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INEQUALITIES CORRESPONDING TO THE CLASSICAL JENSEN'S INEQUALITY

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Abstract. In this paper some integral inequalities are proved in probability spaces, which go back to some discrete variants of the Jensen's inequality. Especially, we refine the classical Jensen's inequality. Convergence results corresponding to the inequalities are also studied.

1. Introduction and the main results

The classical Jensen's inequality says (see [3]):

THEOREM A. *Let $I \subset R$ be an interval, and let $q : I \to R$ be a convex function on I. Let (X, A, μ) be a probability space, and let $f : X \to I$ be a μ-integrable function over X. Then $\int_X f d\mu \in I$. If $q \circ f$ is μ-integrable over X, then*

$$q\left(\int_X f d\mu\right) \leqslant \int_X q \circ f d\mu. \tag{1}$$

The following discrete Jensen's inequality is also well known (see [5]).

THEOREM B. *Let C be a convex subset of a real vector space V, and let $q : C \to \mathbb{R}$ be a convex function. If $p_1, \ldots, p_k$ are nonnegative numbers with $p_1 + \ldots + p_k = 1$, and $v_1, \ldots, v_k \in C$, then*

$$q\left(\sum_{i=1}^{k} p_i v_i\right) \leqslant \sum_{i=1}^{k} p_i q(v_i).$$

Generalizations of these inequalities have been investigated by many authors, and they have important applications.

The following refinements of Theorem B are proved in [4] and in [2], respectively.

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THEOREM C. *Let C be a convex subset of a real vector space V, and let $q: C \to \mathbb{R}$ be a convex function. If $r_1, \ldots, r_k$ are nonnegative numbers with $r_1 + \ldots + r_k = 1$, and $v_1, \ldots, v_k \in C$, then*

$$q\left(\sum_{i=1}^{k} r_i v_i\right) \leqslant \sum_{i_1,\ldots,i_{n+1}=1}^{k} r_{i_1} \ldots r_{i_{n+1}} q\left(\frac{v_{i_1} + \ldots + v_{i_{n+1}}}{n+1}\right) \tag{2}$$

$$\leqslant \sum_{i_1,\ldots,i_n=1}^{k} r_{i_1} \ldots r_{i_n} q\left(\frac{v_{i_1} + \ldots + v_{i_n}}{n}\right) \leqslant \sum_{i=1}^{k} r_i q(v_i), \quad n \geqslant 1.$$

THEOREM D. *Let C be a convex subset of a real vector space V, and let $q: C \to \mathbb{R}$ be a convex function. Let $r_1, \ldots, r_k$ be nonnegative numbers with $r_1 + \ldots + r_k = 1$, and let $v_1, \ldots, v_k \in C$. If $p_1, \ldots, p_n$ are nonnegative numbers with $p_1 + \ldots + p_n = 1$, then*

$$q\left(\sum_{i=1}^{k} r_i v_i\right) \leqslant \sum_{i_1,\ldots,i_n=1}^{k} r_{i_1} \ldots r_{i_n} q\left(\frac{v_{i_1} + \ldots + v_{i_n}}{n}\right) \tag{3}$$

$$\leqslant \sum_{i_1,\ldots,i_n=1}^{k} r_{i_1} \ldots r_{i_n} q\left(p_1 v_{i_1} + \ldots p_n v_{i_n}\right) \leqslant \sum_{i=1}^{k} r_i q(v_i), \quad 1 \leqslant n \leqslant k.$$

Inspired by (2) and (3), the aim of this paper is to establish some new inequalities in a measure theoretical setting. We have some refinements of (1) from the results.

Let $\mathbb{N} := \{1, 2, \ldots\}$. Let the index set T be either $\{1, \ldots, n\}$ with $n \in \mathbb{N}$, or $\mathbb{N}$. Suppose we are given a family $\{A_i \mid i \in T\}$ of non-empty sets. If S is a non-empty subset of T, then the projection mapping $pr_S^T : \underset{i \in T}{\times} A_i \to \underset{i \in S}{\times} A_i$ is defined by associating with every point of $\underset{i \in T}{\times} A_i$ its restriction to S. We write for short pr_i^n for $pr_{\{i\}}^T$ $(i \in T = \{1, \ldots, n\})$ and pr_i^∞ for $pr_{\{i\}}^T$ $(i \in T = \mathbb{N})$. Similarly, if $T = \mathbb{N}$, then $pr_{1\ldots k}^\infty$ means $pr_{\{1,\ldots,k\}}^T$ $(k \in \mathbb{N})$.

