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A GENERALIZATION OF MUIRHEAD'S INEQUALITY

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Abstract. We give a proof of a generalization of Muirhead's Inequality and informally explain its application in establishing an instantial relevance principle in Polyadic Inductive Logic.

Introduction

The purpose of this short paper is to give a proof of a generalization of Muirhead's Inequality which is needed to establish a natural instantial relevance principle in Polyadic Inductive Logic.

We start by briefly explaining what this principle is in very informal terms. For a fuller account of the relevant area see [1], [2], [4], [5], [8], [9]. (The reader can of course, without technical loss, blithely skip this account and jump straight to the statement of Theorem 1.)

Consider an ostensibly 'rational' agent who will receive some classifying data concerning individuals $a_1, a_2, a_3, \ldots, a_q$. The agent wishes to assign a probability to the event that for some particular pairwise disjoint sets $H_1, H_2, \ldots, H_r$ (some possibly empty) with union $\{a_1, a_2, \ldots, a_q\}$ all the individuals in H_j will be identical as far as this data is concerned, for each $j = 1, 2, \ldots, r$. Assuming that the agent is otherwise without any prior knowledge, symmetry considerations[1] whose observance we would take to be a manifestation of 'rationality', suggest that this probability w should simply be a function of the multiset of numbers $n_1, n_2, \ldots, n_r$ where $|H_i| = n_i$, so

$$w = w(\{n_1, n_2, \ldots, n_r\}) = w(\langle n_1, n_2, \ldots, n_r \rangle)$$

where we replace the multiset with the vector $\langle n_1, n_2, \ldots, n_r \rangle$ of members in decreasing order, so we are assuming without loss of generality that $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r \geqslant 0$.

We now ask the question: Under what conditions *must* this agent (bound by the symmetry considerations mentioned above) set

$$w(\langle n_1, n_2, \ldots, n_r \rangle) \geqslant w(\langle m_1, m_2, \ldots, m_r \rangle) \ ?$$

where $\sum_i n_i = \sum_i m_i = q$ and $\vec{n}, \vec{m} \in \mathbb{N}^r$ are decreasing.

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[1] We do not give any details of these symmetry considerations here, see e.g. [9]

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It turns out that if the data simply consists of classifying individuals according to which of some finite set of *unary* relations (i.e. predicates) they do or do not satisfy then the answer to this question, see [10], is that the agent is forced into setting $w(\vec{n}) \geqslant w(\vec{m})$ just if $\vec{n} \succcurlyeq \vec{m}$, meaning that

$$\sum_{j\leqslant i} n_j \geqslant \sum_{j\leqslant i} m_j \quad \text{for all } i = 1,2,\ldots,r.$$

This can be proved by appealing to *Muirhead's Inequality*, see [7], [3, page 45], [6, page 87], which in its basic form asserts that

If $\vec{n},\vec{m} \in \mathbb{N}^r$ are decreasing, $\sum_{i=1}^r m_i = \sum_{i=1}^r n_i$, $\vec{n} \succcurlyeq \vec{m}$ and $0 \leqslant p_1,\ldots,p_r \in \mathbb{R}$ then

$$\sum_{\substack{\sigma \text{ a permutation} \\ \text{of } \{1,\ldots,r\}}} \prod_{j=1}^{r} p_{\sigma(j)}^{n_j} \quad \geqslant \quad \sum_{\substack{\sigma \text{ a permutation} \\ \text{of } \{1,\ldots,r\}}} \prod_{j=1}^{r} p_{\sigma(j)}^{m_j},$$

together with *de Finetti's Representation Theorem for Exchangeable Measures*, which tells us that it is enough (given the above mentioned symmetry considerations) to show this equivalence in a very simple case, namely when the function w is determined by a simple Bernoulli process.

