Journal of
Mathematical
Inequalities
Volume 3, Number 2 (2009), 213–216

SELF–IMPROVEMENT OF THE INEQUALITY BETWEEN ARITHMETIC AND GEOMETRIC MEANS

J. M. ALDAZ

(*Communicated by J. Pečarić*)

Abstract. We show that a simple change of variable allows one to derive from the the AM-GM inequality an improved version of itself. As an application, a refinement of Hölder's inequality for an arbitrary number of functions is obtained.

It is well known that the AM-GM inequality has self-improving properties, that is, it implies better versions of itself. Let $x_i \geqslant 0$ for $i = 1, \dots, n$. The classical, equal weights case, states that

$$\prod_{i=1}^{n} x_i^{1/n} \leqslant \frac{1}{n} \sum_{i=1}^{n} x_i. \tag{1}$$

Let $\alpha_i > 0$ satisfy $\sum_{i=1}^{n} \alpha_i = 1$. Inequality (1) self-improves to the rational weights case simply via repetition of terms, and to the case of real weights α_i just by taking limits. So the general AM-GM inequality

$$\prod_{i=1}^{n} x_i^{\alpha_i} \leqslant \sum_{i=1}^{n} \alpha_i x_i \tag{2}$$

follows. There is a second way in which the AM-GM inequality self-improves. Let $s > 0$ and use the change of variables $x_i = y_i^s$. Substituting in (2) and taking s-th roots we get

$$\prod_{i=1}^{n} y_i^{\alpha_i} \leqslant \left(\sum_{i=1}^{n} \alpha_i y_i^s \right)^{1/s}. \tag{3}$$

Now for $0 < s < 1$, Jensen's inequality tells us that $\left(\sum_{i=1}^{n} \alpha_i y_i^s\right)^{1/s} \leqslant \sum_{i=1}^{n} \alpha_i y_i$ since t^s is concave, and furthermore the inequality is strict unless $y_1 = \dots = y_n$ (this follows from the equality case in Jensen's inequality). So (2) automatically proves a family of better inequalities, namely $M(0) \leqslant M(s)$ if $0 < s < 1$, where M denotes the usual elementary mean. The particular case $s = 1/2$ immediately leads to a natural and useful refinement of (2).

Mathematics subject classification (2000): 26D15.
Keywords and phrases: self-improvement, arithmetic-geometric inequality, Hölder's inequality.
The author was partially supported by Grant MTM2006-13000-C03-03 of the D.G.I. of Spain.

Paper JMI-03-21

THEOREM 1. *For $i=1,\ldots,n$, let $x_i \geqslant 0$, and let $\alpha_i > 0$ satisfy $\sum_{i=1}^n \alpha_i = 1$. Then*

$$\prod_{i=1}^n x_i^{\alpha_i} \leqslant \sum_{i=1}^n \alpha_i x_i - \sum_{i=1}^n \alpha_i \left(x_i^{1/2} - \sum_{k=1}^n \alpha_k x_k^{1/2} \right)^2. \tag{4}$$

Note that the right most term of (4) is the variance $\operatorname{Var}(x^{1/2})$ of the vector $x^{1/2} = (x_1^{1/2},\ldots,x_n^{1/2})$ with respect to the probability $\sum_{i=1}^n \alpha_i \delta_{x_i}$. So a large variance (of $x^{1/2}$) pushes the arithmetic and geometric means apart.

Proof. Recalling that $\operatorname{Var}(X) = E(X^2) - (E(X))^2 = E([X-E(X)]^2)$, and using (3) with $s = 1/2$, we obtain

$$\operatorname{Var}(x^{1/2}) = \sum_{i=1}^n \alpha_i \left(x_i^{1/2} - \sum_{k=1}^n \alpha_k x_k^{1/2} \right)^2 = \sum_{i=1}^n \alpha_i x_i - \left(\sum_{k=1}^n \alpha_k x_k^{1/2} \right)^2 \leqslant \sum_{i=1}^n \alpha_i x_i - \prod_{i=1}^n x_i^{\alpha_i}.$$

□

This refinement of the AM-GM inequality leads to an improvement of Hölder's inequality for several functions.

COROLLARY 2. *For $i=1,\ldots,n$, let $1 < p_i < \infty$ be such that $p_1^{-1} + \cdots + p_n^{-1} = 1$, and let $0 \leqslant f_i \in L^{p_i}$ satisfy $\|f_i\|_{p_i} > 0$. Then*

$$\left\| \prod_{i=1}^n f_i \right\|_1 \leqslant \prod_{i=1}^n \|f_i\|_{p_i} \left(1 - \sum_{i=1}^n \frac{1}{p_i} \left\| \frac{f_i^{p_i/2}}{\|f_i\|_{p_i}^{p_i/2}} - \sum_{k=1}^n \frac{1}{p_k} \frac{f_k^{p_k/2}}{\|f_k\|_{p_k}^{p_k/2}} \right\|_2^2 \right). \tag{5}$$

Proof. Set $\alpha_i = p_i^{-1}$ and $x_i = f_i^{p_i}(u)/\|f_i\|_{p_i}^{p_i}$ in (4). To obtain (5), integrate and multiply both sides by $\prod_{i=1}^n \|f_i\|_{p_i}$. □

