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AN ESTIMATE OF THE COMMUTATIVITY OF C^2–FUNCTIONS AND PROBABILITY MEASURES

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Abstract. In [1], an estimate of the difference of the two sides of the Jensen's inequality with respect to probability measures was given, which is a special case of a Cauchy type mean value theorem (cf. [3, 4, 5]). Without Cauchy type mean value theorem, we give an estimate of the commutativity of C^2-function and probability measure. The purpose of this paper is to determine the equality condition for the estimate above.

1. Introduction and main result

Recently A. M. Fink [1] considered how much the difference of the two sides of the Jensen inequality might be. Although Fink dealt with probability measures and end positive measures, we will focus on attention on probability measure. In this case, his result reads as follows:

THEOREM A. *Let μ be a probability measure on a finite closed interval $[a,b]$ and $f \in L^\infty_{\mathbb{R}}([a,b],\mu)$ with $\alpha \leqslant f \leqslant \beta$. If φ is a convex C^2-function on $[\alpha,\beta]$, then*

$$\int_a^b \varphi(f)\,d\mu - \varphi\left(\int_a^b f\,d\mu\right) \leqslant \frac{1}{2}\max_{\alpha\leqslant t\leqslant\beta}\varphi''(t)\left\{\int_a^b f^2\,d\mu - \left(\int_a^b f\,d\mu\right)^2\right\}. \tag{1}$$

When $\varphi(t) = t^2/2$, the equality holds.

It seems that the inequality (1) is well-known. In fact, A. McD. Mercer proved the discrete version of inequality (1). Moreover, (1) is a special case of [4, Theorem 6] by J. E. Pečarić, I. Perić and H. M. Srivastava (cf. [5, Theorem 2]). Fink only gave an example that the equality holds in (1). Applying the argument in [1], we will determine the equality condition for a generalized inequality of (1). The following is our main result.

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THEOREM 1.1. *Let $C_{\mathbb{R}}(X)$ be the space of all real-valued continuous functions on a compact Hausdorff space X. Let μ be a probability measure on X and $f \in C_{\mathbb{R}}(X)$ with $\alpha = \text{ess inf}_{x\in X} f(x)$ and $\beta = \text{ess sup}_{x\in X} f(x)$ with respect to μ. If φ is a complex-valued C^2-function on $[\alpha,\beta]$ and if ψ is a convex or concave C^2-function on $[\alpha,\beta]$ such that $|\varphi''(t)| \leqslant A|\psi''(t)|$ $(\forall t \in [\alpha,\beta])$ for some positive constant $A < \infty$, then*

$$\left|\int_X \varphi(f)\,d\mu - \varphi\left(\int_X f\,d\mu\right)\right| \leqslant K\left|\int_X \psi(f)\,d\mu - \psi\left(\int_X f\,d\mu\right)\right| \tag{2}$$

holds, where

$$K = \sup\left\{\frac{|\varphi''(t)|}{|\psi''(t)|} : t \in [\alpha,\beta] \text{ with } \psi''(t) \neq 0\right\}.$$

The equality holds in (2) *if and only if $\varphi(t) = a\psi(t) + bt + c$ $(\forall t \in [\alpha,\beta])$ for some a, b, $c \in \mathbb{C}$.*

Although the inequality (2) is a direct consequence of [4, Theorem 6], we will give another proof of it in order to determine the equality condition for (2).

LEMMA 1.2. *Let (Ω,μ) be a probability space and $f \in L^1(\Omega,\mu)$. Then the following are equivalent.*

(i) $\displaystyle\left|\int_\Omega f\,d\mu\right| = \int_\Omega |f|\,d\mu.$

(ii) There exists $c \in \mathbb{C}$ such that $|c| = 1$ and $f(w) = c|f(w)|$ for μ-a.e. $w \in \Omega$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $\left|\int_\Omega f\,d\mu\right| = \int_\Omega |f|\,d\mu$. If $\int_\Omega f\,d\mu = 0$, then by hypothesis $|f(w)| = 0$ for μ-a.e. $w \in \Omega$. So, it is enough to consider the case where $\int_\Omega f\,d\mu \neq 0$. Take $\theta \in [0,2\pi)$ so that $\int_\Omega f\,d\mu = e^{i\theta}\left|\int_\Omega f\,d\mu\right|$. We have that

$$\begin{aligned}\left|\int_\Omega f\,d\mu\right| &= \int_\Omega e^{-i\theta} f\,d\mu = \int_\Omega \mathrm{Re}\,(e^{-i\theta} f)\,d\mu \\ &\leqslant \int_\Omega |\mathrm{Re}\,(e^{-i\theta} f)|\,d\mu \leqslant \int_\Omega |f|\,d\mu.\end{aligned}$$

It follows that

$$\int_\Omega \mathrm{Re}\,(e^{-i\theta} f)\,d\mu = \int_\Omega |f|\,d\mu,$$

and so $|f(w)| = \mathrm{Re}\,(e^{-i\theta} f(w))$ for μ-a.e. $w \in \Omega$. Since $|e^{-i\theta} f(w)| = |f(w)|$, we have that $|f(w)| = e^{-i\theta} f(w)$ for μ-a.e. $w \in \Omega$. This implies that $f(w) = c|f(w)|$ for μ-a.e. $w \in \Omega$, where $c = e^{i\theta}$.

