Journal of
Mathematical
Inequalities
Volume 3, Number 2 (2009), 217–225

SCHUR CONVEXITY AND SCHUR–GEOMETRICALLY CONCAVITY OF GENERALIZED EXPONENT MEAN

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(*Communicated by G. Toader*)

Abstract. The monotonicity, the Schur-convexity and the Schur-geometrically convexity with variables (x,y) in $\mathbb{R}^2_{++}$ for fixed a of the generalized exponent mean $I_a(x,y)$ is proved. Besides, the monotonicity with parameters a in $\mathbb{R}$ for fixed (x,y) of $I_a(x,y)$ is discussed by using the hyperbolic composite function. Furthermore, some new inequalities are obtained.

1. Introduction

Throughout the paper we denote the set of the real numbers, the nonnegative real numbers and the positive real numbers by $\mathbb{R},\mathbb{R}_+$ and $\mathbb{R}_{++}$ respectively.

Let $(a,b)\in\mathbb{R}^2$, $(x,y)\in\mathbb{R}^2_{++}$. The extended mean (or Stolarsky mean) of (x,y) is defined in [1, p. 43] as

$$E(a,b;x,y)=\begin{cases}\left(\dfrac{b}{a}\cdot\dfrac{y^a-x^a}{y^b-x^b}\right)^{1/(a-b)}, & ab(a-b)(x-y)\neq 0,\\ \left(\dfrac{1}{a}\cdot\dfrac{y^a-x^a}{\ln y-\ln x}\right)^{1/a}, & a(x-y)\neq 0,\ b=0;\\ \dfrac{1}{e^{1/a}}\left(\dfrac{x^{x^a}}{y^{y^a}}\right)^{1/(x^a-y^a)}, & a(x-y)\neq 0,\ a=b;\\ \sqrt{xy}, & a=b=0,\ x\neq y;\\ x, & x=y.\end{cases}$$

In particular, for $a\neq 0$,

$$E(a,a;x,y)=\begin{cases}\dfrac{1}{e^{1/a}}\left(\dfrac{x^{x^a}}{y^{y^a}}\right)^{1/(x^a-y^a)}, & x\neq y;\\ x, & x=y\end{cases}$$

is called the generalized exponent or identric mean, in symbols $I_a(x,y)$.

The Schur-convexity of the extended mean $E(r,s;x,y)$ with (x,y) was discussed in [2] and the following conclusion is obtained:

Mathematics subject classification (2000): 26D15, 26A51.

Keywords and phrases: Generalized exponent mean, monotonicity, Schur-convexity, Schur-geometrically concavity inequality, hyperbolic function.

The authors are supported in part by the Scientific Research Common Program of Beijing Municipal Commission of Education (KM200611417009).

THEOREM A. *For fixed* $(a,b) \in \mathbb{R}^2$,

(*i*) *if* $2 < 2a < b$ *or* $2 \leqslant 2b \leqslant a$, *then* $E(a,b;x,y)$ *is Schur-convex with* (x,y) *on* $\mathbb{R}^2_{++}$,

(*ii*) *if* $(a,b) \in \{a < b \leqslant 2a, 0 < a \leqslant 1\} \cup \{b < a \leqslant 2b, 0 < b \leqslant 1\} \cup \{0 < b < a \leqslant 1\} \cup \{0 < a < b \leqslant 1\} \cup \{b \leqslant 2a < 0\} \cup \{a \leqslant 2b < 0\}$, *then* $E(a,b;x,y)$ *is Schur-concave with* (x,y) *on* $\mathbb{R}^2_{++}$.

But this conclusion is not related to the case $a = b$. In other words, the Schur-convexity of the generalized exponent mean $I_a(x,y)$ with (x,y) is not discussed in [2].

In this paper, the monotonicity, the Schur-convexity and the Schur-geometrically convexity with variables (x,y) in $\mathbb{R}^2_{++}$ for fixed a of the generalized exponent mean $I_a(x,y)$ is proved. Besides, the monotonicity with parameters a in $\mathbb{R}$ for fixed (x,y) of $I_a(x,y)$ is discussed by using the hyperbolic composite function. Furthermore, some new inequalities are obtained.

