'_j(x_j))^{1-p_j} f_j^{p_j}(x_j) dx_j \right\}^{\frac{q}{p_j}}, \end{aligned} \tag{3.4}$$

where the constant factors $\left[\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \right]^q$ is the best possible.

Proof. Set

$$f_n(x) = \frac{u'_n(x_n)}{(u_n(x_n))^{1-q\phi_n}} \left[\int_{a_1}^{b_1} \dots \int_{a_{n-1}}^{b_{n-1}} \frac{1}{(\sum_{i=1}^n u_i(x_i))^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^{q-1}.$$

By (3.1), we have

$$\begin{aligned}
 0 &< \int_{a_n}^{b_n} (u_n(x_n))^{p_n(1-\phi_n)-1} (u'_n(x_n))^{1-p_n} f_n^{p_n}(x_n) dx_n \\
 &= \int_{a_n}^{b_n} \frac{u'_n(x_n)}{(u_n(x_n))^{1-q\phi_n}} \\
 &\quad \times \left[\int_{a_1}^{b_1} \dots \int_{a_{n-1}}^{b_{n-1}} \frac{1}{(\sum_{i=1}^n u_i(x_i))^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^q dx_n \\
 &= \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \frac{1}{(\sum_{i=1}^n u_i(x_i))^\lambda} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n \\
 &\leq \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \prod_{j=1}^n \left\{ \int_{a_j}^{b_j} (u_j(x_j))^{p_j(1-\phi_j)-1} (u'_j(x_j))^{1-p_j} f_j^{p_j}(x_j) dx_j \right\}^{\frac{1}{p_j}}
 \end{aligned} \tag{3.5}$$

then

$$\begin{aligned}
 0 &< \left\{ \int_{a_n}^{b_n} (u_n(x_n))^{p_n(1-\phi_n)-1} (u'_n(x_n))^{1-p_n} f_n^{p_n}(x_n) dx_n \right\}^{\frac{1}{q}} \\
 &= \left\{ \int_{a_n}^{b_n} \frac{u'_n(x_n)}{(u_n(x_n))^{1-q\phi_n}} \left[\int_{a_1}^{b_1} \dots \int_{a_{n-1}}^{b_{n-1}} \frac{1}{(\sum_{i=1}^n u_i(x_i))^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^q dx_n \right\}^{\frac{1}{q}} \\
 &\leq \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \prod_{j=1}^n \left\{ \int_{a_j}^{b_j} (u_j(x_j))^{p_j(1-\phi_j)-1} (u'_j(x_j))^{1-p_j} f_j^{p_j}(x_j) dx_j \right\}^{\frac{1}{p_j}} \\
 &< \infty.
 \end{aligned} \tag{3.6}$$

It follows that (3.5) takes the form of strict inequality by using (3.1); so, does (3.6). Hence we can get (3.4).

On the other hand, if (3.4) holds, then by Hölder's inequality, we have

$$\begin{aligned}
 &\int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \frac{1}{(\sum_{i=1}^n u_i(x_i))^\lambda} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n \\
 &= \int_{a_n}^{b_n} \left\{ \frac{(u'_n(x_n))^{\frac{1}{q}}}{(u_n(x_n))^{\frac{1}{q}-\phi_n}} \int_{a_1}^{b_1} \dots \int_{a_{n-1}}^{b_{n-1}} \frac{1}{(\sum_{i=1}^n u_i(x_i))^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right\} \\
 &\quad \times \left\{ \frac{(u_n(x_n))^{\frac{1}{q}-\phi_n}}{(u'_n(x_n))^{\frac{1}{q}}} f_n(x_n) \right\} dx_n \\
 &\leq \left\{ \int_{a_n}^{b_n} \frac{u'_n(x_n)}{(u_n(x_n))^{1-q\phi_n}} \left[\int_{a_1}^{b_1} \dots \int_{a_{n-1}}^{b_{n-1}} \frac{1}{(\sum_{i=1}^n u_i(x_i))^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^q dx_n \right\}^{\frac{1}{q}} \\
 &\quad \times \left\{ \int_{a_n}^{b_n} (u_n(x_n))^{p_n(1-\phi_n)-1} (u'_n(x_n))^{1-p_n} f_n^{p_n}(x_n) dx_n \right\}^{\frac{1}{p_n}}
 \end{aligned}$$

Hence by (3.4), (3.1) yields. Thus it follows that (3.1) and (3.4) are equivalent. Since the constant in (3.1) is the best possible, hence the constant in (3.4) is the best possible. The theorem is proved.