(ii) $\Rightarrow$ (i) If there exists $c \in \mathbb{C}$ such that $|c| = 1$ and $f(w) = c|f(w)|$ for μ-a.e. $w \in \Omega$, then we have

$$\left|\int_\Omega f\,d\mu\right| = \left|\int_\Omega c|f|\,d\mu\right| = \int_\Omega |f|\,d\mu.$$

This completes the proof. □

Proof of Theorem 1.1. Let η be a C^2-function on $[\alpha,\beta]$. For each $p,q \in [\alpha,\beta]$, we have

$$\eta(q)-\eta(p)=\int_p^q \eta'(t)\,dt=(q-p)\eta'(p)+\int_p^q (q-t)\eta''(t)\,dt.$$

Set $M=\int_X f\,d\mu$, then we have $\alpha \leqslant M \leqslant \beta$. Since $f(x)\in[\alpha,\beta]$, we have for each $x\in X$ that

$$\eta(f(x))-\eta(M)=(f(x)-M)\eta'(M)+\int_M^{f(x)}(f(x)-t)\eta''(t)\,dt. \tag{3}$$

Integrating both sides of (3) with respect to μ, we get

$$\int_X \eta(f(x))\,d\mu(x)-\eta(M)=\int_X d\mu(x)\int_M^{f(x)}(f(x)-t)\eta''(t)\,dt. \tag{4}$$

Set, for each C^2-function η on $[\alpha,\beta]$,

$$J_\eta=\left|\int_X \eta(f(x))\,d\mu(x)-\eta\left(\int_X f\,d\mu\right)\right|.$$

Under this notation, it is enough to prove that

$$J_\varphi \leqslant KJ_\psi. \tag{5}$$

Recall that $M=\int_X f\,d\mu$. By (4), applied to $\eta=\varphi$, we have that

$$J_\varphi=\left|\int_X d\mu(x)\int_M^{f(x)}(f(x)-t)\varphi''(t)\,dt\right|. \tag{6}$$

Set

$$X^-=\{x\in X: f(x)<M\} \quad \text{and} \quad X^+=\{x\in X: M\leqslant f(x)\}.$$

We first consider the case when ψ is a convex function. Then $\psi''(t)\geqslant 0$ for every $t\in[\alpha,\beta]$. Thus, $|\varphi''(t)|\leqslant K\psi''(t)$ for every $t\in[\alpha,\beta]$. We have

$$\begin{aligned}
J_\varphi &= \left|\int_{X^-} d\mu\int_M^{f(x)}(f(x)-t)\varphi''(t)\,dt+\int_{X^+} d\mu\int_M^{f(x)}(f(x)-t)\varphi''(t)\,dt\right| && (7)\\
&\leqslant \int_{X^-} d\mu\left|\int_M^{f(x)}(f(x)-t)\varphi''(t)\,dt\right|+\int_{X^+} d\mu\left|\int_M^{f(x)}(f(x)-t)\varphi''(t)\,dt\right| \\
&\leqslant \int_{X^-} d\mu\int_{f(x)}^{M}(t-f(x))|\varphi''(t)|\,dt+\int_{X^+} d\mu\int_M^{f(x)}(f(x)-t)|\varphi''(t)|\,dt && (8)\\
&\leqslant K\int_{X^-} d\mu\int_{f(x)}^{M}(t-f(x))\psi''(t)\,dt+K\int_{X^+} d\mu\int_M^{f(x)}(f(x)-t)\psi''(t)\,dt && (9)\\
&= K\int_{X^-} d\mu\int_M^{f(x)}(f(x)-t)\psi''(t)\,dt+K\int_{X^+} d\mu\int_M^{f(x)}(f(x)-t)\psi''(t)\,dt
\end{aligned}$$

$$= K\left|\int_X d\mu(x)\int_M^{f(x)}(f(x)-t)\psi''(t)\,dt\right| = KJ_\psi.$$

We thus get the inequality (5).