Although our notation and terminology is mostly standard, we recall a few facts from measure and integration theory (we refer to [3]).

Let $(Y_i, \mathscr{B}_i, \nu_i)$, $i \in T$ be probability spaces, where either $T := \{1, \ldots, n\}$ with $n \in \mathbb{N}$, or $T := \mathbb{N}$. The product of these spaces is denoted by $(Y^T, \mathscr{B}^T, \nu^T)$. This means that $Y^T := \underset{i \in T}{\times} Y_i$, and $\mathscr{B}^T$ is the smallest σ-algebra in Y such that each $pr_{\{i\}}^T$ is $\mathscr{B}^T - \mathscr{B}_i$ measurable $(i \in T)$. If $T = \{1, \ldots, n\}$, then ν^T is the only measure on $\mathscr{B}^T$ which satisfies

$$\nu^T (B_1 \times \ldots \times B_n) = \nu_1(B_1) \ldots \nu_n(B_n)$$

for every $B_i \in \mathscr{B}_i$. If $T = \mathbb{N}$, then ν^T is the unique measure on $\mathscr{B}^T$ such that the image measure of ν^T under the projection mapping $pr_{1\ldots k}^\infty$ is the product of the measures $\nu_1, \ldots, \nu_k$ $(k \in \mathbb{N})$.

It is obvious that $(Y^T, \mathscr{B}^T, \nu^T)$ is also a probability space.

The n-fold ($n \geqslant 1$ or $n = \infty$) product of the probability space $(X, \mathscr{A}, \mu)$ is denoted by $(X^n, \mathscr{A}^n, \mu^n)$. We suppose that the μ-integrability of a function $g: X \to \mathbb{R}$ over X implies the measurability of g.

Our first result corresponds to (3).

THEOREM 1.1. *Let $I \subset \mathbb{R}$ be an interval, and let $q : I \to \mathbb{R}$ be a convex function on I. Let $(Y_i, \mathscr{B}_i, \nu_i)$, $i \in T := \{1, \ldots, n\}$ be probability spaces, and let $f_i : Y_i \to I$ be a ν_i-integrable function over Y_i $(i = 1, \ldots, n)$. Suppose that $p_1, \ldots, p_n$ are nonnegative numbers with $p_1 + \ldots + p_n = 1$. If $q \circ f_i$ is ν_i-integrable over Y_i $(i = 1, \ldots, n)$, then*

$$q\left(\sum_{i=1}^{n} p_i \int_{Y_i} f_i d\nu_i\right) \leqslant \int_{Y^T} q\left(\sum_{i=1}^{n} p_i f_i(y_i)\right) d\nu^T(y_1, \ldots, y_n) \leqslant \sum_{i=1}^{n} p_i \int_{Y_i} q \circ f_i d\nu_i. \tag{4}$$

The next result concerns the asymptotic behaviour of the sequence

$$\int_{Y^{\{1,\ldots,n\}}} q\left(\frac{1}{n}\sum_{i=1}^{n} f_i(y_i)\right) d\nu^{\{1,\ldots,n\}}(y_1, \ldots, y_n), \quad n \in \mathbb{N}, \tag{5}$$

in some cases. (5) corresponds to the middle member of (4).