4. Some particular results

In this section, we shall derive the inequalities, for which $u_j(x)$'s are power function, logarithm function, exponential function, trigonometric function and inverse trigonometric function. Also we consider the different parameters ϕ_j .

Power function

Taking $u_j(x) = (x - a_j)^{\alpha_j}$ in Theorem 3.1 and Theorem 3.2, we obtain the following result.

THEOREM 4.1. *Let $a_j \in \mathbb{R}$, $\alpha_j > 0$ ($j = 1, 2, \dots, n$). If $f_i \geq 0$, satisfy*

$$0 < \int_{a_j}^{\infty} (x - a_j)^{p_j(1-\alpha_j\phi_j)-1} f_j^{p_j}(x) dx < \infty,$$

then we have the following two equivalent inequalities:

$$\begin{aligned} & \int_{a_1}^{\infty} \dots \int_{a_n}^{\infty} \frac{1}{(\sum_{i=1}^n (x_i - a_i)^{\alpha_i})^\lambda} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n \\ & < \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \left(\Gamma(\phi_j) \alpha_j^{\frac{1}{p_j}-1} \right) \prod_{j=1}^n \left\{ \int_{a_j}^{\infty} (x_j - a_j)^{p_j(1-\alpha_j\phi_j)-1} f_j^{p_j}(x_j) dx_j \right\}^{\frac{1}{p_j}} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} & \int_{a_n}^{\infty} (x_n - a_n)^{q\alpha_n\phi_n-1} \\ & \times \left[\int_{a_1}^{\infty} \dots \int_{a_{n-1}}^{\infty} \frac{1}{(\sum_{i=1}^n (x_i - a_i)^{\alpha_i})^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^q dx_n \\ & < \left[\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \left(\Gamma(\phi_j) \alpha_j^{\frac{1}{p_j}-1} \right) \right]^{q-1} \prod_{j=1}^{n-1} \left\{ \int_{a_j}^{\infty} (x_j - a_j)^{p_j(1-\alpha_j\phi_j)-1} f_j^{p_j}(x_j) dx_j \right\}^{\frac{q}{p_j}}, \end{aligned} \tag{4.2}$$

where the constant factors $\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \left(\Gamma(\phi_j) \alpha_j^{\frac{1}{p_j}-1} \right)$ and $\left[\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \left(\Gamma(\phi_j) \alpha_j^{\frac{1}{p_j}-1} \right) \right]^q$ are the best possible.

EXAMPLE 4.1.1. Setting $\phi_j = \frac{\lambda}{p_j}$ in Theorem 4.1, we have the following inequality: Let $a_j \in \mathbb{R}$, $\lambda > 0$, $\alpha_j > 0$ ($j = 1, 2, \dots, n$). If $f_i \geq 0$, satisfy

$$0 < \int_{a_j}^{\infty} (x - a_j)^{p_j-\lambda\alpha_j-1} f_j^{p_j}(x) dx < \infty,$$

then we have the following two equivalent inequalities:

$$\int_{a_1}^{\infty} \dots \int_{a_n}^{\infty} \frac{1}{(\sum_{i=1}^n (x_i - a_i)\alpha_i)^\lambda} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n$$

$$< \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \left(\Gamma\left(\frac{\lambda}{p_j}\right) \alpha_j^{\frac{1}{p_j}-1} \right) \prod_{j=1}^n \left\{ \int_{a_j}^{\infty} (x_j - a_j)^{p_j-\lambda\alpha_j-1} f_j^{p_j}(x_j) dx_j \right\}^{\frac{1}{p_j}} \tag{4.3}$$