If φ is of the form $\varphi(t) = a\psi(t)+bt+c \ \ (t\in[\alpha,\beta])$ for some $a,\ b,\ c\in\mathbb{C}$, then by a simple calculation we see that the equality holds in (2). Conversely, suppose that the equality holds in (2). We will prove that $\varphi(t) = a\psi(t)+bt+c \ \ (t\in[\alpha,\beta])$ for some $a,\ b,\ c\in\mathbb{C}$. By (8), we have that

$$\left|\int_M^{f(x)}(f(x)-t)\varphi''(t)\,dt\right| = \int_{f(x)}^M (t-f(x))|\varphi''(t)|\,dt \tag{10}$$

for μ-a.e. $x\in X^-$, and that

$$\left|\int_M^{f(x)}(f(x)-t)\varphi''(t)\,dt\right| = \int_M^{f(x)}(f(x)-t)|\varphi''(t)|\,dt \tag{11}$$

for μ-a.e. $x\in X^+$. On the other hand, by (9), we have that

$$\int_{f(x)}^M (t-f(x))|\varphi''(t)|\,dt = K\int_{f(x)}^M (t-f(x))\psi''(t)\,dt \tag{12}$$

for μ-a.e. $x\in X^-$, and that

$$\int_M^{f(x)}(f(x)-t)|\varphi''(t)|\,dt = K\int_M^{f(x)}(f(x)-t)\psi''(t)\,dt \tag{13}$$

for μ-a.e. $x\in X^+$. By normalizing the Lebesgue measure dt, we may apply Lemma 1.2 to (10). Then we see that, for μ-a.e. $x\in X^-$, there exists $\gamma(x)\in\mathbb{C}$ such that $|\gamma(x)|=1$ and that

$$(t-f(x))\varphi''(t) = \gamma(x)(t-f(x))|\varphi''(t)|$$

for dt-a.e. $t\in[f(x),M]$. By the continuity of φ'', we see that there exists $N\subset X^-$ with $\mu(N)=0$ such that

$$\varphi''(t) = \gamma(x)|\varphi''(t)| \qquad (\forall t\in[f(x),M]) \tag{14}$$

for every $x\in X^-\setminus N$. We will show that there exists $\gamma_0\in\mathbb{C}$ with $|\gamma_0|=1$ such that

$$\varphi''(t) = \gamma_0 K\psi''(t) \qquad (\forall t\in[\alpha,M]). \tag{15}$$

To do this, set

$$N_0 = \{x\in X^-\setminus N : \varphi''(t)=0 \quad (\forall t\in[f(x),M])\}.$$

We first consider the case where $N_0 = X^-\setminus N$. Then, by hypothesis, $\varphi''(t)=0$ $(\forall t\in[f(x),M])$ for every $x\in X^-\setminus N$. This implies that $\varphi''(t)=0$ for every $t\in[\alpha,M]$. It follows from (12) that $\psi''(t)=0$ for every $t\in[\alpha,M]$, that is, (15) holds if $N_0 = X^-\setminus N$.

We next consider the case where $N_0 \subsetneq X^- \setminus N$. Take $x_0 \in (X^- \setminus N) \setminus N_0$. By the definition of N_0, there exists $t_0 \in [f(x_0), M]$ such that $\varphi''(t_0) \neq 0$. Since $x_0 \in X^- \setminus N$, it follows from (14) that

$$\varphi''(t_0) = \gamma(x_0)|\varphi''(t_0)|. \tag{16}$$

Then we have $\gamma(x_0) = \varphi''(t_0)/|\varphi''(t_0)|$. Set $\gamma_0 = \gamma(x_0)$, then by (14), we have

$$\varphi''(t) = \gamma_0|\varphi''(t)| \qquad (\forall t \in [f(x_0), M]). \tag{17}$$

We will prove that

$$\varphi''(t) = \gamma_0|\varphi''(t)| \qquad (\forall t \in [f(x), M]) \tag{18}$$

for every $x \in X^- \setminus N$. Take $x \in X^- \setminus N$ arbitrarily. If $f(x_0) \leqslant f(x)$, then $[f(x), M] \subset [f(x_0), M]$, and so by (17), we have (18). If $f(x) < f(x_0)$, then $t_0 \in [f(x), M]$. Recall that $\varphi''(t_0) \neq 0$. It follows from (14) that

$$\gamma(x) = \frac{\varphi''(t_0)}{|\varphi''(t_0)|} = \gamma(x_0) = \gamma_0,$$

and so (18) holds. Since $x \in X^- \setminus N$ was arbitrary, we have proved that (18) holds for every $x \in X^- \setminus N$. This implies that

$$\varphi''(t) = \gamma_0|\varphi''(t)| \qquad (\forall t \in [\alpha, M]). \tag{19}$$

On the other hand, it follows from (12) that

$$|\varphi''(t)| = K\psi''(t) \qquad (\forall t \in [f(x), M])$$

for μ-a.e. $x \in X^-$, and so

$$|\varphi''(t)| = K\psi''(t) \qquad (\forall t \in [\alpha, M]).$$

By (19), we have (15) even if $N_0 \subsetneq X^- \setminus N$.