THEOREM 1.2. *Let $I \subset \mathbb{R}$ be an interval, and let $q : I \to \mathbb{R}$ be a convex and bounded function on I. Let $(Y_i, \mathscr{B}_i, \nu_i)$, $i \in \mathbb{N}$ be probability spaces, and let $f_i : Y_i \to I$ be a square ν_i-integrable function over Y_i $(i \in \mathbb{N})$ such that*

$$\int_{Y_i} f_i d\nu_i = \int_{Y_1} f_1 d\nu_1, \quad i \in \mathbb{N},$$

and

$$\sum_{i=1}^{\infty} \frac{1}{i^2} \int_{Y_i} f_i^2 d\nu_i < \infty.$$

Then

$$\lim_{n\to\infty} \int_{Y^{\{1,\ldots,n\}}} q\left(\frac{1}{n}\sum_{i=1}^{n} f_i(y_i)\right) d\nu^{\{1,\ldots,n\}}(y_1, \ldots, y_n) = q\left(\int_{Y_1} f_1 d\nu_1\right).$$

The third main result refines (1), and corresponds to (2) and (3).

THEOREM 1.3. *Let $I \subset \mathbb{R}$ be an interval, and let $q : I \to \mathbb{R}$ be a convex function on I. Let $(X, \mathscr{A}, \mu)$ be a probability space, and let $f : X \to I$ be a μ-integrable function over X such that $q \circ f$ is also μ-integrable over X. Suppose that $p_1, \ldots, p_n$ are nonnegative numbers with $p_1 + \ldots + p_n = 1$. Then*

(a)

$$q\left(\int_X f d\mu\right) \leqslant \int_{X^n} q\left(\sum_{i=1}^{n} p_i f(x_i)\right) d\mu^n(x_1, \ldots, x_n) \leqslant \int_X q \circ f d\mu. \tag{6}$$

(b)

$$\int\limits_{X^{n+1}} q\left(\frac{1}{n+1}\sum_{i=1}^{n+1} f(x_i)\right) d\mu^{n+1}(x_1,\dots,x_{n+1}) \tag{7}$$
$$\leqslant \int\limits_{X^n} q\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) d\mu^{n}(x_1,\dots,x_n) \leqslant \int\limits_{X^n} q\left(\sum_{i=1}^{n} p_i f(x_i)\right) d\mu^{n}(x_1,\dots,x_n).$$

(c) If q is bounded, then

$$\lim_{n\to\infty}\int\limits_{X^n} q\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) d\mu^{n}(x_1,\dots,x_n) = q\left(\int\limits_{X} f d\mu\right). \tag{8}$$

Suppose $V := \mathbb{R}$ and $C := I$ in Theorem C and Theorem D. Then the inequalities (2) and (3) can be obtained easily from the inequalities (4)–(7), but (4)–(7) are more general.

As an application, we study some discrete inequalities.

2. Preliminaries

In order to prove the main results as transparent as possible, we begin some preparatory lemmas.

First, we investigate the integrability properties of some functions.

LEMMA 2.1. *Let $I \subset \mathbb{R}$ be an interval, and let $q : I \to \mathbb{R}$ be a convex function on I. Let $(Y_i, \mathscr{B}_i, \nu_i)$, $i \in T := \{1,\dots,n\}$ be probability spaces, and let the function $f_i : Y_i \to I$ be ν_i-integrable over Y_i $(i = 1,\dots,n)$.*

(a) The function $f_i \circ pr_i^n$ is ν^T-integrable over Y^T, and

$$\int\limits_{Y^T} f_i \circ pr_i^n d\nu^T = \int\limits_{Y_i} f_i d\nu_i, \quad i = 1,\dots,n. \tag{9}$$

(b) If $q \circ f_i$ is also ν_i-integrable over Y_i $(i = 1,\dots,n)$, then the function

$$q \circ \left(\sum_{i=1}^{n} p_i (f_i \circ pr_i^n)\right), \quad \textit{where} \quad p_i \geqslant 0 \quad (i = 1,\dots,n), \quad \sum_{i=1}^{n} p_i = 1 \tag{10}$$

is ν^T-integrable over Y^T.

Proof. (a) Since the image measure of ν^T under the mapping pr_i^n is ν_i $(i = 1,\dots,n)$, and f_i is ν_i-integrable over Y_i $(i = 1,\dots,n)$, we therefore get from the general transformation theorem for integrals (see [3]) that $f_i \circ pr_i^n$ is ν^T-integrable over Y^T $(i = 1,\dots,n)$, and (9) holds.