and

$$\int_{a_n}^{\infty} (x_n - a_n)^{(q-1)\lambda\alpha_n-1}$$

$$\times \left[\int_{a_1}^{\infty} \dots \int_{a_{n-1}}^{\infty} \frac{1}{(\sum_{i=1}^n (x_i - a_i)\alpha_i)^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^q dx_n$$

$$< \left[\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \left(\Gamma\left(\frac{\lambda}{p_j}\right) \alpha_j^{\frac{1}{p_j}-1} \right) \right]^{q-1} \prod_{j=1}^{n-1} \left\{ \int_{a_j}^{\infty} (x_j - a_j)^{p_j-\lambda\alpha_j-1} f_j^{p_j}(x_j) dx_j \right\}^{\frac{q}{p_j}}, \tag{4.4}$$

where the constant factors $\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma\left(\frac{\lambda}{p_j}\right)$ and $\left[\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \left(\Gamma\left(\frac{\lambda}{p_j}\right) \alpha_j^{\frac{1}{p_j}-1} \right) \right]^q$ are the best possible.

REMARK 1. Taking $a_j = 0, \alpha_j = 1 (j = 1, 2, \dots, n)$, in (4.3), we get (1.5) and in (4.4), we get an equivalent inequality of (1.5).

EXAMPLE 4.1.2. Setting $\phi_j = 1 + \frac{\lambda-n}{p_j}$ in Theorem 4.1, we have the following inequality: Let $a_j \in \mathbb{R}, \lambda > n - \min_{1 \leq i \leq n} \{p_i\}, \alpha_j > 0 (j = 1, 2, \dots, n)$. If $f_i \geq 0$, satisfy

$$0 < \int_{a_j}^{\infty} (x - a_j)^{p_j-\alpha_j(p_j+\lambda-n)-1} f_j^{p_j}(x) dx < \infty,$$

then we have the following two equivalent inequalities:

$$\int_{a_1}^{\infty} \dots \int_{a_n}^{\infty} \frac{1}{(\sum_{i=1}^n (x_i - a_i)\alpha_i)^\lambda} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n$$

$$< k_\lambda(\alpha, p) \prod_{j=1}^n \left\{ \int_{a_j}^{\infty} (x_j - a_j)^{p_j-\alpha_j(p_j+\lambda-n)-1} f_j^{p_j}(x_j) dx_j \right\}^{\frac{1}{p_j}} \tag{4.5}$$

and

$$\int_{a_n}^{\infty} (x_n - a_n)^{\alpha_n(q-1)(p_n+\lambda-n)-1}$$

$$\times \left[\int_{a_1}^{\infty} \dots \int_{a_{n-1}}^{\infty} \frac{1}{(\sum_{i=1}^n (x_i - a_i)\alpha_i)^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^q dx_n$$

$$< [k_\lambda(\alpha, p)]^q \prod_{j=1}^{n-1} \left\{ \int_{a_j}^{\infty} (x_j - a_j)^{p_j-\alpha_j(p_j+\lambda-n)-1} f_j^{p_j}(x_j) dx_j \right\}^{\frac{q}{p_j}}, \tag{4.6}$$

where $k_\lambda(\alpha, p) = \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \left(\Gamma\left(\frac{p_j + \lambda - n}{p_j}\right) \alpha_j^{\frac{1}{p_j} - 1} \right)$ and the constant factors $k_\lambda(\alpha, p)$ and $[k_\lambda(\alpha, p)]^q$ are the best possible.

REMARK 2. Taking $a_j = \alpha, \alpha_j = 1 (j = 1, 2, \dots, n)$, in (4.5), we get (1.3) and in (4.6), we get an equivalent inequality of (1.3).