In the same way, we see that there exists $\gamma_1 \in \mathbb{C}$ with $|\gamma_1| = 1$ such that $\varphi''(t) = \gamma_1 K\psi''(t)$ for all $t \in [M, \beta]$. Thus we can write

$$\varphi''(t) = \begin{cases} \gamma_0 K\psi''(t) \text{ if } & t \in [\alpha, M] \\ \gamma_1 K\psi''(t) \text{ if } & t \in [M, \beta] \end{cases}. \tag{20}$$

We will prove that

$$\varphi''(t) = \gamma K\psi''(t) \qquad (\forall t \in [\alpha, \beta]) \tag{21}$$

for some $\gamma \in \mathbb{C}$ with $|\gamma| = 1$. By (20), we see that if ψ'' is identically 0 on $[\alpha, M]$, then so is φ''. This implies that $\varphi''(t) = \gamma_1 K\psi''(t)$ for every $t \in [\alpha, \beta]$. Similarly to the above, we have that $\varphi''(t) = \gamma_0 K\psi''(t)$ for every $t \in [\alpha, \beta]$ if $\psi'' = 0$ on $[M, \beta]$. Finally, we consider the case where $\psi''(t_1) \neq 0$ and $\psi''(t_2) \neq 0$ for some $t_1 \in [\alpha, M]$ and $t_2 \in [M, \beta]$. Set

$$r = \int_{X^-} d\mu \int_{f(x)}^{M} (t - f(x))\psi''(t)\, dt$$

and

$$s = \int_{X^+} d\mu \int_M^{f(x)} (f(x)-t)\psi''(t)\,dt.$$

Then we have $r \neq 0$ and $s \neq 0$ since ψ'' is not identically 0 on $[\alpha, M]$ and $[M, \beta]$. By (7) and (20), we have

$$J_\varphi = K|\gamma_0 r + \gamma_1 s|.$$

On the other hand, $J_\psi = r+s$ by definition and (4). Recall, by hypothesis, that $J_\varphi = KJ_\psi$. This implies that

$$K|\gamma_0 r + \gamma_1 s| = K(r+s).$$

If $K \neq 0$, then it is easy to see that $\gamma_0 = \gamma_1$. In this case, it follows from (20) that $\varphi''(t) = \gamma_0 K\psi''(t)$ for every $t \in [\alpha, \beta]$. On the other hand, if $K = 0$, then by (20), we have (21). In any case, we conclude that there exists $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ such that (21) holds. Now it is obvious that $\varphi(t) = a\psi(t) + bt + c$ $(\forall t \in [\alpha, \beta])$ for some $a, b, c \in \mathbb{C}$.

If ψ is concave, then the above proof works for $-\psi$ because $-\psi$ is convex. This completes the proof. □

REMARK 1.1. In Theorem 1.1, the convexity or concavity assumption for ψ is essential. In fact, let $X = [0,1]$ and μ a probability measure on X such that $\mu(0) = \mu(1) = 1/2$. Let $f(x) = 3x-2$ for $x \in X$, $\varphi(t) = t^4/4 + t^3/6$ and $\psi(t) = t^3 + t^2$ for $t \in [-2,1]$. Then ψ is neither convex nor concave. We have that $\varphi''(t) = t(3t+1)$ and $\psi''(t) = 2(3t+1)$, and so

$$\sup_{\substack{t\in[-2,1] \\ \psi''(t)\neq 0}} \left|\frac{\varphi''(t)}{\psi''(t)}\right| = \sup_{t\in[-2,1]} \frac{|t|}{2} = 1.$$

On the other hand, by a simple calculation, we see that

$$\int_X \varphi(f)\,d\mu - \varphi\left(\int_X f\,d\mu\right) = \frac{37}{24} + \frac{1}{192} = \frac{297}{192}$$

and

$$\left|\int_X \psi(f)\,d\mu - \psi\left(\int_X f\,d\mu\right)\right| = \frac{9}{8} < \frac{297}{192}.$$

Thus the inequality (2) does not hold.