The above question then is relatively easily answered when the data is expressed in terms of purely unary relations. Complications however arise when the data about a_i also relates it to other a_j, i.e. when the data involves relationships which are not purely unary. Nevertheless, an appropriate version of de Finetti's Representation Theorem has recently become available, see [5], and in consequence all that is required (in the main case) is to prove the following generalization of Muirhead's Theorem:

THEOREM 1. *If $\vec{n},\vec{m} \in \mathbb{N}^r$ are decreasing, $\sum_{i=1}^r m_i = \sum_{i=1}^r n_i$, $\vec{n} \succcurlyeq \vec{m}$ and $0 \leqslant p_1,\ldots,p_k \in \mathbb{R}$ then*

$$\sum_{\substack{\{S_1,\ldots,S_r\} \\ \text{a partition of} \\ \{1,\ldots,k\}}} \prod_{j=1}^{r} \left(\sum_{i\in S_j} p_i\right)^{\Box n_j} \geqslant \sum_{\substack{\{S_1,\ldots,S_r\} \\ \text{a partition of} \\ \{1,\ldots,k\}}} \prod_{j=1}^{r} \left(\sum_{i\in S_j} p_i\right)^{\Box m_j}, \tag{1}$$

where the $\Box$ in $(\sum_{i\in S_j} p_i)^{\Box m_j}$ etc. indicates that in the expansion of this power we only count those terms which have a non-zero power of p_i for each $i \in S_j$, etc., and in

$$\{S_1,\ldots,S_r\} \text{ a partition of } \{1,\ldots,k\}$$

we allow that some of the S_i may be empty.

Before proving the theorem we make several remarks.

For S and m such that $m < |S|$ we have $\left(\sum_{i\in S} p_i\right)^{\Box m} = 0$. Consequently, both sides of (1) are zero when $k > \sum_{i=1}^r m_i$.

As usual we take sums over the empty set to be 0, so

$$\left(\sum_{i\in\emptyset} p_i\right)^{\Box m} = \begin{cases} 1 \text{ if } m = 0, \\ 0 \text{ if } m > 0. \end{cases}$$

It follows that

$$\sum_{\substack{\{S_1,\ldots,S_r\} \\ a\ partition\ of \\ \{1,\ldots,k\}}} \prod_{j=1}^{r} \left(\sum_{i\in S_j} p_i \right)^{\square m_j} \tag{2}$$

must be zero when k is strictly less than the number of non-zero entries in $\vec{m}$, and a partition $S_1,\ldots,S_r$ can contribute to (2) only when $m_j \geqslant |S_j|$ for all $j = 1,\ldots,r$, and $m_j = 0$ for all j for which S_j is empty. Similarly for $\vec{n}$.

Provided that $\vec{n},\vec{m}$ have no coordinates equal to zero, if $k = r$ then (1) reduces to the basic Muirhead Inequality.

In the next section we prove a key lemma from which this inequality follows.

The Proof

LEMMA 2. *Let $m,n,k \in \mathbb{N}$, $n > m \geqslant 0$, $X = \{1,\ldots,k\}$, $0 \leqslant p_1,\ldots,p_k \in \mathbb{R}$ and*

$$P(n,m) = \sum_{Q\subseteq X} \left(\sum_{i\in Q} p_i \right)^{\square n} \left(\sum_{i\in X-Q} p_i \right)^{\square m}.$$

Then

$$P(n+1,m) \geqslant P(n,m+1). \tag{3}$$

Proof. We remark that the cases of $k < 2$ and $k > n+m+1$ are trivial.

Otherwise consider the terms $p_{i_1}p_{i_2}\cdots p_{i_{n+m+1}}$ which appear in the formal expansion of $P(n+1,m)$, where

$$p_{i_1}p_{i_2}\cdots p_{i_{n+1}} \in \left(\sum_{i\in Q} p_i \right)^{\square(n+1)}, \qquad p_{i_{n+2}}p_{i_{n+3}}\cdots p_{i_{n+m+1}} \in \left(\sum_{i\in X-Q} p_i \right)^{\square m}$$

for some $Q \subseteq X$, and where by the *formal expansion* we understand the sum of products of the p_i resulting from multiplying out

$$\underbrace{\left(\sum_{i\in Q} p_i \right) \times \cdots \times \left(\sum_{i\in Q} p_i \right)}_{(n+1)\ \text{times}} \times \underbrace{\left(\sum_{i\in X-Q} p_i \right) \times \cdots \times \left(\sum_{i\in X-Q} p_i \right)}_{m\ \text{times}},$$

keeping the p_i in order (so not collecting powers of the same p_i) and discarding the products that do not contain each p_i at least once. We can partition these terms according to the cases:

(a1) $i_{n+1} \notin \{i_1,i_2,\ldots,i_n\}$,

(a2 G,T) $i_{n+1} \in \{i_1,i_2,\ldots,i_n\}$, $G = \{i_{m+1},i_{m+2},\ldots,,i_n\} - \{i_1,i_2,\ldots,i_m\}$ and $T = \{\, j \,|\, m < j \leqslant n,\ i_j \in G\}$, for $G \subseteq X$, $T \subseteq \{m+1,m+2,\ldots,n\}$.

Similarly by considering the terms $p_{i_1}p_{i_2}\cdots p_{i_{n+m+1}}$ in the formal expansion of $P(n,m+1)$, where

$$p_{i_1}p_{i_2}\cdots p_{i_n}\in\left(\sum_{i\in Q}p_i\right)^{\square n},\quad p_{i_{n+1}}p_{i_{n+2}}\cdots p_{i_{n+m+1}}\in\left(\sum_{i\in X-Q}p_i\right)^{\square(m+1)}$$

for some $Q\subseteq X$, we see that we can partition them according to the cases:

(b1) $i_{n+1}\notin\{i_{n+2},i_{n+3},\dots,i_{n+m+1}\}$,

(b2 G,T) $i_{n+1}\in\{i_{n+2},i_{n+3},\dots,i_{n+m+1}\}$, $G=\{i_{m+1},i_{m+2},\dots,,i_n\}-\{i_1,i_2,\dots,i_m\}$ and $T=\{j\,|\,m<j\leqslant n,\ i_j\in G\}$, for $G\subseteq X,\ T\subseteq\{m+1,m+2,\dots,n\}$.

We now compare $P(n+1,m)$ and $P(n,m+1)$ according to this partition. The terms from cases (a1), (b1) clearly cancel out. Now fix G and T. Then the contribution from case (a2 G,T) to $P(n+1,m)$ is

$$\sum_{Q\subseteq X-G}\left(\sum_{i\in Q}p_i\right)^{\square m}\left(\sum_{i\in G}p_i\right)^{\square|T|}\left(\sum_{i\in Q}p_i\right)^{n-m-|T|}\left(\sum_{i\in Q\cup G}p_i\right)\left(\sum_{i\in X-Q-G}p_i\right)^{\square m}\tag{4}$$

whilst the contribution from case (b2 G,T) to $P(n,m+1)$ is

$$\sum_{Q\subseteq X-G}\left(\sum_{i\in Q}p_i\right)^{\square m}\left(\sum_{i\in G}p_i\right)^{\square|T|}\left(\sum_{i\in Q}p_i\right)^{n-m-|T|}\left(\sum_{i\in X-G-Q}p_i\right)\left(\sum_{i\in X-G-Q}p_i\right)^{\square m}.\tag{5}$$

Taking the difference (4) − (5) gives

$$\sum_{Q\subseteq X-G}\left(\sum_{i\in Q}p_i\right)^{\square m}\left(\sum_{i\in G}p_i\right)^{\square|T|}\left(\sum_{i\in Q}p_i\right)^{n-m-|T|}\left(\sum_{i\in G}p_i\right)\left(\sum_{i\in X-G-Q}p_i\right)^{\square m},$$

which is non-negative, together with

$$\sum_{Q\subseteq X-G}\left(\sum_{i\in Q}p_i\right)^{\square m}\left(\sum_{i\in G}p_i\right)^{\square|T|}\left(\sum_{i\in X-G-Q}p_i\right)^{\square m}\times$$
$$\left\{\left(\sum_{i\in Q}p_i\right)^{n+1-m-|T|}-\left(\sum_{i\in Q}p_i\right)^{n-m-|T|}\left(\sum_{i\in X-G-Q}p_i\right)\right\}.\tag{6}$$