(b) Since the range of f_i is a subset of I $(i = 1,\ldots,n)$, the properties of the numbers $p_1,\ldots,p_n$ in (10) imply that the range of the function

$$\sum_{i=1}^{n} p_i \left(f_i \circ pr_i^n\right)$$

is a subset of I too. Thus the domain of the function (10) is Y^T.

The function q, being convex on I, is lower semicontinuous on I, and therefore q is measurable on I.

The measurability of the function (10) now follows from the above statements.

Let a be a fixed interior point of I. Now the convexity of q on I insures that

$$q(t) \geqslant q(a) + q'_+(a)(t-a), \quad t \in I,$$

where $q'_+(a)$ denotes the right-hand derivative of q at a. Using the previous inequality, and the convexity of q again, we have

$$\begin{aligned} q(a) + q'_+(a)\left(\sum_{i=1}^{n} p_i f_i(y_i) - a\right) &\leqslant q\left(\sum_{i=1}^{n} p_i f_i(y_i)\right) \\ &\leqslant \sum_{i=1}^{n} p_i q(f_i(y_i)), \quad (y_1,\ldots,y_n) \in Y^T. \end{aligned} \tag{11}$$

By what has already been proved in (a) the lower bound for the function (10) in (11) is ν^T-integrable over Y^T. The condition on $q \circ f_i$ ensures that f_i can be replaced by $q \circ f_i$ in (a), and thus the upper bound for the function (10) in (11) is ν^T-integrable over Y^T too. These and the inequality (11), together with the measurability of the function (10) on Y^T imply the ν^T-integrability of the function (10) on Y^T.

The proof is now complete. □

We derive an analog of Lemma 2.1 for a sequence of probability spaces.

LEMMA 2.2. *Let $I \subset \mathbb{R}$ be a bounded interval, and let $q : I \to \mathbb{R}$ be a convex and bounded function on I. Let $(Y_i, \mathscr{B}_i, \nu_i)$, $i \in T := \mathbb{N}$ be probability spaces, and let the function $f_i : Y_i \to I$ be ν_i-integrable over Y_i $(i \in \mathbb{N})$.*

(a) The function $f_i \circ pr_i^\infty$ is ν^T-integrable over Y^T, and

$$\int_{Y^T} f_i \circ pr_i^\infty d\nu^T = \int_{Y_i} f_i d\nu_i, \quad i \in \mathbb{N}.$$

(b) If $q \circ f_i$ is also ν_i-integrable over Y_i $(i \in \mathbb{N})$, then for every $n \in \mathbb{N}$

$$\int_{Y^T} q \circ \left(\frac{1}{n}\sum_{i=1}^{n} f_i \circ pr_i^\infty\right) d\nu^T = \int_{Y^{\{1,\ldots,n\}}} q \circ \left(\frac{1}{n}\sum_{i=1}^{n} f_i \circ pr_i^n\right) d\nu^{\{1,\ldots,n\}}. \tag{12}$$

Proof. (a) We argue as in the proof of Lemma 2.1 (a).

(b) According to Lemma 2.1 (b), the function

$$q \circ \left(\frac{1}{n} \sum_{i=1}^{n} f_i \circ pr_i^n \right)$$

is $\nu^{\{1,\dots,n\}}$-integrable over $Y^{\{1,\dots,n\}}$ $(n \in \mathbb{N})$. By the definition of ν^T, the image measure of ν^T under the mapping $pr_{1\dots n}^{\infty}$ is $\nu^{\{1,\dots,n\}}$ $(n \in \mathbb{N})$. Therefore the general transformation theorem for integrals gives (12).

The result is completely proved. □

The next result is simple to prove but useful.