2. Applications

COROLLARY 2.1. *Let $L^\infty_{\mathbb{R}}(\Omega, \nu)$ be the set of all essentially bounded real-valued measurable functions on a probability space (Ω, ν). Let $f \in L^\infty_{\mathbb{R}}(\Omega, \nu)$ with $\alpha =$ ess $\inf_{w\in\Omega} f(w)$ and $\beta =$ ess $\sup_{w\in\Omega} f(w)$. If φ is a complex-valued C^2-function on*

$[\alpha,\beta]$ and if ψ is a convex or concave C^2-function on $[\alpha,\beta]$ such that $|\varphi''(t)| \leqslant A|\psi''(t)|$ $(\forall t \in [\alpha,\beta])$ for some constant $A<\infty$, then

$$\left|\int_\Omega \varphi(f)\,d\nu - \varphi\left(\int_\Omega f\,d\nu\right)\right| \leqslant K\left|\int_\Omega \psi(f)\,d\nu - \psi\left(\int_\Omega f\,d\nu\right)\right| \tag{22}$$

holds, where

$$K = \sup\left\{\frac{|\varphi''(t)|}{|\psi''(t)|} : t\in[\alpha,\beta] \text{ with } \psi''(t)\neq 0\right\}.$$

The equality holds in (22) if and only if $\varphi(t) = a\psi(t)+bt+c$ $(\forall t\in[\alpha,\beta])$ for some $a,\ b,\ c\in\mathbb{C}$.

Proof. Let X be the maximal ideal space of the commutative Banach algebra $L^\infty(\Omega,\nu)$. Then X is a compact Hausdorff space and $L^\infty(\Omega,\nu)$ is isometrically isomorphic to $C(X)$, the Banach algebra of all complex-valued continuous functions on X (cf. [2, Theorem 4.3.1]). By the Riesz representation theorem (cf. [6, Theorem 2.14]), there exists a probability measure μ on X such that

$$\int_X \hat{g}\,d\mu = \int_\Omega g\,d\nu \qquad (\forall g\in L^\infty(\Omega,\nu)), \tag{23}$$

where $\hat{g}$ is the Gelfand transform of g. Note that μ has the following property:

$$\mu(G)\neq 0 \quad \text{for every non-empty open subset } G \text{ of } X. \tag{$*$}$$

In fact, if $\mu(G)=0$ for some non-empty open subset G of X, then take a non-zero positive function $h\in L^\infty(\Omega,\nu)$ so that $\operatorname{supp}\hat{h}\subset G$. By (23), we have $\int_\Omega h\,d\nu = \int_X \hat{h}\,d\mu = 0$, and so $h(w)=0$ for ν-a.e. $w\in\Omega$. This implies that $\hat{h}=0$ on X, which is impossible. Thus, μ satisfies $(*)$. Set $\hat{\alpha} = \operatorname{ess\,inf}_{x\in X}\hat{f}(x)$ and $\hat{\beta} = \operatorname{ess\,sup}_{x\in X}\hat{f}(x)$ with respect to μ. We will show that $\alpha=\hat{\alpha}$ and $\beta=\hat{\beta}$. Since $\alpha\leqslant f(w)$ for ν-a.e. $w\in\Omega$, we have $\alpha\leqslant\hat{f}(x)$ for all $x\in X$. That is, $\alpha\leqslant\hat{\alpha}$. Suppose that $\alpha<\hat{\alpha}$. There exists $\delta>0$ such that $\alpha+\delta<\hat{\alpha}$. By definition, we have $\alpha+\delta\leqslant\hat{f}(x)$ for μ-a.e. $x\in X$. It follows from $(*)$ that $\alpha+\delta\leqslant\hat{f}(x)$ for every $x\in X$, and so $\alpha+\delta\leqslant f(w)$ for ν-a.e. $w\in\Omega$, a contradiction. This implies that $\alpha=\hat{\alpha}$. In the same way, we see that $\beta=\hat{\beta}$. By Theorem 1.1, the inequality (2) holds for $\varphi(\hat{f})$, $\hat{f}$ and $\psi(\hat{f})$ instead of $\varphi(f)$, f and $\psi(f)$, respectively. By (23), we have (22). □

COROLLARY 2.2. *Let X be a compact Hausdorff space and μ a probability measure on X. Let $f\in C_{\mathbb{R}}(X)$ with $\alpha = \operatorname{ess\,inf}_{x\in X} f(x) > 0$ and $\beta = \operatorname{ess\,sup}_{x\in X} f(x)$ with respect to μ. Set, for each $p\in\mathbb{R}$,*

$$\psi_p(t) = \begin{cases} t^p & \text{if} \quad p\in\mathbb{R}\setminus\{0,1\} \\ \log t & \text{if} \quad p=0 \\ t\log t & \text{if} \quad p=1 \end{cases} \qquad (\forall t\in[\alpha,\beta]).$$

Let φ be a complex-valued C^2-function on $[\alpha,\beta]$. We define a mapping $\Phi\colon\mathbb{R}\to\mathbb{R}$ by

$$\Phi(p) = \max_{t\in[\alpha,\beta]}\left|\frac{\varphi''(t)}{\psi_p''(t)}\right| \left|\int_X \psi_p(f)\,d\mu - \psi_p\left(\int_X f\,d\mu\right)\right| \qquad (\forall p\in\mathbb{R}).$$

Then the following are true.

