Journal of
Mathematical
Inequalities
Volume 3, Number 2 (2009), 155–159

A p–FREE ℓ^p–INEQUALITY

GRAHAME BENNETT

(*Communicated by P. Mercer*)

Abstract. We show how certain simple ℓ^p-inequalities may be proved by "ignoring the p."

We prove the following result, which confirms a conjecture made in [3].

THEOREM. *Suppose that a,b,c,d,w,x,y,z are positive numbers. Then the inequality*

$$a^p+b^p+c^p+d^p \leqslant w^p+x^p+y^p+z^p \tag{1}$$

is valid whenever $|p|\geqslant 1$, and it reverses direction whenever $|p|\leqslant 1$, if and only if the following five conditions are satisfied:

$$a+b+c+d=w+x+y+z \tag{2}$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{1}{w}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \tag{3}$$

$$abcd=wxyz \tag{4}$$

$$\max\{a,b,c,d\}\leqslant\max\{w,x,y,z\} \tag{5}$$

and

$$\min\{a,b,c,d\}\geqslant\min\{w,x,y,z\}. \tag{6}$$

Moreover, inequality (1) is then strict, except when $p=-1,0$ or 1, or the sets $\{a,b,c,d\}$ and $\{w,x,y,z\}$ coincide.

The theorem is an example of a *p–free ℓ^p–inequality.* We have here an "ℓ^p–inequality", (1), valid for all real values of p (in the directions indicated), yet the inequality is equivalent to certain hypotheses, (2)–(6), that make no explicit mention of p; that are, in effect, "p–free."

Mathematics subject classification (2000): 26D15.
Keywords and phrases: ℓ^p-inequality, majorization, higher order convexity.

Paper JMI-03-15

COROLLARY. (cf. Lemma 1 and Theorem 4 of [1]). *Suppose that* p *is fixed. Then the sequence*

$$\frac{(n+1)^p}{(n+1)^p-n^p} \qquad (n=1,2,\dots) \tag{7}$$

is convex if $p \geqslant 1$ *or* $-1 \leqslant p < 0$ *and concave if* $0 < p \leqslant 1$ *or* $p \leqslant -1$.

Proof. The assertion of the corollary is equivalent to the inequality

$$\begin{aligned} 2\left[(n+1)^2(n+3)\right]^p + 2\left[n(n+2)^2\right]^p \leqslant \left[(n+1)(n+2)^2\right]^p + [n(n+2)(n+3)]^p \\ + \left[(n+1)^2(n+2)\right]^p + [n(n+1)(n+3)]^p \end{aligned} \tag{8}$$

being valid whenever $|p| \geqslant 1$, and reversing direction whenever $|p| \leqslant 1$. □

Proof of theorem. (Necessity). If inequality (1) holds as stated, there must be equality when $p = \pm 1$, so (2) and (3) are guaranteed. To deduce (4), (5) and (6), we first rephrase (1) in terms of L^p*–means,*

$$\left(\frac{a^p+b^p+c^p+d^p}{4}\right)^{\frac{1}{p}} \leqslant \left(\frac{w^p+x^p+y^p+z^p}{4}\right)^{\frac{1}{p}}. \tag{9}$$

It is clear that (9) is valid whenever $p \geqslant 1$ or $-1 \leqslant p < 0$ and that the inequality reverses direction whenever $p \leqslant -1$ or $0 < p \leqslant 1$. Making $p \to \infty$ in (9), the means approach the corresponding maxima ([6], §2,3.4) forcing (5) to hold. (6) follows similarly by making $p \to -\infty$. To prove (4), we make $p \to 0$ in (9), whereupon the L^p–means are replaced by the corresponding geometric means ([6], §2.3.3). When $p \to 0^-$ we deduce that

$$(abcd)^{\frac{1}{4}} \leqslant (wxyz)^{\frac{1}{4}}, \tag{10}$$

and, when $p \to 0^+$, that (10) is reversed.