Summing over $Q\subseteq X-G$ in (6) is the same as summing over the complements Q' of Q in $X-G$, so (6) equals

$$\sum_{Q'\subseteq X-G}\left(\sum_{i\in X-G-Q'}p_i\right)^{\square m}\left(\sum_{i\in G}p_i\right)^{\square|T|}\left(\sum_{i\in Q'}p_i\right)^{\square m}\times$$

$$\left\{\left(\sum_{i\in X-G-Q'} p_i\right)^{n+1-m-|T|} - \left(\sum_{i\in X-G-Q'} p_i\right)^{n-m-|T|}\left(\sum_{i\in Q'} p_i\right)\right\}. \qquad (7)$$

Consequently, it is enough to show that the sum of (6) and (7) is non-negative. Writing Q in place of Q' in (7) we see that the sum of (6) and (7) is

$$\sum_{Q\subseteq X-G}\left(\sum_{i\in Q} p_i\right)^{\square m}\left(\sum_{i\in G} p_i\right)^{\square |T|}\left(\sum_{i\in X-G-Q} p_i\right)^{\square m}\times$$

$$\left(A^{n+1-m-|T|}-A^{n-m-|T|}B+B^{n+1-m-|T|}-B^{n-m-|T|}A\right),$$

where $A=\sum_{i\in Q} p_i$, $B=\sum_{i\in X-G-Q} p_i$, which is non-negative since

$$\left(A^{n+1-m-|T|}-A^{n-m-|T|}B+B^{n+1-m-|T|}-B^{n-m-|T|}A\right)$$
$$=\left(A^{n-m-|T|}-B^{n-m-|T|}\right)(A-B)\geqslant 0. \quad \square$$

COROLLARY 3. *Let* $\vec{n},\vec{m}\in\mathbb{N}^r$ *be decreasing and* $1\leqslant i<j\leqslant r$. *Suppose that* $n_l=m_l$ *for* $l\neq i,j,\ 1\leqslant l\leqslant r,\ m_i=n_i-1\geqslant m_j=n_j+1$. *Then for* $0\leqslant p_1,\ldots,p_k\in\mathbb{R}$,

$$\sum_{\substack{\{S_1,\ldots,S_r\}\\ \text{a partition of}\\ \{1,\ldots,k\}}}\prod_{j=1}^{r}\left(\sum_{i\in S_j} p_i\right)^{\square n_j}\geqslant\sum_{\substack{\{S_1,\ldots,S_r\}\\ \text{a partition of}\\ \{1,\ldots,k\}}}\prod_{j=1}^{r}\left(\sum_{i\in S_j} p_i\right)^{\square m_j}.$$

Proof. Simply consider this inequality in the case where we fix the S_l for $l\neq i,j$ and set $S_i\cup S_j=X$ and $S_i=Q$ as in Lemma 2. $\square$

For $\vec{n},\vec{m}$ as in the statement of Corollary 3 we say that $\vec{m}$ is an immediate $\preccurlyeq$-predecessor of $\vec{n}$.

Proof of Theorem 1. By a version of Muirhead's lemma (see [7], alternatively see [10, Theorem 2], and [11]) it follows that for $\vec{n},\vec{m}\in\mathbb{N}^r$ decreasing with the same sum, $\vec{m}\preccurlyeq\vec{n}$ just if there is a finite sequence of decreasing vectors $\vec{n}^1,\vec{n}^2,\ldots,\vec{n}^s$ each with this same sum such that $\vec{n}^1=\vec{m}$, $\vec{n}^s=\vec{n}$ and for $i=1,\ldots,s-1$ $\vec{n}^i$ is an immediate $\preccurlyeq$-predecessor of $\vec{n}^{i+1}$. The result now follows by Corollary 3. $\square$

Having derived Lemma 2 there seems to be some interest in giving an equivalent version which avoids the use of the $\square$ and at the same time provides an inequality of an (apparently) previously unstudied form.