2. Definitions and Lemmas

We need the following definitions and lemmas.

DEFINITION 1. ([3, 4]) Let $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n) \in \mathbb{R}^n$.

(*i*) x is said to be majorized by y (in symbols $x \prec y$) if $\sum_{i=1}^k x_{[i]} \leqslant \sum_{i=1}^k y_{[i]}$ for $k = 1,2,\ldots,n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geqslant \cdots \geqslant x_{[n]}$ and $y_{[1]} \geqslant \cdots \geqslant y_{[n]}$ are rearrangements of x and y in a descending order.

(*ii*) $x \geqslant y$ means $x_i \geqslant y_i$ for all $i = 1,2,\ldots,n$. Let $\Omega \subset \mathbb{R}^n$. The function $\varphi\colon \Omega \to \mathbb{R}$ is said to be increasing if $x \geqslant y$ implies $\varphi(x) \geqslant \varphi(y)$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

(*iii*) $\Omega \subset \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1,\ldots,\alpha x_n + \beta y_n) \in \Omega$ for every x and $y \in \Omega$, where α and $\beta \in [0,1]$ with $\alpha + \beta = 1$.

(*iv*) let $\Omega \subset \mathbb{R}^n$. The function $\varphi\colon \Omega \to \mathbb{R}$ be said to be a Schur-convex function on Ω if $x \prec y$ on Ω implies $\varphi(x) \leqslant \varphi(y)$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex.

DEFINITION 2. ([5, 6]) Let $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n) \in \mathbb{R}^n_{++}$.

(*i*) $\Omega \subset \mathbb{R}^n_{++}$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta,\ldots,x_n^\alpha y_n^\beta) \in \Omega$ for all x and $y \in \Omega$, where α and $\beta \in [0,1]$ with $\alpha + \beta = 1$.

(*ii*) Let $\Omega \subset \mathbb{R}^n_{++}$. The function $\varphi\colon \Omega \to \mathbb{R}_+$ is said to be Schur-geometrically convex function on Ω if $(\ln x_1,\ldots,\ln x_n) \prec (\ln y_1,\ldots,\ln y_n)$ on Ω implies $\varphi(x) \leqslant \varphi(y)$. The function φ is said to be a Schur-geometrically concave on Ω if and only if $-\varphi$ is Schur-geometrically convex.

DEFINITION 3. ([4]) (i) $\Omega \subset \mathbb{R}^n$ is called symmetric set, if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P.

(ii) The function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\varphi(Px) = \varphi(x)$ for all $x \in \Omega$.

LEMMA 1. ([3, 4]) *A function $\varphi(x)$ is increasing if and only if $\nabla\varphi(x) \geqslant 0$ for $x \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is an open set, $\varphi : \Omega \to R$ is differentiable, and*

$$\nabla\varphi(x) = \left(\frac{\partial\varphi(x)}{\partial x_1}, \dots, \frac{\partial\varphi(x)}{\partial x_n}\right) \in \mathbb{R}^n.$$

LEMMA 2. ([3, 4]) *Let $\Omega \subset \mathbb{R}^n$ be a symmetric set and with a nonempty interior Ω^0, $\varphi : \Omega \to \mathbb{R}$ be a continuous on Ω and differentiable in Ω^0. Then φ is the $Schur-convex(Schur-concave) function$, if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geqslant 0 (\leqslant 0)$$

holds for any $x = (x_1, x_2, \cdots, x_n) \in \Omega^0$.

LEMMA 3. ([5, p. 108]) *Let $\Omega \subset \mathbb{R}^n_{++}$ be symmetric with a nonempty interior geometrically convex set. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial\varphi}{\partial x_1} - x_2 \frac{\partial\varphi}{\partial x_2}\right) \geqslant 0 (\leqslant 0)$$

holds for any $x = (x_1, x_2, \cdots, x_n) \in \Omega^0$, then φ is a Schur-geometrically convex (Schur-geometrically concave) function.