LEMMA 2.3. *Let $(X, \mathscr{A}, \mu)$ be a probability space, and let $f : X^n \to \mathbb{R}$ be a μ^n-integrable function.*

(a) Let π be a permutation of the numbers $1, \dots, n$, and let the mapping $T : X^n \to X^n$ be defined by

$$T(x_1, \dots, x_n) := (x_{\pi(1)}, \dots, x_{\pi(n)}).$$

Then the function $f \circ T$ is μ^n-integrable on X^n and

$$\int_{X^n} f \circ T d\mu^n = \int_{X^n} f d\mu^n.$$

(b) If $1 \leqslant i < n$, then

$$\int_{X^n} f d\mu^n = \int_{X^{n-1}} \left(\int_X f(x_1, \dots, x_n) d\mu(x_i) \right) d\mu^{n-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

Proof. (a) Since the image of μ^n under the mapping T is μ^n, the result follows from the general transformation theorem for integrals.

(b) We have only to apply (a) and the Fubini's theorem. □

The last result that we discuss corresponds to the laws of large numbers.

LEMMA 2.4. *Let $I \subset \mathbb{R}$ be an interval, and let $p : I \to \mathbb{R}$ be a function on I, which is continuous at every interior point of I, and bounded on I.*

(a) Let $(Y_i, \mathscr{B}_i, \nu_i)$, $i \in T := \mathbb{N}$ be probability spaces, and let $f_i : Y_i \to I$ be a square ν_i-integrable function over Y_i $(i \in \mathbb{N})$ such that

$$\int_{Y_i} f_i d\nu_i = \int_{Y_1} f_1 d\nu_1, \quad i \in \mathbb{N},$$

and

$$\sum_{i=1}^{\infty} \frac{1}{i^2} \int_{Y_i} f_i^2 d\nu_i < \infty. \tag{13}$$

Then

$$\lim_{n\to\infty} \int\limits_{Y^{\{1,\dots,n\}}} p\left(\frac{1}{n}\sum_{i=1}^{n} f_i(y_i)\right) d\nu^{\{1,\dots,n\}}(y_1,\dots,y_n) = p\left(\int\limits_{Y_1} f_1 d\nu_1\right).$$

(b) Let $(X,\mathscr{A},\mu)$ be a probability space, and let $f : X \to I$ be a μ-integrable function over X. Then

$$\lim_{n\to\infty} \int\limits_{X^n} p\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) d\mu^n(x_1,\dots,x_n) = p\left(\int\limits_{X} f d\mu\right).$$

Proof. (a) Since f_i is a square ν_i-integrable function over Y_i, f_i is ν_i-integrable over Y_i $(i \in \mathbb{N})$. From Lemma 2.2 (a) we obtain that $f_i \circ pr_i^\infty$ and $f_i^2 \circ pr_i^\infty$ $(i \in \mathbb{N})$ are ν^T-integrable over Y^T, and

$$\int\limits_{Y^T} f_i \circ pr_i^\infty d\nu^T = \int\limits_{Y_i} f_i d\nu_i = \int\limits_{Y_1} f_1 d\nu_1, \tag{14}$$

and

$$\int\limits_{Y^T} f_i^2 \circ pr_i^\infty d\nu^T = \int\limits_{Y_i} f_i^2 d\nu_i. \tag{15}$$

If $V(f_i \circ pr_i^\infty)$ means the variance of the random variable $f_i \circ pr_i^\infty$ $(i \in \mathbb{N})$, then by using (14), (15) and (13), we have that

$$\sum_{i=1}^{\infty} \frac{V(f_i \circ pr_i^\infty)}{i^2} = \sum_{i=1}^{\infty} \frac{1}{i^2}\left(\int\limits_{Y_i} f_i^2 d\nu_i - \left(\int\limits_{Y_1} f_1 d\nu_1\right)^2\right) < \infty.$$

It is easy to verify that the random variables $f_i \circ pr_i^\infty$ $(i \in \mathbb{N})$ are independent. Now, Kolmogorov's criterion (see [1]) implies that the sequence $(f_i \circ pr_i^\infty)_{i\in\mathbb{N}}$ of random variables obeys the strong law of large numbers, that is

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f_i \circ pr_i^\infty = \int\limits_{Y_1} f_1 d\nu_1 \quad \nu^T\text{-almost everywhere on } Y^T. \tag{16}$$