(i) Φ *is a continuous function on* $\mathbb{R}$ *such that*

$$\left|\int_X \varphi(f)\,d\mu - \varphi\left(\int_X f\,d\mu\right)\right| \leqslant \Phi(p) \qquad (\forall p \in \mathbb{R}). \tag{24}$$

(ii) For each $p \in \mathbb{R}$,

$$\Phi(2) \leqslant \left(\frac{\beta}{\alpha}\right)^{|2-p|} \Phi(p). \tag{25}$$

Proof. (i) Since $\alpha > 0$, we have, for each $p \in \mathbb{R}$, that ψ_p is a convex or concave function with $|\psi_p''(t)| > 0$ for every $t \in [\alpha, \beta]$. Thus, we see that, for each $p \in \mathbb{R}$, there exists a positive constant $A_p < \infty$ such that $|\varphi''(t)| \leqslant A_p|\psi_p''(t)|$ for every $t \in [\alpha, \beta]$. By Theorem 1.1, we have (24). It remains to be proved that Φ is continuous on $\mathbb{R}$. To do this, set, for each $p \in \mathbb{R}$,

$$\xi(p) = \max_{t\in[\alpha,\beta]} \frac{|\varphi''(t)|}{t^{p-2}} \qquad \text{and} \qquad \zeta(p) = \max\{\alpha^p, \beta^p, \alpha^{-p}, \beta^{-p}\}.$$

Then ζ is a continuous function on $\mathbb{R}$ with $\zeta(p) > 1$ for every $p \in \mathbb{R}$. Firstly, we show that ξ is continuous on $\mathbb{R}$. Take $p_0 \in \mathbb{R}$ arbitrarily. Note that

$$\xi(p) \leqslant \xi(p_0)\zeta(p_0 - p) \tag{26}$$

for every $p \in \mathbb{R}$. In fact, we have, for each $p \in \mathbb{R}$, that

$$\xi(p) = \max_{t\in[\alpha,\beta]} \frac{|\varphi''(t)|}{t^{p-2}} = \max_{t\in[\alpha,\beta]} \frac{|\varphi''(t)|}{t^{p_0-2}} t^{p_0-p} \leqslant \xi(p_0)\zeta(p_0 - p)$$

as required. This implies that

$$\xi(p) - \xi(p_0) \leqslant (\zeta(p_0 - p) - 1)\xi(p_0)$$

for every $p \in \mathbb{R}$. In the same way, we also have

$$\xi(p_0) - \xi(p) \leqslant (\zeta(p_0 - p) - 1)\xi(p),$$

which implies that

$$|\xi(p) - \xi(p_0)| \leqslant (\zeta(p_0 - p) - 1)\max\{\xi(p), \xi(p_0)\} \tag{27}$$

for every $p \in \mathbb{R}$. Take $\varepsilon > 0$ arbitrarily. Since ζ is continuous with $\zeta(0) = 1$, there exists $\delta > 0$ such that $0 < \zeta(p - p_0) - 1 < \varepsilon$ for every $p \in \mathbb{R}$ with $|p - p_0| \leqslant \delta$. By (27), we have, for each $p \in \mathbb{R}$ with $|p - p_0| \leqslant \delta$, that

$$|\xi(p) - \xi(p_0)| < \varepsilon \max\{\xi(p), \xi(p_0)\} \leqslant \varepsilon \sup_{q\in[p_0-\delta, p_0+\delta]} \xi(q).$$

Since ζ is continuous, it follows from (26) that

$$\sup_{q\in[p_0-\delta,p_0+\delta]} \xi(q) \leqslant \xi(p_0) \max_{q\in[p_0-\delta,p_0+\delta]} \zeta(p_0-q) < \infty.$$

This shows that ξ is continuous at p_0. Since $p_0 \in \mathbb{R}$ was arbitrary, we have that ξ is continuous on $\mathbb{R}$.

Set, for each $p \in \mathbb{R}$,

$$\eta(p) = \left| \int_X f^p \, d\mu - \left(\int_X f \, d\mu \right)^p \right|.$$

By the Lebesgue dominated convergent theorem, we see that the function $p \mapsto \int_X f^p \, d\mu$ $(p \in \mathbb{R})$ is continuous, and hence η is continuous on $\mathbb{R}$. Secondly, we show that Φ is continuous on $\mathbb{R} \setminus \{0,1\}$. For $p \in \mathbb{R} \setminus \{0,1\}$, we can write

$$\begin{aligned}\Phi(p) &= \frac{1}{|p(p-1)|} \max_{t\in[\alpha,\beta]} \frac{|\varphi''(t)|}{t^{p-2}} \left| \int_X f^p \, d\mu - \left(\int_X f \, d\mu \right)^p \right| \\ &= \frac{1}{|p(p-1)|} \xi(p)\eta(p) \end{aligned} \tag{28}$$

by definition. Since ξ and η are continuous, we have that Φ is continuous on $\mathbb{R} \setminus \{0,1\}$.

