

SCHUR COMPLEMENTS AND DETERMINANT INEQUALITIES

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Abstract. This paper is focused on the applications of Schur complements to determinant inequalities. It presents a monotonic characterization of Schur complements in the Löwner partial ordering sense such that a new proof of the Hadamard-Fischer-Koteljanski inequality is obtained. Meanwhile, it presents matrix identities and determinant inequalities involving positive semidefinite matrices and extends the Hua Loo-keng determinant inequality by the technique of Schur complements.

1. Introduction

Let M be an $n \times n$ Hermitian matrix partitioned as

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad (1)$$

in which M_{11} is square and nonsingular $k \times k$ block with $1 \leq k < n$. The Schur complement of M_{11} in M is defined and denoted by

$$M/M_{11} = M_{22} - M_{21}M_{11}^{-1}M_{12}.$$

The Schur complement is a basic tool in many areas of matrix analysis, and is a rich source of matrix inequalities. The idea of using the Schur complement technique to deal with linear systems and matrix problems is classical. A famous determinant identity presented by Schur 1917 was referred to as the formula of Schur (see [8]).

LEMMA 1.1. *Let M be a square matrix partitioned as in (1). If M_{11} is nonsingular, then*

$$|M| = |M_{11}| \cdot |M/M_{11}|.$$

The following useful formula, due to Babachiewicz [1], presents the inverse of a matrix in terms of Schur complements.

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THEOREM 1.2. *Let M be a square matrix partitioned as in (1) and suppose M is nonsingular. Then M/M_{11} is nonsingular and*

$$M^{-1} = \begin{pmatrix} M_{11}^{-1} + M_{11}^{-1}M_{12}(M/M_{11})^{-1}M_{21}M_{11}^{-1} & -M_{11}^{-1}M_{12}(M/M_{11})^{-1} \\ -(M/M_{11})^{-1}M_{21}M_{11}^{-1} & (M/M_{11})^{-1} \end{pmatrix}.$$

A fundamental and very useful fact is the following Schur complement lemma (see [4]).

LEMMA 1.3. *Let M be a square matrix partitioned as in (1). On the assumption that M_{11} is positive definite, M is positive semidefinite if and only if the Schur complement M/M_{11} is positive semidefinite.*

LEMMA 1.4. *Let A and B be two positive semidefinite Hermitian matrices with the same size. Then*

$$|A + B| \geq |A| + |B|.$$

LEMMA 1.5. *Let A and B be two positive definite Hermitian matrices with $A \geq B$. Then $A^{-1} \leq B^{-1}$ and $|A| \geq |B|$.*

THEOREM 1.6. *Let M be a positive semidefinite Hermitian matrix with the partition (1). Assume that $M_{11}, M_{12}, M_{21}, M_{22}$ are four square matrices with the same size. Then*

$$|M_{11}| \cdot |M_{22}| - |M_{21}|^2 \geq |M| \geq 0.$$

Proof: Without loss of generalization, we assume that M_{11} is positive definite (otherwise, replaced M_{11} by $M_{11} + \varepsilon I$, where ε is an enough small positive number and I is an unit matrix with the size of M_{11}). By the use of Lemma 1.1 and 1.4, we have

$$\begin{aligned} |M_{22}| &\geq |M_{22} - M_{21}M_{11}^{-1}M_{12}| + |M_{21}M_{11}^{-1}M_{12}| \\ &= \frac{|M|}{|M_{11}|} + \frac{|M_{21}|^2}{|M_{11}|}, \end{aligned}$$

which implies the desired result. \square

This paper is focused on the applications of Schur complements to determinant inequalities. Section 2 presents a monotonic characterization of Schur complements for principle submatrices of a positive definite matrix and applies it to prove the Hadamard-Fischer-Koteljanski inequality. Section 3 gives matrix identities and inequalities to extend the Hua Loo-keng determinant inequality.

2. A monotonic characterization of Schur complements and its applications

THEOREM 2.1. *Let A be a Hermitian matrix partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}. \tag{2}$$

If the submatrix B of A defined by

$$B = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is positive definite, then there is the Löwner partial ordering

$$A_{32}A_{22}^{-1}A_{23} \leq (A_{31}, A_{32}) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix}, \quad (3)$$

or equivalently,

$$A_{33} - A_{32}A_{22}^{-1}A_{23} \geq A_{33} - (A_{31}, A_{32}) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix}. \quad (4)$$