If $\int\limits_{Y_1} f_1 d\nu_1$ is an interior point of I, then the continuity of p at $\int\limits_{Y_1} f_1 d\nu_1$ and (16) yield that

$$\lim_{n\to\infty} p \circ \left(\frac{1}{n}\sum_{i=1}^{n} f_i \circ pr_i^\infty\right) = p\left(\int\limits_{Y_1} f_1 d\nu_1\right) \quad \nu^T\text{-almost everywhere on } Y^T. \tag{17}$$

Suppose $\int\limits_{Y_1} f_1 d\nu_1$ is either the left-hand endpoint or the right-hand endpoint of I. In either case

$$f_i = \int\limits_{Y_1} f_1 d\nu_1 \quad \nu_i\text{-almost everywhere on } Y_i \quad i \in \mathbb{N},$$

hence

$$f_i \circ pr_i^\infty = \int\limits_{Y_1} f_1 d\nu_1 \quad \nu^T\text{-almost everywhere on } Y^T, \quad i \in \mathbb{N},$$

and this justifies (17).

Since p is bounded on I, it follows from (17) and Lebesgue's convergence theorem that

$$\lim_{n\to\infty} \int\limits_{Y^T} p \circ \left(\frac{1}{n} \sum_{i=1}^n f_i \circ pr_i^\infty \right) d\nu^T = p \left(\int\limits_{Y_1} f_1 d\nu_1 \right),$$

thus we can apply Lemma 2.2 (b).

(b) In this case the proof of (a) also works if instead of Kolmogorov's criterion Kolmogorov's law of large numbers (see [1]) is used, since the random variables $f \circ pr_i^\infty$ $(i \in \mathbb{N})$ are independent, ν^T-integrable over Y^T, and identically distributed.

The whole theorem is proved. □

3. Proofs of the main results

Let us now prove Theorem 1.1.

Proof. By Lemma 2.1 (a) and (b), the functions

$$\sum_{i=1}^n p_i (f_i \circ pr_i^n) \quad \text{and} \quad q \circ \left(\sum_{i=1}^n p_i (f_i \circ pr_i^n) \right)$$

are ν^T-integrable over Y^T. An application of Theorem A now yields that

$$q \left(\int\limits_{Y^T} \left(\sum_{i=1}^n p_i (f_i \circ pr_i^n) \right) d\nu^T \right) \leqslant \int\limits_{Y^T} \left(q \circ \left(\sum_{i=1}^n p_i (f_i \circ pr_i^n) \right) d\nu^T \right).$$

The first inequality in (4) follows from this, by (9).

It remains to prove the second inequality in (4). By Lemma 2.1 (a) (replaced f_i by $q \circ f_i$), the function $q \circ (f_i \circ pr_i^n)$ is ν^T-integrable over Y^T $(i = 1, \ldots, n)$, and

$$\int\limits_{Y^T} q \circ (f_i \circ pr_i^n) d\nu^T = \int\limits_{Y_i} q \circ f_i d\nu_i.$$

Applying this and taking account of the convexity of q on I, we calculate

$$\int_{Y^T} q\left(\sum_{i=1}^{n} p_i f_i(y_i)\right) d\nu^T(y_1,\ldots,y_n) \leqslant \int_{Y^T} \sum_{i=1}^{n} p_i q(f_i(y_i)) d\nu^T(y_1,\ldots,y_n)$$
$$= \sum_{i=1}^{n} p_i \int_{Y_i} q \circ f_i d\nu_i,$$

and this completes the proof. $\square$

Next, we prove Theorem 1.2.

Proof. This is an immediate consequence of Lemma 2.2 (b), since q is continuous at every interior point of I. $\square$

We are now in a position to prove Theorem 1.3.