(Sufficiency). This, of course, is the gist of the theorem: inequality (1) holds for all real p if it holds at just five "points", $p = 0$, ± 1 and $\pm\infty$.

We assume throughout that the sets $\{a,b,c,d\}$ and $\{w,x,y,z\}$ are disjoint; otherwise they coincide and the theorem is then trivial. (If they have a point in common, say $d = z$, then the polynomial

$$(t-a)(t-b)(t-c) = t^3 - (a+b+c)t^2 + \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)abct - abc$$

is the same as $(t-w)(t-x)(t-y)$ by hypotheses (2), (3) and (4).)

For the remainder of the proof we assume, as we may, that the sets $\{a,b,c,d\}$ and $\{w,x,y,z\}$ are each arranged in descending order:

$$a \geqslant b \geqslant c \geqslant d \quad \text{and} \quad w \geqslant x \geqslant y \geqslant z. \tag{11}$$

It then follows, in fact, that

$$w > a \geqslant b > x \geqslant y > c \geqslant d > z. \tag{12}$$

The first and last inequalities are consequences of hypothesis (5) and (6); strictness therein follows from the discussion above.

We justify the third inequality by assuming the opposite,

$$b \leqslant x, \tag{13}$$

and showing that this leads to a contradiction. It follows that

$$\begin{aligned}
a &\leqslant w && \text{by (5)} && (14)\\
a+b &\leqslant w+x && \text{by (13) and (14)} && (15)\\
a+b+c &\leqslant w+x+y && \text{by (2) and (6)} && (16)\\
a+b+c+d &= w+x+y+z && \text{by (2).} && (17)
\end{aligned}$$

These inequalities, in conjunction with (11), are precisely the ones stipulated by Hardy, Littlewood and Pólya ([6], page 45) in order that the majorization

$$(a,b,c,d) \prec (w,x,y,z) \tag{18}$$

be valid. We deduce from Theorem 108 of [6] that the inequality

$$\varphi(a)+\varphi(b)+\varphi(c)+\varphi(d) < \varphi(w)+\varphi(x)+\varphi(y)+\varphi(z) \tag{19}$$

holds whenever $\varphi : [z,w] \to \mathbb{R}$ is strictly convex. But this assertion is contradicted by hypothesis (3) when we take $\varphi(t) = 1/t$ in (19).

A similar argument, via the majorization

$$\left(\frac{1}{d},\frac{1}{c},\frac{1}{b},\frac{1}{a}\right) \prec \left(\frac{1}{z},\frac{1}{y},\frac{1}{x},\frac{1}{w}\right), \tag{20}$$

justifies the fifth inequality, $y > c$, in (12).

We deduce from (12) that

$$\int_z^d \frac{g(t)}{t^2}\,dt + \int_x^b \frac{g(t)}{t^2}\,dt \leqslant \int_c^y \frac{g(t)}{t^2}\,dt + \int_a^w \frac{g(t)}{t^2}\,dt \tag{21}$$

is valid whenever $g : [z,w] \to \mathbb{R}$ is a *3–convex function* (i.e. the third-order divided differences, $g[\alpha,\beta,\gamma,\delta]$, are all non-negative).

To see this, we consider the quadratic function, Q, that agrees with g at d,y and b. It is known that Q alternates successively **above** and below g on the intervals $[\mathbf{z},\mathbf{d}]$, $[d,y]$, $[\mathbf{y},\mathbf{b}]$ and $[b,w]$. (This striking observation, due to Bullen ([5], Theorem 5), is the analogue, for 3–convex functions, of the familiar fact that *the graph of a convex function lies always beneath its chords.*) In particular, we have

$$g(t) \leqslant Q(t) \quad \text{if } t \in [z,d] \text{ or } t \in [x,b]$$

and

$$g(t) \geqslant Q(t) \quad \text{if } t \in [c,y] \text{ or } t \in [a,w].$$

It suffices, therefore, to prove inequality (21) with g replaced by Q. But (21) is trivially satisfied by *any* quadratic function (being then an identity), courtesy of hypotheses (2), (3) and (4).