Proof: Since the submatrix B of A is a positive definite Hermitian matrix, from Lemma 1.3, for any positive number $\varepsilon > 0$, the matrix

$$\begin{pmatrix} \frac{1}{\varepsilon}A_{11} & A_{12} \\ A_{21} & \varepsilon A_{22} \end{pmatrix} = \begin{pmatrix} (1 + \frac{1}{\varepsilon})A_{11} & 0 \\ 0 & (1 + \varepsilon)A_{22} \end{pmatrix} - \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is positive definite, which means that

$$\begin{pmatrix} (1 + \frac{1}{\varepsilon})A_{11} & 0 \\ 0 & (1 + \varepsilon)A_{22} \end{pmatrix} > \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

It shows, from Lemma 1.5, that

$$\begin{pmatrix} (1 + \frac{1}{\varepsilon})A_{11} & 0 \\ 0 & (1 + \varepsilon)A_{22} \end{pmatrix}^{-1} \leq \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1}.$$

Multiplying by (A_{31}, A_{32}) and its transpose of both sides, respectively, yields

$$(A_{31}, A_{32}) \begin{pmatrix} (1 + \frac{1}{\varepsilon})A_{11} & 0 \\ 0 & (1 + \varepsilon)A_{22} \end{pmatrix}^{-1} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix} \leq (A_{31}, A_{32}) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix},$$

or equivalently,

$$\frac{\varepsilon}{1 + \varepsilon} A_{31}A_{11}^{-1}A_{13} + \frac{1}{1 + \varepsilon} A_{32}A_{22}^{-1}A_{23} \leq (A_{31}, A_{32}) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix}.$$

Since $\frac{\varepsilon}{1 + \varepsilon} A_{31}A_{11}^{-1}A_{13}$ is positive semidefinite, then

$$\frac{1}{1 + \varepsilon} A_{32}A_{22}^{-1}A_{23} \leq (A_{31}, A_{32}) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix},$$

which implies the inequality (3) if ε tends to 0. \square

The inequality (4) reveals the monotonic characterization of Schur complements in the Löwner partial ordering.

In 1893, Hadamard [2] discovered a fundamental fact about positive definite matrices, viz., that, for such $A = (a_{ij})$,

$$|A| \leq a_{11}a_{22} \cdots a_{nn} \tag{5}$$

with equality if and only if A is diagonal.

Let α and β be given index sets, i.e., subsets of $\{1, 2, \dots, n\}$. Let $A(\alpha)$ denote the submatrix of A with rows and columns indexed by α . If α is empty, we define $A(\alpha) = 1$. An improvement of (5) is the following Hadamard-Fischer inequality [3]

$$|A(\alpha \cup \beta)| \leq |A(\alpha)| \cdot |A(\beta)|,$$

where $\alpha \cap \beta$ is an empty index set. One version of extended Hadamard-Fischer inequalities is the following Hadamard-Fischer-Koteljanski inequality (see [6]).

THEOREM 2.2. *Let A be an $n \times n$ positive definite Hermitian matrix, $\alpha, \beta \subset \{1, \dots, n\}$. Then*

$$|A(\alpha \cup \beta)| \leq \frac{|A(\alpha)| \cdot |A(\beta)|}{|A(\alpha \cap \beta)|}. \tag{6}$$

Proof: We firstly prove (6) for the case $\alpha \cap \beta \neq \emptyset$. Without loss of generalization, we assume that $A(\alpha \cup \beta) = A$ with the partition (2), $A(\alpha \cap \beta) = A_{22}$ and

$$A(\alpha) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

$$A(\beta) = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}.$$

Using Lemma 1.1, we have

$$|A(\alpha \cup \beta)| = |A(\alpha)| \cdot \left| A_{33} - (A_{31}, A_{32}) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix} \right|,$$

$$|A(\beta)| = |A(\alpha \cap \beta)| \cdot |A_{33} - A_{32}A_{22}^{-1}A_{23}|.$$

By the virtue of Theorem 2.1 and Lemma 1.5, we obtain the inequality (6) for the case $\alpha \cap \beta \neq \emptyset$.

If $\alpha \cap \beta = \emptyset$, the desired result follows from the former fact if we replace A_{22} by the identity matrix, A_{12}, A_{21}, A_{23} and A_{32} by zero matrices, respectively. \square

3. More inequality via positive semidefinite matrices

In this section we present some determinant inequalities involving positive semidefinite matrices.