LEMMA 4. *Let $x \leqslant y$, $u(t) = tx + (1-t)y$, $v(t) = ty + (1-t)x$. If $1/2 \leqslant t_2 \leqslant t_1 \leqslant 1$ or $0 \leqslant t_1 \leqslant t_2 \leqslant 1/2$, then*

$$(u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (x, y). \tag{1}$$

Proof. Case 1. When $1/2 \leqslant t_2 \leqslant t_1 \leqslant 1$, it is easy to see that $u(t_1) \geqslant v(t_1)$, $u(t_2) \geqslant v(t_2)$, $u(t_1) \geqslant u(t_2)$ and $u(t_2) + v(t_2) = u(t_1) + v(t_1) = x + y$, that is (1) holds.

Case 2. When $0 \leqslant t_1 \leqslant t_2 \leqslant 1$, then $1/2 \leqslant 1 - t_2 \leqslant 1 - t_1 \leqslant 1$, by the Case 1, it follows

$$(u(1-t_2), v(1-t_2)) \prec (u(1-t_1), v(1-t_1)),$$

i.e. $(u(t_2), v(t_2)) \prec (u(t_1), v(t_1))$. $\square$

LEMMA 5. ([4, 7]) *Let $0 \leqslant x \leqslant y$, $c \geqslant 0$. Then*

$$\left(\frac{x+c}{x+y+2c}, \frac{y+c}{x+y+2c}\right) \prec \left(\frac{x}{x+y}, \frac{y}{x+y}\right). \tag{2}$$

LEMMA 6. *For x in $\mathbb{R}$ with $x \neq 0$, we have*

$$\sinh^2 x > x^2. \tag{3}$$

Proof. Let $f(x) = \sinh^2 x - x^2$. Then $f'(x) = \sinh 2x - 2x$. Since $f''(x) = 2(\cosh 2x - 1) > 0$ for $x \in \mathbb{R}$ with $x \neq 0$, $f'(x)$ is strictly increasing. It follows that $f'(x) > f'(0) = 0$, so $f(x) > f(0) = 0$ for $x > 0$. As $f(-x) = f(x)$, (3) holds for any $x \in \mathbb{R}$ with $x \neq 0$. □

LEMMA 7. *Let (x,y) and $(a,b) \in \mathbb{R}^2_{++}$ with $x < y$, $a < b$, $a + b = 1$. Then*

$$ax + by > \frac{x+y}{2}, \tag{4}$$

$$bx + ay < \frac{x+y}{2}, \tag{5}$$

Proof. As

$$\begin{aligned} ax + by - \frac{x+y}{2} &= \left(a - \frac{1}{2}\right)x + \left(b - \frac{1}{2}\right)y \\ &= \left(1 - b - \frac{1}{2}\right)x + \left(b - \frac{1}{2}\right)y = -\left(b - \frac{1}{2}\right)x + \left(b - \frac{1}{2}\right)y \\ &= \left(b - \frac{1}{2}\right)(y - x) > 0, \end{aligned}$$

(4) holds. (5) can be proved similarly. □

LEMMA 8. *Let $(x,y) \in \mathbb{R}^2_{++}$ and $(a,b) \in \mathbb{R}^2$ with $ab(a-b)(x-y) \neq 0$. Then*

$$E(a,b;x,y) = \sqrt{xy}\left(\frac{b\sinh(a\ln\sqrt{u})}{a\sinh(b\ln\sqrt{u})}\right)^{\frac{1}{a-b}}, \tag{6}$$

where $u = y/x$.