EXAMPLE 4.1.3. Setting $\phi_j = A_j + 1$ in Theorem 4.1, we have the following inequality: Let $a_j \in \mathbb{R}, A_j > -1, \sum_{j=1}^n A_j = \lambda - n, \alpha_j > 0 (j = 1, 2, \dots, n)$. If $f_i \geq 0$, satisfy

$$0 < \int_{a_j}^{\infty} (x - a_j)^{p_j(1 - \alpha_j(A_j + 1)) - 1} f_j^{p_j}(x) dx < \infty,$$

then we have the following two equivalent inequalities:

$$\begin{aligned} & \int_{a_1}^{\infty} \dots \int_{a_n}^{\infty} \frac{1}{(\sum_{i=1}^n (x_i - a_i)^{\alpha_i})^\lambda} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n \\ & < [\tilde{k}_\lambda(\alpha, A) \prod_{j=1}^n \left\{ \int_{a_j}^{\infty} (x_j - a_j)^{p_j(1 - \alpha_j(A_j + 1)) - 1} f_j^{p_j}(x_j) dx_j \right\}^{\frac{1}{p_j}}] \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} & \int_{a_n}^{\infty} (x_n - a_n)^{q\alpha_n(A_n + 1) - 1} \\ & \times \left[\int_{a_1}^{\infty} \dots \int_{a_{n-1}}^{\infty} \frac{1}{(\sum_{i=1}^n (x_i - a_i)^{\alpha_i})^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^q dx_n \\ & < [\tilde{k}_\lambda(\alpha, A)]^q \prod_{j=1}^{n-1} \left\{ \int_{a_j}^{\infty} (x_j - a_j)^{p_j(1 - \alpha_j(A_j + 1)) - 1} f_j^{p_j}(x_j) dx_j \right\}^{\frac{q}{p_j}}, \end{aligned} \tag{4.8}$$

where $\tilde{k}_\lambda(\alpha, A) = \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \left(\Gamma(A_j + 1) \alpha_j^{\frac{1}{p_j} - 1} \right)$ and the constant factors $\tilde{k}_\lambda(\alpha, A)$ and $[\tilde{k}_\lambda(\alpha, A)]^q$ are the best possible.

REMARK 3. Taking $a_j = 0, \alpha_j = 1 (j = 1, 2, \dots, n)$, in (4.7), we get (1.6) and in (4.8), we get an equivalent inequality of (1.6).

REMARK 4. Further new inequalities are derived from Theorem-4.1, by taking parameters (a) $\phi_j = \frac{\lambda + n}{p_j} - 1$; (b) $\phi_j = \frac{\lambda - 1}{n} + \frac{1}{p_j}$; (c) $\phi_j = \frac{\lambda + 1}{n} - \frac{1}{p_j}$; (d) $\phi_j = \frac{\lambda}{n} - \frac{n}{p_j} + 1$; (e) $\phi_j = \frac{\lambda}{n} + \frac{n}{p_j} - 1$ and others.

Logarithm function

Taking $u_j(x) = \ln x$ in Theorem 3.1 and Theorem 3.2, we obtain the following result.

THEOREM 4.2. If $f_i \geq 0$ ($j = 1, 2, \dots, n$), satisfy

$$0 < \int_1^\infty (\ln x)^{p_j(1-\phi_j)-1} x^{p_j-1} f_j^{p_j}(x) dx < \infty,$$

then we have the following two equivalent inequalities:

$$\begin{aligned} & \int_1^\infty \dots \int_1^\infty \frac{1}{(\ln \prod_{i=1}^n x_i)^\lambda} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n \\ & < \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \prod_{j=1}^n \left\{ \int_1^\infty (\ln x)^{p_j(1-\phi_j)-1} x^{p_j-1} f_j^{p_j}(x) dx \right\}^{\frac{1}{p_j}} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} & \int_1^\infty \frac{(\ln x_n)^{q\phi_n-1}}{x_n} \left[\int_1^\infty \dots \int_1^\infty \frac{1}{(\ln \prod_{i=1}^n x_i)^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^q dx_n \\ & < \left[\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \right]^q \prod_{j=1}^{n-1} \left\{ \int_1^\infty (\ln x)^{p_j(1-\phi_j)-1} x^{p_j-1} f_j^{p_j}(x) dx \right\}^{\frac{q}{p_j}}, \end{aligned} \tag{4.10}$$

where the constant factors $\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j)$ and $\left[\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \right]^q$ are the best possible.