THEOREM 3.1. *Let X, Y, A, B, C, D be square matrices with the same size. If X, Y be positive definite, then*

$$|AXA^* + BYB^*| \cdot |CXC^* + DYD^*| \geq |AXC^* + BYD^*|^2. \tag{7}$$

Proof: Since the matrix

$$\begin{aligned} & \begin{pmatrix} AXA^* + BYB^* & AXC^* + BYD^* \\ CXA^* + DYB^* & CXC^* + DYD^* \end{pmatrix} \\ &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \end{aligned} \tag{8}$$

is positive semidefinite, then the desired result is valid by the use of Lemma 1.6 for the (2,2) block of the left hand side of (8). \square

The inequality (7) contains some useful determinant inequalities. If $A = D = I$, for instance, then

$$|X + BYB^*| \cdot |Y + CXC^*| \geq |XC^* + BY|^2,$$

which implies that

$$|I + BB^*| \cdot |I + CC^*| \geq |B + C|^2.$$

On the other hand, if $X = Y = I$, the equality (7) becomes

$$|AA^* + BB^*| \cdot |CC^* + DD^*| \geq |AC^* + BD^*|^2.$$

THEOREM 3.2. *Let X, B, A be square matrices of the same size. Then*

$$\begin{aligned} XX^* - BB^* &= (X - BA^*)(I - A^*A)^{-1}(X - BA^*)^* \\ &\quad - (B - XA)(I - A^*A)^{-1}(B - XA)^*. \end{aligned} \tag{9}$$

Proof: If we assume that $X - BA^*$ is nonsingular, then the matrix

$$P = \begin{pmatrix} I & A^* \\ B & X \end{pmatrix} \begin{pmatrix} I & -B^* \\ -A & X^* \end{pmatrix} = \begin{pmatrix} I - A^*A & A^*X^* - B^* \\ B - XA & XX^* - BB^* \end{pmatrix}$$

is nonsingular with the inverse

$$\begin{aligned} P^{-1} &= \begin{pmatrix} I & -B^* \\ -A & X^* \end{pmatrix}^{-1} \begin{pmatrix} I & A^* \\ B & X \end{pmatrix}^{-1} \\ &= \begin{pmatrix} I + B^*(X^* - AB^*)^{-1}A & B^*(X^* - AB^*)^{-1} \\ (X^* - AB^*)^{-1}A & (X^* - AB^*)^{-1} \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} I + A^*(X - BA^*)^{-1}B & -A^*(X - BA^*)^{-1} \\ -(X - BA^*)^{-1}B & (X - BA^*)^{-1} \end{pmatrix}. \end{aligned}$$

Compute the (2,2) block of this product and use Theorem 1.2 to get the identity

$$\begin{aligned} (P/(I - A^*A))^{-1} &= -(X^* - AB^*)^{-1}AA^*(X - BA^*)^{-1} \\ &\quad + (X^* - AB^*)^{-1}(X - BA^*)^{-1} \\ &= (X^* - AB^*)^{-1}(I - AA^*)(X - BA^*)^{-1}. \end{aligned}$$

Taking the inverse of both sides gives

$$P/(I - A^*A) = (X - BA^*)(I - AA^*)^{-1}(X - BA^*)^*.$$

On the other hand, we can compute directly the Schur complement of $I - A^*A$ in P :

$$P/(I - A^*A) = XX^* - BB^* + (B - XA)(I - A^*A)^{-1}(B - XA)^*.$$

The asserted identity results from equating these two representations for the Schur complement $P/(I - A^*A)$.

If $X - BA^*$ is singlar, the desired equality follows form a continuity argument, that is, replace X with $X + \varepsilon I$ and let $\varepsilon \rightarrow 0$. \square

Since both terms on the right side of (9) are positive semidefinite on the assumption of $I - A^*A > 0$, we obtain a matrix inequality in the L\"{o}wner partial ordering by omitting the second term, i.e.,

$$XX^* - BB^* \leq (X - BA^*)(I - A^*A)^{-1}(X - BA^*)^*.$$

THEOREM 3.3. *Let X, B, A be square matrices of the same size. If both $I - AA^*$ and $XX^* - BB^*$ are positive definite, then*

$$|I - AA^*| \cdot |XX^* - BB^*| + |B - XA|^2 \leq |X - BA^*|^2. \tag{10}$$

Proof: It is a directly result of both Theorem 3.2 and Lemma 1.4 for the fact $|I - A^*A| = |I - AA^*|$. \square

A special case of (10) is the Hua Loo-keng determinant inequality [5]

$$|I - A^*A| \cdot |I - BB^*| \leq |I - BA^*|^2.$$

More details can be found in Marshall and Olkin [7].

A similar result of Theorem 3.2 can be found in Fuzhen Zhang [4].

THEOREM 3.4. *Let A, B, X be square matrices of the same size. Then*

$$AA^* + BB^* = (B + AX)(I + X^*X)^{-1}(B + AX)^* + (A - BX)(I + X^*X)^{-1}(A - BX)^*.$$

Two directly results of theorem 3.4 are

$$AA^* + BB^* \geq (B + AX)(I + X^*X)^{-1}(B + AX)^*,$$

and

$$|AA^* + BB^*| \cdot |I + X^*X| \geq |B + AX|^2.$$

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