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SCHUR COMPLEMENTS AND DETERMINANT INEQUALITIES

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Abstract. This paper is focused on the applications of Schur complements to determinant inequalities. It presents a monotonic characterization of Schur complements in the Löwner partial ordering sense such that a new proof of the Hadamard-Fischer-Koteljanski inequality is obtained. Meanwhile, it presents matrix identities and determinant inequalities involving positive semidefinite matrices and extends the Hua Loo-keng determinant inequality by the technique of Schur complements.

1. Introduction

Let M be an $n\times n$ Hermitian matrix partitioned as

$$M=\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \tag{1}$$

in which M_{11} is square and nonsingular $k\times k$ block with $1\leqslant k<n$. The Schur complement of M_{11} in M is defined and denoted by

$$M/M_{11}=M_{22}-M_{21}M_{11}^{-1}M_{12}.$$

The Schur complement is a basic tool in many areas of matrix analysis, and is a rich source of matrix inequalities. The idea of using the Schur complement technique to deal with linear systems and matrix problems is classical. A famous determinant identity presented by Schur 1917 was referred to as the formula of Schur (see [8]).

LEMMA 1.1. *Let M be a square matrix partitioned as in* (1)*. If M_{11} is nonsingular, then*

$$|M|=|M_{11}|\cdot|M/M_{11}|.$$

The following useful formula, due to Babachiewicz [1], presents the inverse of a matrix in terms of Schur complements.

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THEOREM 1.2. *Let M be a square matrix partitioned as in* (1) *and suppose M is nonsinglar. Then M/M_{11} is nonsinglar and*

$$M^{-1} = \begin{pmatrix} M_{11}^{-1} + M_{11}^{-1}M_{12}(M/M_{11})^{-1}M_{21}M_{11}^{-1} & -M_{11}^{-1}M_{12}(M/M_{11})^{-1} \\ -(M/M_{11})^{-1}M_{21}M_{11}^{-1} & (M/M_{11})^{-1} \end{pmatrix}.$$

A fundamental and very useful fact is the following Schur complement lemma (see [4]).

LEMMA 1.3. *Let M be a square matrix partitioned as in* (1)*. On the assumption that M_{11} is positive definite, M is positive semidefinite if and only if the Schur complement M/M_{11} is positive semidefinite.*

LEMMA 1.4. *Let A and B be two positive semidefinite Hermitian matrices with the same size. Then*

$$|A+B| \geqslant |A| + |B|.$$

LEMMA 1.5. *Let A and B be two positive definite Hermitian matrices with $A \geqslant B$. Then $A^{-1} \leqslant B^{-1}$ and $|A| \geqslant |B|$.*

THEOREM 1.6. *Let M be a positive semidefinite Hermitian matrix with the partition* (1)*. Assume that $M_{11}, M_{12}, M_{21}, M_{22}$ are four square matrices with the same size. Then*

$$|M_{11}| \cdot |M_{22}| - |M_{21}|^2 \geqslant |M| \geqslant 0.$$

Proof: Without loss of generalization, we assume that M_{11} is positive definite (otherwise, replaced M_{11} by $M_{11} + \varepsilon I$, where ε is an enough small positive number and I is an unit matrix with the size of M_{11}). By the use of Lemma 1.1 and 1.4, we have

$$\begin{aligned} |M_{22}| &\geqslant |M_{22} - M_{21}M_{11}^{-1}M_{12}| + |M_{21}M_{11}^{-1}M_{12}| \\ &= \frac{|M|}{|M_{11}|} + \frac{|M_{21}|^2}{|M_{11}|}, \end{aligned}$$

which implies the desired result. □

This paper is focused on the applications of Schur complements to determinant inequalities. Section 2 presents a monotonic characterization of Schur complements for principle submatrices of a positive definite matrix and applies it to prove the Hadamard-Fischer-Koteljanski inequality. Section 3 gives matrix identities and inequalities to extend the Hua Loo-keng determinant inequality.

2. A monotonic characterization of Schur complements and its applications

THEOREM 2.1. *Let A be a Hermitian matrix partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}. \tag{2}$$

If the submatrix B of A defined by

$$B = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is positive definite, then there is the Löwner partial ordering

$$A_{32}A_{22}^{-1}A_{23} \leqslant (A_{31}, A_{32}) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix}, \tag{3}$$

or equivalently,

$$A_{33} - A_{32}A_{22}^{-1}A_{23} \geqslant A_{33} - (A_{31}, A_{32}) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix}. \tag{4}$$