COROLLARY 4. *Let* $0\leqslant p_1,\ldots,p_k\in\mathbb{R}$ *and* $n\geqslant m\geqslant 0$. *Then*

$$\sum_{\substack{V,T\subseteq\{1,2,\ldots,k\}\\ V\cap T=\emptyset}}(-2)^{k-|V\cup T|}\left(\sum_{i\in V} p_i\right)^{n}\left(\sum_{i\in T} p_i\right)^{m}\left(\sum_{i\in V} p_i-\sum_{i\in T} p_i\right)\geqslant 0.$$

Proof. By induction on $|Q|$ we can show that for $Q \subseteq X = \{1,2,\dots,k\}$ we have

$$\left(\sum_{i\in Q} p_i\right)^{\Box n} = \sum_{V\subseteq Q} (-1)^{|Q|-|V|}\left(\sum_{i\in V} p_i\right)^{n}. \tag{8}$$

In more detail this is clear for $Q=\emptyset$ and if it holds for all proper subsets of Q then

$$\begin{aligned}
\left(\sum_{i\in Q} p_i\right)^{\Box n} &= \left(\sum_{i\in Q} p_i\right)^{n} - \sum_{Z\subset Q}\left(\sum_{i\in Z} p_i\right)^{\Box n}\\
&= \left(\sum_{i\in Q} p_i\right)^{n} - \sum_{Z\subset Q}\sum_{V\subseteq Z}(-1)^{|Z|-|V|}\left(\sum_{i\in V} p_i\right)^{n}\\
&= \left(\sum_{i\in Q} p_i\right)^{n} - \sum_{V\subset Q}\sum_{V\subseteq Z\subset Q}(-1)^{|Z|-|V|}\left(\sum_{i\in V} p_i\right)^{n}\\
&= \left(\sum_{i\in Q} p_i\right)^{n} + \sum_{V\subset Q}(-1)^{|Q|-|V|}\left(\sum_{i\in V} p_i\right)^{n}\\
&= \sum_{V\subseteq Q}(-1)^{|Q|-|V|}\left(\sum_{i\in V} p_i\right)^{n}
\end{aligned}$$

since for $V\subset Q$,

$$0=\sum_{V\subseteq Z\subseteq Q}(-1)^{|Z|-|V|} = (-1)^{|Q|-|V|} + \sum_{V\subseteq Z\subset Q}(-1)^{|Z|-|V|}.$$

Hence

$$\begin{aligned}
&\sum_{Q\subseteq X}\left(\sum_{i\in Q} p_i\right)^{\Box n}\left(\sum_{j\in X-Q} p_j\right)^{\Box m}\\
&\quad= \sum_{Q\subseteq X}\left(\sum_{V\subseteq Q}\left(\sum_{i\in V} p_i\right)^{n}(-1)^{|Q|-|V|}\right)\left(\sum_{T\subseteq X-Q}\left(\sum_{j\in T} p_j\right)^{m}(-1)^{k-|Q|-|T|}\right)\\
&\quad= \sum_{\substack{V,T\subseteq X\\ V\cap T=\emptyset}}\left(\sum_{i\in V} p_1\right)^{n}\left(\sum_{j\in T} p_j\right)^{m}\sum_{\substack{V\subseteq Q\\ T\subseteq X-Q}}(-1)^{|Q|-|V|+k-|Q|-|T|}\\
&\quad= \sum_{\substack{V.T\subseteq X\\ V\cap T=\emptyset}} 2^{k-|V|-|T|}(-1)^{k-|V|-|T|}\left(\sum_{i\in V} p_i\right)^{n}\left(\sum_{j\in T} p_j\right)^{m}.
\end{aligned}$$

It follows that the inequality $P(n+1,m)\geqslant P(n,m+1)$ is equivalent to

$$\sum_{\substack{V,T\subseteq X\\ V\cap T=\emptyset}}(-2)^{k-|V|-|T|}\left(\sum_{i\in V} p_i\right)^{n+1}\left(\sum_{j\in T} p_j\right)^{m}$$

$$\geqslant \sum_{\substack{V,T\subseteq X\\ V\cap T=\emptyset}} (-2)^{k-|V|-|T|}\left(\sum_{i\in V} p_i\right)^n \left(\sum_{j\in T} p_j\right)^{m+1}.$$

and the required result follows. □

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