Proof. (a) This is an immediate consequence of Theorem 1.1.
(b) Since q is convex on I

$$\int_{X^{n+1}} q\left(\frac{1}{n+1}\sum_{i=1}^{n+1} f(x_i)\right) d\mu^{n+1}(x_1,\ldots,x_{n+1})$$
$$= \int_{X^{n+1}} q\left(\frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{1}{n}\sum_{j\in\{1,\ldots,n+1\}\setminus\{i\}} f(x_j)\right)\right) d\mu^{n+1}(x_1,\ldots,x_{n+1})$$
$$\leqslant \int_{X^{n+1}} \frac{1}{n+1}\sum_{i=1}^{n+1} q\left(\frac{1}{n}\sum_{j\in\{1,\ldots,n+1\}\setminus\{i\}} f(x_j)\right) d\mu^{n+1}(x_1,\ldots,x_{n+1}).$$

By the Fubini's theorem (see Lemma 2.3 (b)), the right-hand side of the previous inequality can be written in the form

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\int_{X^n}\left(\int_X q\left(\frac{1}{n}\sum_{j\in\{1,\ldots,n+1\}\setminus\{i\}} f(x_j)\right) d\mu(x_i)\right) d\mu^n(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1})$$
$$= \frac{1}{n+1}\sum_{i=1}^{n+1}\int_{X^n} q\left(\frac{1}{n}\sum_{j\in\{1,\ldots,n+1\}\setminus\{i\}} f(x_j)\right) d\mu^n(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1})$$
$$= \int_{X^n} q\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) d\mu^n(x_1,\ldots,x_n),$$

confirming the first inequality in (7).

Finally, we prove the second inequality in (7). Let $\pi_i(j)$ be the unique number from $\{1,\ldots,n\}$ for which

$$\pi_i(j) \equiv i+j-1 \pmod{n}, \quad i,j=1,\ldots,n.$$

Then the functions π_i $(i=1,\dots,n)$ are permutations of the numbers $1,\dots,n$. Clearly, $\sum_{j=1}^{n} p_{\pi_i(j)} = 1$ $(i=1,\dots,n)$, and $\pi_i(j)=\pi_j(i)$, $i,j=1,\dots,n$.

The convexity of q on I implies that

$$\int_{X^n} q\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) d\mu^n(x_1,\dots,x_n) \tag{18}$$

$$= \int_{X^n} q\left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} p_{\pi_i(j)} f(x_i)\right)\right) d\mu^n(x_1,\dots,x_n)$$

$$= \int_{X^n} q\left(\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i=1}^{n} p_{\pi_j(i)} f(x_i)\right)\right) d\mu^n(x_1,\dots,x_n)$$

$$\leqslant \int_{X^n} \sum_{j=1}^{n} \frac{1}{n} q\left(\sum_{i=1}^{n} p_{\pi_j(i)} f(x_i)\right) d\mu^n(x_1,\dots,x_n)$$

$$= \sum_{j=1}^{n} \frac{1}{n} \int_{X^n} q\left(\sum_{i=1}^{n} p_{\pi_j(i)} f(x_i)\right) d\mu^n(x_1,\dots,x_n).$$

It follows from Lemma 2.3 (a) that

$$\int_{X^n} q\left(\sum_{i=1}^{n} p_{\pi_j(i)} f(x_i)\right) d\mu^n(x_1,\dots,x_n)$$

$$= \int_{X^n} q\left(\sum_{i=1}^{n} p_i f(x_i)\right) d\mu^n(x_1,\dots,x_n), \quad (j=1,\dots,n).$$

This fact and (18) yields the result, bringing the proof to an end.

(c) It comes from Lemma 2.4 (b), since q is continuous at every interior point of I. $\square$

4. An application

Let $I \subset \mathbb{R}$ be an interval, and let $q : I \to \mathbb{R}$ be a convex function on I. Suppose $Y_i := \{1,\dots,k_i\}$ $(i \in \mathbb{N})$, $\mathscr{B}_i$ is the power set of Y_i $(i \in \mathbb{N})$, and $\nu_i(\{j\}) := r_{ij} \geqslant 0$ $(i \in \mathbb{N},\ j=1,\dots,k_i)$ such that $\sum_{j=1}^{k_i} r_{ij} = 1$ $(i \in \mathbb{N})$. Let $f_i(j) := v_{ij} \in I$ $(i \in \mathbb{N},\ j = 1,\dots,k_i)$.