Applying (21) to the function

$$g(t) = \begin{cases} pt^{p+1} & \text{if } |p| \geqslant 1 \\ -pt^{p+1} & \text{if } |p| \leqslant 1 \end{cases} \tag{22}$$

(which is 3–convex since $g'''(t) \geqslant 0$), we obtain the inequality

$$d^p - z^p + b^p - x^p \leqslant y^p - c^p + w^p - a^p \tag{23}$$

when $|p| \geqslant 1$ and its reversal when $|p| \leqslant 1$. This completes our proof of sufficiency.

The last sentence of the theorem, concerning cases of equality in (1), is justified by observing that the function (22) is strictly 3–convex except when $p = -1$, or 0 or 1. [A simple modification of Bullen's proof shows that the graph of Q lies strictly above that of g on the intervals $[z,d)$ and (y,b), and strictly below on (d,y) and $(b,w]$, whenever g is strictly 3–convex. It follows that inequality (21), and hence also (23), is then strict.] □

Our proof shows rather more than has been stated. Indeed, (21) suggests a new kind of Majorization, one that is not considered in [3]. *The inequality*

$$f(a) + f(b) + f(c) + f(d) \leqslant f(w) + f(x) + f(y) + f(z) \tag{24}$$

is valid for all functions f such that

$$\left(t^2 f'(t)\right)' \quad \textit{is convex} \tag{25}$$

if and only if hypotheses (2)—(6) are in effect.

This is seen by replacing g in (21) by $t^2 f'(t)$, and using the characterization of 3–convex functions given in Proposition 1 of [3].

There are other p–free ℓ^p–inequalities in the literature. The simplest example, via the Theory of Majorization, is described on page 820 of [1]. A second example, with applications to Moment Theory, is given in [2], while a third, solving a problem of Knuth, appears in [4]. These all deal with 3-tuples, however, and they do not extend to higher dimensions. Some interest attaches, therefore, to the fact that our theorem applies to 4-tuples.

It was natural, in seeking such a theorem, to focus attention on the main result of [1]: *the sequence*

$$\frac{1^p + 2^p + \cdots + n^p}{n^p} \qquad (n = 1, 2, \ldots) \tag{26}$$

is convex if $p \geqslant 1$ or $p \leqslant 0$, concave otherwise. (The inequalities implicit in (26), after all, show that the Theory of Majorization produces p–free ℓ^p–inequalities *only* for 1–, 2– and 3–tuples. See section 5 of [1].)

This led us back to our corollary, which is an important component in the proof of (26). (See sections 2 and 3 of [1].) It is easy to check that hypotheses (2)–(6) are satisfied by the variables in (8); indeed, it was this observation that led us to the formulation of our conjecture. The corollary, of course, may be proved independently (see Lemma 1 of [1], where it appears as a calculus exercise). But our theorem is a vastly more general result, and the proof given here extends readily to provide p–free ℓ^p–inequalities in any dimension.

REFERENCES

[1] G. BENNETT, *Sums of powers and the meaning of ℓ^p*, Houston J. Math., **32**, 3 (2006), 801–831.
[2] G. BENNETT, *An inequality from Moment Theory*, Positivity, **11**, 2 (2007), 231–238.
[3] G. BENNETT, *Some forms of Majorization*, to appear in Houston J. Math.
[4] G. BENNETT, *A problem of Knuth*, submitted to Amer. Math. Monthly.
[5] P.S. BULLEN, *A criterion for n–convexity*, Pacific J. Math., **36** (1971), 81–98.
[6] G.H. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities. 2nd edition*, Cambridge University Press, Cambridge, 1967.

(Received October 20, 2008)

Grahame Bennett
Department of Mathematics
Indiana University
Bloomington, IN 47405
U. S. A.
e-mail: bennettg@indiana.edu