Thirdly, we will prove the continuity of Φ at $p=0$. Note, by definition, that

$$\Phi(0) = \max_{t\in[\alpha,\beta]} t^2|\varphi''(t)| \left| \int_X \log f \, d\mu - \log \int_X f \, d\mu \right|.$$

By the Lebesgue dominated convergent theorem, we have

$$\begin{aligned}\lim_{p\to 0} \frac{\eta(p)}{|p|} &= \lim_{p\to 0} \frac{1}{|p|} \left| \int_X f^p \, d\mu - \left(\int_X f \, d\mu \right)^p \right| \\ &= \lim_{p\to 0} \left| \int_X \frac{f^p - 1}{p} \, d\mu + \frac{1}{p} \left\{ 1 - \left(\int_X f \, d\mu \right)^p \right\} \right| \\ &= \left| \int_X \log f \, d\mu - \log \int_X f \, d\mu \right|.\end{aligned}$$

It follows from (28) that

$$\lim_{p\to 0} \Phi(p) = \lim_{p\to 0} \frac{\xi(p)}{|p-1|} \frac{\eta(p)}{|p|} = \xi(0) \lim_{p\to 0} \frac{\eta(p)}{|p|} = \Phi(0).$$

We thus conclude that Φ is continuous at $p=0$.

Finally, we show that Φ is continuous at $p=1$. In the same way as above, we see that

$$\lim_{p\to 1} \frac{\eta(p)}{|p-1|} = \left| \int_X f \log f \, d\mu - \int_X f \, d\mu \log \int_X f \, d\mu \right|$$

and that

$$\lim_{p\to 1}\Phi(p)=\lim_{p\to 1}\frac{\xi(p)}{|p|}\frac{\eta(p)}{|p-1|}=\xi(1)\lim_{p\to 1}\frac{\eta(p)}{|p-1|}=\Phi(1),$$

which proves the continuity of Φ at $p=1$.

(ii) Let $p\in\mathbb{R}\setminus\{0,1\}$. Taking $\varphi(t)=t^2$ in (24), we have

$$\left|\int_X f^2\,d\mu-\left(\int_X f\,d\mu\right)^2\right|\leqslant\max_{t\in[\alpha,\beta]}\frac{2t^{2-p}}{|p(p-1)|}\left|\int_X f^p\,d\mu-\left(\int_X f\,d\mu\right)^p\right|. \tag{29}$$

It follows that

$$\begin{aligned}\Phi(2)&=\max_{t\in[\alpha,\beta]}\frac{|\varphi''(t)|}{2}\left|\int_X f^2\,d\mu-\left(\int_X f\,d\mu\right)^2\right|\\&\leqslant\max_{t\in[\alpha,\beta]}\frac{|\varphi''(t)|}{2}\max_{t\in[\alpha,\beta]}\frac{2t^{2-p}}{|p(p-1)|}\left|\int_X f^p\,d\mu-\left(\int_X f\,d\mu\right)^p\right|\\&=\frac{1}{|p(p-1)|}\max_{t\in[\alpha,\beta]}|\varphi''(t)|\max_{t\in[\alpha,\beta]}t^{2-p}\left|\int_X f^p\,d\mu-\left(\int_X f\,d\mu\right)^p\right|.\end{aligned} \tag{30}$$

To prove (25), we will show that

$$\max_{t\in[\alpha,\beta]}|\varphi''(t)|\max_{t\in[\alpha,\beta]}t^{2-p}\leqslant\left(\frac{\beta}{\alpha}\right)^{|p-2|}\max_{t\in[\alpha,\beta]}\frac{|\varphi''(t)|}{t^{p-2}}. \tag{31}$$

Note that

$$\min\{\alpha^{2-p},\beta^{2-p}\}\max_{t\in[\alpha,\beta]}|\varphi''(t)|\leqslant\max_{t\in[\alpha,\beta]}\frac{|\varphi''(t)|}{t^{p-2}}. \tag{32}$$

In fact, for each $t\in[\alpha,\beta]$, we have

$$\min\{\alpha^{2-p},\beta^{2-p}\}\,|\varphi''(t)|\leqslant t^{2-p}\,|\varphi''(t)|=\frac{|\varphi''(t)|}{t^{p-2}},$$

which proves (32). It follows from (32) that

$$\begin{aligned}\max_{t\in[\alpha,\beta]}|\varphi''(t)|\max_{t\in[\alpha,\beta]}t^{2-p}&\leqslant\frac{1}{\min\{\alpha^{2-p},\beta^{2-p}\}}\max_{t\in[\alpha,\beta]}\frac{|\varphi''(t)|}{t^{p-2}}\max_{t\in[\alpha,\beta]}t^{2-p}\\&=\frac{\max\{\alpha^{2-p},\beta^{2-p}\}}{\min\{\alpha^{2-p},\beta^{2-p}\}}\max_{t\in[\alpha,\beta]}\frac{|\varphi''(t)|}{t^{p-2}}\\&=\left(\frac{\beta}{\alpha}\right)^{|2-p|}\max_{t\in[\alpha,\beta]}\frac{|\varphi''(t)|}{t^{p-2}}.\end{aligned}$$