Proof: Since the submatrix B of A is a positive definite Hermitian matrix, from Lemma 1.3, for any positive number $\varepsilon > 0$, the matrix

$$\begin{pmatrix} \frac{1}{\varepsilon}A_{11} & A_{12} \\ A_{21} & \varepsilon A_{22} \end{pmatrix} = \begin{pmatrix} (1+\frac{1}{\varepsilon})A_{11} & 0 \\ 0 & (1+\varepsilon)A_{22} \end{pmatrix} - \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is positive definite, which means that

$$\begin{pmatrix} (1+\frac{1}{\varepsilon})A_{11} & 0 \\ 0 & (1+\varepsilon)A_{22} \end{pmatrix} > \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

It shows, from Lemma 1.5, that

$$\begin{pmatrix} (1+\frac{1}{\varepsilon})A_{11} & 0 \\ 0 & (1+\varepsilon)A_{22} \end{pmatrix}^{-1} \leqslant \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1}.$$

Multiplying by (A_{31}, A_{32}) and its transpose of both sides, respectively, yields

$$(A_{31}, A_{32}) \begin{pmatrix} (1+\frac{1}{\varepsilon})A_{11} & 0 \\ 0 & (1+\varepsilon)A_{22} \end{pmatrix}^{-1} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix} \leqslant (A_{31}, A_{32}) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix},$$

or equivalently,

$$\frac{\varepsilon}{1+\varepsilon}A_{31}A_{11}^{-1}A_{13} + \frac{1}{1+\varepsilon}A_{32}A_{22}^{-1}A_{23} \leqslant (A_{31}, A_{32}) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix}.$$

Since $\frac{\varepsilon}{1+\varepsilon}A_{31}A_{11}^{-1}A_{13}$ is positive semidefinite, then

$$\frac{1}{1+\varepsilon}A_{32}A_{22}^{-1}A_{23} \leqslant (A_{31}, A_{32}) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix},$$

which implies the inequality (3) if ε tends to 0. □

The inequality (4) reveals the monotonic characterization of Schur complements in the Löwner partial ordering.

In 1893, Hadamard [2] discovered a fundamental fact about positive definite matrices, viz., that, for such $A = (a_{ij})$,

$$|A| \leqslant a_{11}a_{22}\cdots a_{nn} \tag{5}$$

with equality if and only if A is diagonal.

Let α and β be given index sets, i.e., subsets of $\{1,2,\cdots,n\}$. Let $A(\alpha)$ denote the submatrix of A with rows and columns indexed by α. If α is empty, we define $A(\alpha) = 1$. An improvement of (5) is the following Hadamard-Fischer inequality [3]

$$|A(\alpha\cup\beta)| \leqslant |A(\alpha)|\cdot|A(\beta)|,$$

where $\alpha\cap\beta$ is an empty index set. One version of extended Hadamard-Fischer inequalities is the following Hadamard-Fischer-Koteljanski inequality (see [6]).

THEOREM 2.2. *Let A be an $n\times n$ positive definite Hermitian matrix, $\alpha,\beta\subset\{1,\cdots,n\}$. Then*

$$|A(\alpha\cup\beta)| \leqslant \frac{|A(\alpha)|\cdot|A(\beta)|}{|A(\alpha\cap\beta)|}. \tag{6}$$

Proof: We firstly prove (6) for the case $\alpha\cap\beta\neq\emptyset$. Without loss of generalization, we assume that $A(\alpha\cup\beta) = A$ with the partition (2), $A(\alpha\cap\beta) = A_{22}$ and

$$A(\alpha) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$
$$A(\beta) = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}.$$

Using Lemma 1.1, we have

$$|A(\alpha\cup\beta)| = |A(\alpha)|\cdot\left|A_{33} - (A_{31},A_{32})\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1}\begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix}\right|,$$
$$|A(\beta)| = |A(\alpha\cap\beta)|\cdot|A_{33} - A_{32}A_{22}^{-1}A_{23}|.$$

By the virtue of Theorem 2.1 and Lemma 1.5, we obtain the inequality (6) for the case $\alpha\cap\beta\neq\emptyset$.

If $\alpha\cap\beta = \emptyset$, the desired result follows from the former fact if we replace A_{22} by the identity matrix, A_{12},A_{21},A_{23} and A_{32} by zero matrices, respectively. $\square$

3. More inequality via positive semidefinite matrices

In this section we present some determinant inequalities involving positive semidefinite matrices.