Proof. Without loss of generality, we may assume $0 < x < y$. Then

$$\begin{aligned} E(a,b;x,y) &= \left(\frac{b}{a} \cdot \frac{y^a - x^a}{y^b - x^b}\right)^{\frac{1}{a-b}} = \left(\frac{b}{a} \cdot \frac{u^a - 1}{u^b - 1} x^{a-b}\right)^{1/(a-b)} \\ &= x\left(\frac{b}{a} \cdot \frac{e^{2a\ln\sqrt{u}} - 1}{e^{2b\ln\sqrt{u}} - 1}\right)^{\frac{1}{a-b}} = x\left(\frac{b}{a} \cdot \frac{\frac{e^{2a\ln\sqrt{u}}-1}{2e^{a\ln\sqrt{u}}}}{\frac{e^{2b\ln\sqrt{u}}-1}{2e^{b\ln\sqrt{u}}}} e^{(a-b)\ln\sqrt{u}}\right)^{\frac{1}{a-b}} \\ &= x\sqrt{u}\left(\frac{b\sinh(a\ln\sqrt{u})}{a\sinh(b\ln\sqrt{u})}\right)^{\frac{1}{a-b}} = \sqrt{xy}\left(\frac{b\sinh(a\ln\sqrt{u})}{a\sinh(b\ln\sqrt{u})}\right)^{\frac{1}{a-b}}. \end{aligned}$$

□

LEMMA 9. *Let* $(x,y)\in\mathbb{R}^2_{++}$ *with* $x\neq y$, *and let* $a\in\mathbb{R}$ *with* $a\neq 0$. *Then*

$$I_a(x,y)=\sqrt{xy}\exp\left\{\frac{t}{\tanh(at)}-\frac{1}{a}\right\} \tag{7}$$

where $t=\ln\sqrt{u}$, $u=y/x$.

Proof. For $b\in\mathbb{R}$ with $b\neq a$, let

$$v=\frac{b\sinh(a\ln\sqrt{u})-a\sinh(b\ln\sqrt{u})}{a\sinh(b\ln\sqrt{u})}.$$

Then from Lemma 8 we have

$$\begin{aligned}
I_a(x,y)&=\lim_{b\to a}E(a,b;x,y)=\lim_{b\to a}\sqrt{xy}\left(\frac{b\sinh(a\ln\sqrt{u})}{a\sinh(b\ln\sqrt{u})}\right)^{\frac{1}{a-b}}\\
&=\sqrt{xy}\lim_{b\to a}(1+v)^{\frac{1}{a-b}}\\
&=\sqrt{xy}\lim_{b\to a}\left[(1+v)^{\frac{1}{v}}\right]^{\frac{b\sinh(a\ln\sqrt{u})-a\sinh(b\ln\sqrt{u})}{a-b}\frac{1}{a\sinh(b\ln\sqrt{u})}}\\
&=\sqrt{xy}\exp\left\{\lim_{b\to a}\frac{b\sinh(a\ln\sqrt{u})-a\sinh(b\ln\sqrt{u})}{a-b}\frac{1}{a\sinh(b\ln\sqrt{u})}\right\}\\
&=\sqrt{xy}\exp\left\{\frac{1}{a\sinh(a\ln\sqrt{u})}\lim_{b\to a}\frac{b\sinh(a\ln\sqrt{u})-a\sinh(b\ln\sqrt{u})}{a-b}\right\}\\
&=\sqrt{xy}\exp\left\{\frac{1}{a\sinh(a\ln\sqrt{u})}\lim_{b\to a}\frac{\sinh(a\ln\sqrt{u})-a(\ln\sqrt{u})\cosh(b\ln\sqrt{u})}{-1}\right\}\\
&=\sqrt{xy}\exp\left\{\frac{a(\ln\sqrt{u})\cosh(a\ln\sqrt{u})-\sinh(a\ln\sqrt{u})}{a\sinh(a\ln\sqrt{u})}\right\}\\
&=\sqrt{xy}\exp\left\{\frac{(at)\cosh(at)-\sinh(at)}{a\sinh(at)}\right\}\\
&=\sqrt{xy}\exp\left\{\frac{t}{\tanh(at)}-\frac{1}{a}\right\}. \qquad\square
\end{aligned}$$

3. Main results and their proofs

THEOREM 1. *For fixed* $(x,y)\in\mathbb{R}^2_{++}$, $I_a(x,y)$ *is increasing with* a *on* $\mathbb{R}$.