EXAMPLE 4.2.1. Setting $\lambda = n - 1$, $\phi_j = 1 - \frac{1}{p_j}$ and replacing $f_j(x)$ by $\frac{f_j(x)}{x}$ in Theorem 4.2, we have the following inequality: If $f_i \geq 0$ ($j = 1, 2, \dots, n$), satisfy

$$0 < \int_1^\infty \frac{f_j^{p_j}(x)}{x} dx < \infty,$$

then we have the following two equivalent inequalities:

$$\begin{aligned} & \int_1^\infty \dots \int_1^\infty \frac{1}{(\prod_{i=1}^n x_i) (\ln \prod_{i=1}^n x_i)^{n-1}} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n \\ & < \frac{1}{\Gamma(n-1)} \prod_{j=1}^n \Gamma(1 - \frac{1}{p_j}) \prod_{j=1}^n \left\{ \int_1^\infty \frac{f_j^{p_j}(x)}{x} dx \right\}^{\frac{1}{p_j}} \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} & \int_1^\infty \left[\int_1^\infty \dots \int_1^\infty \frac{1}{(\prod_{i=1}^n x_i) (\ln \prod_{i=1}^n x_i)^{n-1}} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^q dx_n \\ & < \left[\frac{1}{\Gamma(n-1)} \prod_{j=1}^n \Gamma(1 - \frac{1}{p_j}) \right]^q \prod_{j=1}^{n-1} \left\{ \int_1^\infty \frac{f_j^{p_j}(x)}{x} dx \right\}^{\frac{q}{p_j}}, \end{aligned} \tag{4.12}$$

where the constant factors $\frac{1}{\Gamma(n-1)} \prod_{j=1}^n \Gamma(1 - \frac{1}{p_j})$ and $\left[\frac{1}{\Gamma(n-1)} \prod_{j=1}^n \Gamma(1 - \frac{1}{p_j}) \right]^q$ are the best possible.

REMARK 5. For $n = 2$, (4.11) reduces to the following well known Mulholland’s integral inequality (see [1]).

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0$ satisfy $0 < \int_1^\infty \frac{f^p(x)}{x} dx < \infty$ and $0 < \int_1^\infty \frac{g^q(x)}{x} dx < \infty$, then

$$\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{xy \ln xy} dx dy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_1^\infty \frac{f^p(x)}{x} dx \right\}^{\frac{1}{p}} \left\{ \int_1^\infty \frac{g^q(x)}{x} dx \right\}^{\frac{1}{q}}, \tag{4.13}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

Thus (4.11) is a multiple extension of Mulholland’s integral inequality (4.13).

Exponential function

Taking $u_j(x) = a^x, a > 1$ in Theorem 3.1 and Theorem 3.2, we obtain the following result.

THEOREM 4.3. If $a > 1, f_i \geq 0 (j = 1, 2, \dots, n)$, satisfy

$$0 < \int_{-\infty}^\infty a^{x(p_j(2-\phi_j)-2)} f_j^{p_j}(x) dx < \infty,$$

then we have the following two equivalent inequalities:

$$\begin{aligned} & \int_{-\infty}^\infty \dots \int_{-\infty}^\infty \frac{1}{(\sum_{i=1}^n a^{x_i})^\lambda} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n \\ & < \frac{(\ln a)^{n-1}}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \prod_{j=1}^n \left\{ \int_{-\infty}^\infty a^{x(p_j(2-\phi_j)-2)} f_j^{p_j}(x) dx \right\}^{\frac{1}{p_j}} \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} & \int_{-\infty}^\infty a^{q\phi_n x_n} \left[\int_{-\infty}^\infty \dots \int_{-\infty}^\infty \frac{1}{(\sum_{i=1}^n a^{x_i})^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^q dx_n \\ & < \left[\frac{(\ln a)^{n-1}}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \right]^q \prod_{j=1}^{n-1} \left\{ \int_{-\infty}^\infty a^{x(p_j(2-\phi_j)-2)} f_j^{p_j}(x) dx \right\}^{\frac{q}{p_j}}, \end{aligned} \tag{4.15}$$

where the constant factors $\frac{(\ln a)^{n-1}}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j)$ and $\left[\frac{(\ln a)^{n-1}}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \right]^q$ are the best possible.