THEOREM 3.1. *Let X,Y,A,B,C,D be square matrices with the same size. If X,Y be positive definite, then*

$$|AXA^* + BYB^*|\cdot|CXC^* + DYD^*| \geqslant |AXC^* + BYD^*|^2. \tag{7}$$

Proof: Since the matrix

$$\begin{aligned}&\begin{pmatrix} AXA^*+BYB^* & AXC^*+BYD^* \\ CXA^*+DYB^* & CXC^*+DYD^* \end{pmatrix}\\ &=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}\end{aligned} \tag{8}$$

is positive semidefinite, then the desire result is valid by the use of Lemma 1.6 for the (2,2) block of the left hand side of (8). □

The inequality (7) contains some useful determinant inequalities. If $A=D=I$, for instance, then

$$|X+BYB^*|\cdot|Y+CXC^*|\geqslant|XC^*+BY|^2,$$

which implies that

$$|I+BB^*|\cdot|I+CC^*|\geqslant|B+C^*|^2.$$

On the other hand, if $X=Y=I$, the equality (7) becomes

$$|AA^*+BB^*|\cdot|CC^*+DD^*|\geqslant|AC^*+BD^*|^2.$$

THEOREM 3.2. *Let X,B,A be square matrices of the same size. Then*

$$\begin{aligned}XX^*-BB^* &= (X-BA^*)(I-A^*A)^{-1}(X-BA^*)^*\\ &\quad -(B-XA)(I-A^*A)^{-1}(B-XA)^*.\end{aligned} \tag{9}$$

Proof: If we assume that $X-BA^*$ is nonsinglar, then the matrix

$$P=\begin{pmatrix} I & A^* \\ B & X \end{pmatrix}\begin{pmatrix} I & -B^* \\ -A & X^* \end{pmatrix}=\begin{pmatrix} I-A^*A & A^*X^*-B^* \\ B-XA & XX^*-BB^* \end{pmatrix}$$

is nonsinglar with the inverse

$$\begin{aligned}P^{-1}&=\begin{pmatrix} I & -B^* \\ -A & X^* \end{pmatrix}^{-1}\begin{pmatrix} I & A^* \\ B & X \end{pmatrix}^{-1}\\ &=\begin{pmatrix} I+B^*(X^*-AB^*)^{-1}A & B^*(X^*-AB^*)^{-1} \\ (X^*-AB^*)^{-1}A & (X^*-AB^*)^{-1} \end{pmatrix}\\ &\quad\cdot\begin{pmatrix} I+A^*(X-BA^*)^{-1}B & -A^*(X-BA^*)^{-1} \\ -(X-BA^*)^{-1}B & (X-BA^*)^{-1} \end{pmatrix}.\end{aligned}$$

Compute the (2,2) block of this product and use Theorem 1.2 to get the identity

$$\begin{aligned}(P/(I-A^*A))^{-1} &= -(X^*-AB^*)^{-1}AA^*(X-BA^*)^{-1}\\ &\quad +(X^*-AB^*)^{-1}(X-BA^*)^{-1}\\ &= (X^*-AB^*)^{-1}(I-AA^*)(X-BA^*)^{-1}.\end{aligned}$$

Taking the inverse of both sides gives

$$P/(I-A^*A)=(X-BA^*)(I-AA^*)^{-1}(X-BA^*)^*.$$

On the other hand, we can compute directly the Schur complement of $I-A^*A$ in P:

$$P/(I-A^*A) = XX^* - BB^* + (B-XA)(I-A^*A)^{-1}(B-XA)^*.$$

The asserted identity results from equating these two representations for the Schur complement $P/(I-A^*A)$.

If $X-BA^*$ is singlar, the desired equality follows form a continuity argument, that is, replace X with $X+\varepsilon I$ and let $\varepsilon \to 0$. □

Since both terms on the right side of (9) are positive semidefinite on the assumption of $I-A^*A>0$, we obtain a matrix inequality in the L$\ddot{o}$wner partial ordering by omitting the second term, i.e.,

$$XX^* - BB^* \leqslant (X-BA^*)(I-A^*A)^{-1}(X-BA^*)^*.$$

THEOREM 3.3. *Let X,B,A be square matrices of the same size. If both $I-AA^*$ and XX^*-BB^* are positive definite, then*

$$|I-AA^*|\cdot|XX^*-BB^*| + |B-XA|^2 \leqslant |X-BA^*|^2. \tag{10}$$

Proof: It is a directly result of both Theorem 3.2 and Lemma 1.4 for the fact $|I-A^*A| = |I-AA^*|$. □

A special case of (10) is the Hua Loo-keng determinant inequality [5]

$$|I-A^*A|\cdot|I-BB^*| \leqslant |I-BA^*|^2.$$

More details can be found in Marshall and Olkin [7].

A similar result of Theorem 3.2 can be found in Fuzhen Zhang [4].

THEOREM 3.4. *Let A,B,X be square matrices of the same size. Then*

$$\begin{aligned} AA^*+BB^* = &(B+AX)(I+X^*X)^{-1}(B+AX)^* \\ &+(A-BX)(I+X^*X)^{-1}(A-BX)^* \end{aligned}$$

.

Two directly results of theorem 3.4 are

$$AA^*+BB^* \geqslant (B+AX)(I+X^*X)^{-1}(B+AX)^*,$$

and

$$|AA^*+BB^*|\cdot|I+X^*X| \geqslant |B+AX|^2.$$

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