Proof. For $a\neq 0$, set $f(a)=\frac{t}{\tanh(at)}-\frac{1}{a}$, where $t=\ln\sqrt{u}$, $u=y/x$. Then

$$f'(a)=\frac{-t^2}{\tanh^2(at)\cosh^2(at)}+\frac{1}{a^2}=\frac{-t^2}{\sinh^2(at)}+\frac{1}{a^2}=\frac{\sinh^2(at)-(at)^2}{a^2\sinh^2(at)}.$$

Thus from Lemma 6 it follows that $f'(a) > 0$, that is $f(a)$ is increasing on $\mathbb{R}$ with a and

$$I_a(x,y) = \sqrt{xy}\exp\left\{\frac{t}{\tanh(at)} - \frac{1}{a}\right\} = \sqrt{xy}e^{f(a)}$$

is increasing on $\mathbb{R}$ with a. The proof of Theorem 1 is completed. □

THEOREM 2. *For fixed $a \in \mathbb{R}$, $I_a(x,y)$ is increasing with (x,y) on $\mathbb{R}^2_{++}$.*

Proof. Let $A = x^a$, $B = y^a$. Then

$$\ln I_a(x,y) = \frac{x^a \ln x - y^a \ln y}{x^a - y^a} - \frac{1}{a} = \frac{1}{a}\left(\frac{A\ln A - B\ln B}{A-B} - 1\right).$$

$$\begin{aligned}
\frac{\partial \ln I_a}{\partial x} &= \frac{\partial \ln I_a}{\partial A}\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{1}{a}\frac{\partial}{\partial a}\left(\frac{A\ln A - B\ln B}{A-B} - 1\right) a x^{a-1} \\
&= \frac{A}{x}\left[\frac{(A-B) - B(\ln A - \ln B)}{(A-B)^2}\right] \\
&= \frac{A}{x(A-B)}\left(1 - \frac{\ln A - \ln B}{A-B}\cdot B\right) \\
&= \frac{A}{x(A-B)}\left(1 - \frac{B}{\xi}\right) \quad \text{(where } \xi \text{ lies between } A \text{ and } B\text{)} \\
&= \frac{A}{x(A-B)}\frac{\xi - B}{\xi} = \frac{A}{x\xi}\cdot\frac{\xi - B}{A-B} \geqslant 0;
\end{aligned}$$

Similarly can be proved that $\frac{\partial \ln I_a}{\partial y} \geqslant 0$.

By Lemma 1, it follows that $\ln I_a(x,y)$ is increasing with (x,y) on $\mathbb{R}^2_{++}$, and then $I_a(x,y)$ is increasing with (x,y) on $\mathbb{R}^2_{++}$ too.

The proof of Theorem 2 is completed. □

THEOREM 3. *If $0 < a \leqslant 1$, then $I_a(x,y)$ is Schur-concave with (x,y) on $\mathbb{R}^2_{++}$.*

Proof. For $(x,y) \in \mathbb{R}^2_{++}, 0 < a \leqslant 1$, let $A = x^a$, $B = y^a$. When $x \neq y$, we have

$$\frac{\partial \ln I_a}{\partial x} = \frac{A}{x}\cdot\frac{(A-B) - B(\ln A - \ln B)}{(A-B)^2}$$

$$\frac{\partial \ln I_a}{\partial y} = \frac{B}{y}\cdot\frac{A(\ln A - \ln B) - (A-B)}{(A-B)^2}$$