This proves (31), as required. By (30), we have (25) for $p\in\mathbb{R}\setminus\{0,1\}$. Since Φ is continuous, we thus conclude that (25) holds for every $p\in\mathbb{R}$. □

REMARK 2.1. Let X, μ, f, α, β and φ be as in Corollary 2.2. Let ψ be a convex, or concave C^2-function on $[\alpha,\beta]$ with $|\psi''(t)|>0$ for every $t\in[\alpha,\beta]$. By Theorem 1.1, we have

$$\begin{aligned}&\left|\int_X \varphi(f)\,d\mu-\varphi\left(\int_X f\,d\mu\right)\right|\\&\qquad\leqslant \max_{t\in[\alpha,\beta]}|\varphi''(t)|\max_{t\in[\alpha,\beta]}\left|\frac{1}{\psi''(t)}\right|\left|\int_X \psi(f)\,d\mu-\psi\left(\int_X f\,d\mu\right)\right|.\end{aligned}\tag{33}$$

In particular, if we consider the case where $\psi(t)=t^2$, then we have

$$\left|\int_X \varphi(f)\,d\mu-\varphi\left(\int_X f\,d\mu\right)\right|\leqslant \max_{t\in[\alpha,\beta]}|\varphi''(t)|\frac{1}{2}\left|\int_X f^2\,d\mu-\left(\int_X f\,d\mu\right)^2\right|.\tag{34}$$

This is just a slight modification of Fink's estimate (1). On the other hand, taking $\varphi(t)=t^2$ in (33), we have

$$\frac{1}{2}\left|\int_X f^2\,d\mu-\left(\int_X f\,d\mu\right)^2\right|\leqslant \max_{t\in[\alpha,\beta]}\left|\frac{1}{\psi''(t)}\right|\left|\int_X \psi(f)\,d\mu-\psi\left(\int_X f\,d\mu\right)\right|.$$

Moreover, in the above inequality, the equality does not hold unless $\psi(t)=at^2+bt+c$ $(t\in[\alpha,\beta])$ for some $a,b,c\in\mathbb{C}$ by Theorem 1.1. This implies that one can not improve the right hand side of (34) by any ψ with $|\psi''|>0$.

COROLLARY 2.3. *Let $p,q\in\mathbb{R}$ and $x_i,\lambda_i>0$ for $i=1,2,\cdots,n$ with $\sum_{i=1}^n\lambda_i=1$. Set $\alpha=\min_{1\leqslant i\leqslant n}x_i$ and $\beta=\max_{1\leqslant i\leqslant n}x_i$. We have the following.*

$$\begin{aligned}&\left|\sum_{i=1}^n\lambda_i x_i{}^q-\left(\sum_{i=1}^n\lambda_i x_i\right)^q\right|\\&\leqslant\begin{cases}\left|\dfrac{q(q-1)}{p(p-1)}\right|\max\{\alpha^{q-p},\beta^{q-p}\}\left|\displaystyle\sum_{i=1}^n\lambda_i x_i{}^p-\left(\sum_{i=1}^n\lambda_i x_i\right)^p\right| & \text{if}\quad p\neq 0,1\\ |q(q-1)|\max\{\alpha^q,\beta^q\}\left(\log\displaystyle\sum_{i=1}^n\lambda_i x_i-\sum_{i=1}^n\lambda_i\log x_i\right)\\ |q(q-1)|\max\{\alpha^{q-1},\beta^{q-1}\}\left(\displaystyle\sum_{i=1}^n\lambda_i x_i\log x_i-\sum_{i=1}^n\lambda_i x_i\log\sum_{i=1}^n\lambda_i x_i\right),\end{cases}\end{aligned}\tag{35}$$

Proof. Let $\Omega=\{1,2,\cdots,n\}$ and $f(i)=x_i$ for every $i\in\Omega$. Set $\mu=\sum_{i=1}^n\lambda_i\delta_i$, where δ_i is the Dirac measure at i. If $\alpha=\beta$, then the both sides of (35) are 0, and so (35) holds. We consider the case where $\alpha<\beta$. Taking $\varphi(t)=t^q$ for $t\in[\alpha,\beta]$, we have, by (i) of Corollary 2.2, that (35) holds. $\square$

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