Trigonometric function

Taking $u_j(x) = \tan x$ in Theorem 3.1 and Theorem 3.2, we obtain the following result.

THEOREM 4.4. *If $f_i \geq 0$ ($j = 1, 2, \dots, n$), satisfy*

$$0 < \int_0^{\pi/2} (\tan x)^{p_j(1-\phi_j)-1} (\sec x)^{2(1-p_j)} f_j^{p_j}(x) dx < \infty,$$

then we have the following two equivalent inequalities:

$$\begin{aligned} & \int_0^{\pi/2} \dots \int_0^{\pi/2} \frac{1}{(\sum_{i=1}^n \tan x_i)^\lambda} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n \\ & < \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \prod_{j=1}^n \left\{ \int_0^{\pi/2} (\tan x)^{p_j(1-\phi_j)-1} (\sec x)^{2(1-p_j)} f_j^{p_j}(x) dx \right\}^{\frac{1}{p_j}} \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} & \int_0^{\pi/2} \frac{\sec^2 x_n}{(\tan x_n)^{1-q\phi_n}} \\ & \times \left[\int_0^{\pi/2} \dots \int_0^{\pi/2} \frac{1}{(\sum_{i=1}^n \tan x_i)^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^q dx_n \\ & < \left[\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \right]^q \prod_{j=1}^{n-1} \left\{ \int_0^{\pi/2} (\tan x)^{p_j(1-\phi_j)-1} (\sec x)^{2(1-p_j)} f_j^{p_j}(x) dx \right\}^{\frac{q}{p_j}}, \end{aligned} \tag{4.17}$$

where the constant factors $\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j)$ and $\left[\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \right]^q$ are the best possible.

Inverse trigonometric function

Taking $u_j(x) = \arctan x$ in Theorem 3.1 and Theorem 3.2, we obtain the following result.

THEOREM 4.5. *If $f_i \geq 0$ ($j = 1, 2, \dots, n$), satisfy*

$$0 < \int_0^\infty \frac{(\arctan x)^{p_j(1-\phi_j)-1}}{(1+x^2)^{1-p_j}} f_j^{p_j}(x) dx < \infty,$$

then we have the following two equivalent inequalities:

$$\begin{aligned} & \int_0^\infty \dots \int_0^\infty \frac{1}{(\sum_{i=1}^n \arctan x_i)^\lambda} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n \\ & < \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \prod_{j=1}^n \left\{ \int_0^\infty \frac{(\arctan x)^{p_j(1-\phi_j)-1}}{(1+x^2)^{1-p_j}} f_j^{p_j}(x) dx \right\}^{\frac{1}{p_j}} \end{aligned} \tag{4.18}$$

and

$$\int_0^\infty \frac{(\arctan x_n)^{q\phi_n-1}}{1+x_n^2} \times \left[\int_0^\infty \dots \int_0^\infty \frac{1}{(\sum_{i=1}^n \arctan x_i)^\lambda} \prod_{j=1}^{n-1} f_j(x_j) dx_1 \dots dx_{n-1} \right]^q dx_n \quad (4.19)$$

$$< \left[\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \right]^q \prod_{j=1}^{n-1} \left\{ \int_0^\infty \frac{(\arctan x)^{p_j(1-\phi_j)-1}}{(1+x^2)^{1-p_j}} f_j^{p_j}(x) dx \right\}^{\frac{q}{p_j}},$$

where the constant factors $\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j)$ and $\left[\frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma(\phi_j) \right]^q$ are the best possible.

REMARK 6. Taking different parameter values of ϕ_i , as following the Theorem 4.1, in Theorem 4.2 to Theorem 4.5, we get many new inequalities.

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