and then

$$\begin{aligned}
\Delta :&= (x-y)\left(\frac{\partial \ln I_a}{\partial x} - \frac{\partial \ln I_a}{\partial y}\right) \\
&= (x-y)\left[\frac{A}{x}\cdot\frac{(A-B) - B(\ln A - \ln B)}{(A-B)^2} - \frac{B}{y}\cdot\frac{A(\ln A - \ln B) - (A-B)}{(A-B)^2}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{x-y}{(A-B)^2}\left[\frac{A}{x}(A-B)-\frac{AB}{x}(\ln A-\ln B)-\frac{AB}{y}(\ln A-\ln B)+\frac{B}{y}(A-B)\right]\\
&= \frac{x-y}{(A-B)^2}\left[\left(\frac{A}{x}+\frac{B}{y}\right)(A-B)-AB\left(\frac{1}{x}+\frac{1}{y}\right)(\ln A-\ln B)\right]\\
&= \frac{x-y}{A-B}\cdot\frac{\ln A-\ln B}{A-B}\left[\left(\frac{A}{x}+\frac{B}{y}\right)\frac{A-B}{\ln A-\ln B}-AB\left(\frac{1}{x}+\frac{1}{y}\right)\right]\\
&= \frac{x-y}{A-B}\cdot\frac{\ln A-\ln B}{A-B}\left(\frac{A}{x}+\frac{B}{y}\right)\left[\frac{A-B}{\ln A-\ln B}-\frac{\left(\frac{1}{x}+\frac{1}{y}\right)AB}{\frac{A}{x}+\frac{B}{y}}\right]\\
&= \frac{x-y}{A-B}\cdot\frac{\ln A-\ln B}{A-B}\left(\frac{A}{x}+\frac{B}{y}\right)\left(\frac{x^a-y^a}{\ln x^a-\ln y^a}-\frac{y^{a-1}x^a+x^{a-1}y^a}{x^{a-1}+y^{a-1}}\right)\\
&= \frac{x-y}{A-B}\cdot\frac{\ln A-\ln B}{A-B}\left(\frac{A}{x}+\frac{B}{y}\right)\left[L(x^a,y^a)-\left(\frac{y^{a-1}}{x^{a-1}+y^{a-1}}x^a+\frac{x^{a-1}}{x^{a-1}+y^{a-1}}y^a\right)\right]
\end{aligned}
$$

where L denotes the logarithm mean.

Without loss of generality, we may assume $0<x<y$. When $0<a<1$ we have

$$\frac{y^{a-1}}{x^{a-1}+y^{a-1}}<\frac{x^{a-1}}{x^{a-1}+y^{a-1}}$$

and

$$\frac{y^{a-1}}{x^{a-1}+y^{a-1}}+\frac{x^{a-1}}{x^{a-1}+y^{a-1}}=1,$$

and then by Lemma 7, it follows that

$$\frac{y^{a-1}}{x^{a-1}+y^{a-1}}x^a+\frac{x^{a-1}}{x^{a-1}+y^{a-1}}y^a>\frac{x^a+y^a}{2}=A(x^a,y^a).$$

Furthermore notice that $L(x^a,y^a)<A(x^a,y^a)$, we have

$$L(x^a,y^a)-\left(\frac{y^{a-1}}{x^{a-1}+y^{a-1}}x^a+\frac{x^{a-1}}{x^{a-1}+y^{a-1}}y^a\right)<L(x^a,y^a)-A(x^a,y^a)<0.$$

Hence $\Delta<0$ for $0<a<1$. It is easy to see that $\Delta<0$ for $a=1$. By Lemma 2, it follows that for $0<a\leqslant 1$, $\ln I_a(x,y)$ is Schur-concave on $\mathbb{R}^2_{++}$ with (x,y), and then $I_a(x,y)$ is Schur-concave on $\mathbb{R}^2_{++}$ with (x,y) too.

The proof of Theorem 3 is completed. □

THEOREM 4. *If $a>0$, then $I_a(x,y)$ is Schur-geometrically convex with (x,y) on $\mathbb{R}^2_{++}$; If $a<0$, then $I_a(x,y)$ is Schur-geometrically concave with (x,y) on $\mathbb{R}^2_{++}$.*

Proof. For $(x,y)\in\mathbb{R}^2_{++}, a\in\mathbb{R}$, let $A=x^a, B=y^a$. When $x\neq y$, we have

$$\frac{\partial \ln I_a}{\partial x}=\frac{A}{x}\cdot\frac{(A-B)-B(\ln A-\ln B)}{(A-B)^2}$$

$$\frac{\partial \ln I_a}{\partial y} = \frac{B}{y} \cdot \frac{A(\ln A - \ln B) - (A-B)}{(A-B)^2}$$

and then

$$\begin{aligned}
\Lambda := &(x-y)\left(x\frac{\partial \ln I_a}{\partial x} - y\frac{\partial \ln I_a}{\partial y}\right)\\
&= \frac{x-y}{(A-B)^2}\left[A(A-B) - AB(\ln A - \ln B) - AB(\ln A - \ln B) + B(A-B)\right]\\
&= \frac{x-y}{(A-B)^2}\left[(A+B)(A-B) - 2AB(\ln A - \ln B)\right]\\
&= \frac{(x-y)(A+B)(\ln A - \ln B)}{(A-B)^2}\left(\frac{A-B}{\ln A - \ln B} - \frac{2AB}{A+B}\right)\\
&= \frac{(x-y)(A+B)(\ln A - \ln B)}{(A-B)^2}\left(L(A,B) - H(A,B)\right).
\end{aligned}$$

where H denote the harmonic mean.