Suppose that for a fixed $n \in \mathbb{N}$, $p_1,\dots,p_n$ are nonnegative numbers with $p_1 +$

$\ldots + p_n = 1$. Now inequality (4) gives that

$$q\left(\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{k_i} r_{ij} v_{ij}\right)\right) \leqslant \sum_{(j_1,\ldots,j_n)\in Y^{\{1,\ldots,n\}}} q\left(\sum_{i=1}^{n} p_i v_{ij_i}\right) r_{1j_1}\ldots r_{nj_n} \leqslant \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{k_i} r_{ij} q(v_{ij})\right), \tag{19}$$

which generalizes (3) (except the second member).

If q is bounded on I and

$$\sum_{j=1}^{k_i} r_{ij} v_{ij} = m, \quad i \in \mathbb{N}, \tag{20}$$

and

$$\sum_{i=1}^{\infty} \frac{1}{i^2} \left(\sum_{j=1}^{k_i} r_{ij} v_{ij}^2\right) < \infty. \tag{21}$$

then it comes from Theorem 1.2 that

$$\lim_{n\to\infty} \sum_{(j_1,\ldots,j_n)\in Y^{\{1,\ldots,n\}}} q\left(\frac{1}{n}\sum_{i=1}^{n} v_{ij_i}\right) r_{1j_1}\ldots r_{nj_n} = q(m).$$

Especially, suppose $q := -\ln$, $p_i > 0$ $(i = 1,\ldots,n)$, $r_{ij} > 0$ and $v_{ij} > 0$ $(i \in \mathbb{N},$ $j = 1,\ldots,k_i)$. Then (19) yields that

$$-\ln\left(\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{k_i} r_{ij} v_{ij}\right)\right) \leqslant -\ln\left(\prod_{(j_1,\ldots,j_n)\in Y^{\{1,\ldots,n\}}} \left(\sum_{i=1}^{n} p_i v_{ij_i}\right)^{r_{1j_1}\ldots r_{nj_n}}\right) \leqslant -\ln\left(\prod_{i=1}^{n}\left(\left(\prod_{j=1}^{k_i} v_{ij}^{r_{ij}}\right)^{p_i}\right)\right),$$

which shows the next inequality

$$\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{k_i} r_{ij} v_{ij}\right) \geqslant \prod_{(j_1,\ldots,j_n)\in Y^{\{1,\ldots,n\}}} \left(\sum_{i=1}^{n} p_i v_{ij_i}\right)^{r_{1j_1}\ldots r_{nj_n}} \geqslant \prod_{i=1}^{n}\left(\left(\prod_{j=1}^{k_i} v_{ij}^{r_{ij}}\right)^{p_i}\right). \tag{22}$$

Further, if there are positive numbers a and b such that

$$a \leqslant v_{ij} \leqslant b, \quad i \in \mathbb{N}, \quad j = 1,\ldots,k_i,$$

and (20) holds (obviously, (21) satisfies too), then

$$\lim_{n\to\infty} \prod_{(j_1,\ldots,j_n)\in Y^{\{1,\ldots,n\}}} \left(\frac{1}{n}\sum_{i=1}^{n} v_{ij_i}\right)^{r_{1j_1}\ldots r_{nj_n}} = m. \tag{23}$$

REFERENCES

[1] H. BAUER, *Probability theory*, de Gruyter Stud. Math. 23. Walter de Gruyter, Berlin-New York, 1996.
[2] S. S. DRAGOMIR, *A further improvement of Jensen's inequality*, Tamkang J. Math., **25**, 1 (1994), 29–36.
[3] E. HEWITT AND K. R. STROMBERG, *Real and abstract analysis*, Graduate Text in Mathematics 25, Springer-Verlag, Berlin-Heidelberg-New York, 1965.
[4] J. E. PEČARIĆ AND S. S. DRAGOMIR, *A refinements of Jensen inequality and applications*, Studia Univ. Babeş-Bolyai, Mathematica, **24**, 1 (1989), 15–19.
[5] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and new inequalities in analysis*, Kluwer Academic, Dordrecht, 1993.

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