REMARK 3. Inequality (4) was suggested by the following result of D. I. Cartwright and M. J. Field (cf. [5]; cf. also [4] and [6] for additional refinements along these lines). Let $0 < m = \min\{x_1,\ldots,x_n\}$ and let $M = \max\{x_1,\ldots,x_n\}$. Then

$$\frac{1}{2M} \sum_{i=1}^n \alpha_i \left(x_i - \sum_{k=1}^n \alpha_k x_k \right)^2 \leqslant \sum_{i=1}^n \alpha_i x_i - \prod_{i=1}^n x_i^{\alpha_i} \leqslant \frac{1}{2m} \sum_{i=1}^n \alpha_i \left(x_i - \sum_{k=1}^n \alpha_k x_k \right)^2. \tag{6}$$

The motivation to search for variants of (6) comes the fact that it is not well suited to the particular application considered here (refining Hölder's inequality). One would need to assume that $|f_i| \leqslant M$ almost everywhere. We give bounds using the variance of $x^{1/2}$ instead of the variance of x in order to ensure the integrability of the functions involved, and also to obtain the same homogeneity on both sides of (4). On the other hand, it is also well known that Hölder's inequality implies the AM-GM inequality, and hence, refinements of the former also yield refinements of the latter (cf. for instance, [2]). Finally, let us mention that a probabilistic study of the AM-GM inequality is also possible (cf. [3]); among other results, it is shown there that in the equal weights case the ratio of the geometric mean over the arithmetic mean concentrates around $e^{-\gamma}$, where γ is Euler's constant.

REMARK 4. The difference between the arithmetic and geometric means is in general not comparable to $\mathrm{Var}(x^{1/2})$. To see this, it is enough to consider the equal weights case, with $n >> 1$, $x_1 = 0$, and $x_2 = \ldots = x_n = 1$. Or the case where $n = 2$, and one of the weights is much larger than the other. But perhaps it is possible to give an upper bound for $\sum_{i=1}^n \alpha_i x_i - \prod_{i=1}^n x_i^{\alpha_i}$ using $\mathrm{Var}(x^{1/2})$ times some polynomial function of $1/\min_i \alpha_i$. This would lead to the same type of application as above. In fact, for the special case $n = 2$ a two sided, sharper version of (4) appears in Lemma 2.1 of [1]. It is not clear to me how to extend this sharper version to $n > 2$.

REMARK 5. When $n = 2$, inequality (5) reduces to

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q \left(1 - \frac{1}{pq}\left\| \frac{f^{p/2}}{\|f\|_p^{p/2}} - \frac{g^{q/2}}{\|g\|_q^{q/2}} \right\|_2^2\right), \tag{7}$$

where p and q are conjugate exponents, $0 \leqslant f \in L^p$, $0 \leqslant g \in L^q$, $\|f\|_p > 0$, and $\|g\|_q > 0$. In addition to providing a lower bound, with $1/\min\{p,q\}$ instead of $1/(pq)$, Lemma 2.1 of [1] yields a slightly better upper bound: $1/(pq) = 1/(p+q)$ can be replaced by $1/\max\{p,q\}$. But we note that (7) suffices, via the standard argument, to give a refinement of the triangle inequality for L^p spaces, $1 < p < \infty$, which in turn leads to a fairly straightforward proof of uniform convexity in the real valued case (arguing as in [1]). So the self-improving properties of the AM-GM inequality have repercussions beyond what one might expect.

REMARK 6. Note that $f_i^{p_i/2}/\|f_i\|_{p_i}^{p_i/2}$ is just a unit vector in L^2. The strategy underlying inequality (5) is to normalize all functions and map them into L^2, which becomes the common measuring ground where dispersion around the mean is determined. When $n = 2$, the correction term reduces to a function of the angular distance between $f^{p/2}$ and $g^{q/2}$.

REFERENCES

[1] J. M. ALDAZ, *A stability version of Hölder's inequality*, Journal of Mathematical Analysis and Applications, **343**, 2 (2008), 842–852. doi:10.1016/j.jmaa.2008.01.104. Also available at the Mathematics ArXiv: arXiv:math.CA/0710.2307.

[2] J. M. ALDAZ, *A refinement of the inequality between arithmetic and geometric means*, Journal of Mathematical Inequalities, **2**, 4 (2008), 473–477. Also available at the Mathematics ArXiv: arXiv:0811.3145.

[3] J. M. ALDAZ, *Concentration of the ratio between the geometric and arithmetic means*, Journal of Theoretical Probability, to appear. Also available at the Mathematics ArXiv: arXiv:0807.4832. DOI 10.1007/s10959-009-0215-9.

[4] H. ALZER, *A new refinement of the arithmetic mean-geometric mean inequality*, Rocky Mountain J. Math., **27**, 3 (1997), 663–667.

[5] D. I. CARTWRIGHT, M. J. FIELD, *A refinement of the arithmetic mean-geometric mean inequality*, Proc. Amer. Math. Soc., **71**, 1 (1978), 36-38.

[6] A. McD. MERCER, *Bounds for the A-G, A-H, G-H, and a family of inequalities of Ky Fan's type, using a general method*, Journal of Mathematical Analysis and Applications, **243** (2000), 163–173. doi:10.1006/j.jmaa.1999.6688.

(Received December 7, 2008)

J. M. Aldaz
Departamento de Matemáticas
Universidad Autónoma de Madrid
Cantoblanco 28049
Madrid
Spain
e-mail: jesus.munarriz@uam.es