For $(x,y) \in \mathbb{R}^2_{++}$ with $x \neq y$ and $a \in \mathbb{R}$, we have $L(A,B) > H(A,B)$. If $a > 0(< 0)$, then $(x-y)(\ln A - \ln B) = a(x-y)(\ln x - \ln y) > 0(< 0)$, and then $\Lambda > 0(< 0)$. By Lemma 3, it follows that $\ln I_a(x,y)$ is Schur-geometrically convex (concave) on $\mathbb{R}^2_{++}$ with (x,y), and then $I_a(x,y)$ is Schur-geometrically convex (concave) on $\mathbb{R}^2_{++}$ with (x,y) too.

The proof of Theorem 4 is completed. □

4. Applications

THEOREM 5. *Let* $0 < a \leqslant 1$*, and let* $x \leqslant y, u(t) = tx + (1-t)y, v(t) = ty + (1-t)x$*. If* $1/2 \leqslant t_2 \leqslant t_1 \leqslant 1$ *or* $0 \leqslant t_1 \leqslant t_2 \leqslant 1/2$*, then we have*

$$\begin{aligned}
G(x,y) &\leqslant I_a\left(x^{u(t_1)}y^{v(t_1)}, x^{v(t_1)}y^{u(t_1)}\right) \leqslant I_a\left(x^{u(t_2)}y^{v(t_2)}, x^{v(t_2)}y^{u(t_2)}\right)\\
&\leqslant I_a(x,y) \leqslant I_a\left(u(t_2), v(t_2)\right) \leqslant I_a\left(u(t_1), v(t_1)\right) \leqslant A(x,y).
\end{aligned} \tag{8}$$

Proof. Combining Lemma 4 with Theorem 3, we have

$$\begin{aligned}
I_a(x,y) &\leqslant I_a\left(u(t_2), v(t_2)\right) \leqslant I_a\left(u(t_1), v(t_1)\right)\\
&\leqslant I_a\left((x+y)/2, (x+y)/2\right) = A(x,y).
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
(\ln\sqrt{xy}, \ln\sqrt{xy}) &\prec \left(\ln x^{u(t_1)}y^{v(t_1)}, \ln x^{v(t_1)}y^{u(t_1)}\right)\\
&\prec \left(\ln x^{u(t_2)}y^{v(t_2)}, \ln x^{v(t_2)}y^{u(t_2)}\right) \prec (\ln x, \ln y),
\end{aligned}$$

from Theorem 4, it follows

$$G(x,y) = I_a\left(\sqrt{xy}, \sqrt{xy}\right) \leqslant I_a\left(x^{u(t_1)}y^{v(t_1)}, x^{v(t_1)}y^{u(t_1)}\right)$$

$$\leqslant I_a\left(x^{u(t_2)}y^{v(t_2)}, x^{v(t_2)}y^{u(t_2)}\right) \leqslant I_a(x,y).$$

The proof is complete. □

THEOREM 6. *Let* $0 \leqslant x \leqslant y,\ c \geqslant 0,\ 0 < a \leqslant 1$. *Then*

$$I_a\left(\frac{x+c}{x+y+2c}, \frac{y+c}{x+y+2c}\right) \geqslant I_a\left(\frac{x}{x+y}, \frac{y}{x+y}\right). \tag{9}$$

Proof. By Lemma 5 and Theorem 3, it follows that (9) holds.
The proof is complete. □

Acknowledgements. The authors are indebted to the referees for their helpful suggestions.

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(Received September 2, 2008)

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