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ABOUT BERGSTRÖM'S INEQUALITY

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Abstract. In this paper, we generalize identity (3), from where we obtain a rafinement of inequalities (1) and (2).

1. Introduction

The inequality from Theorem 1 is called in the literature Bergström's inequality (see [2], [3], [4], [6]).

THEOREM 1. *If $x_k \in \mathbb{R}$ and $a_k > 0$, $k \in \{1,2,\dots,n\}$, then*

$$\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}+\ldots+\frac{x_n^2}{a_n} \geqslant \frac{(x_1+x_2+\ldots+x_n)^2}{a_1+a_2+\ldots+a_n}, \tag{1}$$

with equality if and only if $\dfrac{x_1}{a_1}=\dfrac{x_2}{a_2}=\ldots=\dfrac{x_n}{a_n}$.

The generalization of Bergström's inequality is contained in the following theorem (see [5]).

THEOREM 2. *If $x_k \in \mathbb{R}$ and $a_k > 0$, $k \in \{1,2,\dots,n\}$, then*

$$\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}+\ldots+\frac{x_n^2}{a_n} \geqslant \frac{(x_1+x_2+\ldots+x_n)^2}{a_1+a_2+\ldots+a_n}+\max_{1\leqslant i<j\leqslant n}\frac{(a_ix_j-a_jx_i)^2}{a_ia_j(a_i+a_j)}. \tag{2}$$

By particularizations in Theorem 2, in paper [5] the refinements are obtained of Cauchy-Schwarz's inequality.

For the complex numbers, it is well-known the identity:

$$\frac{|z_1|^2}{a_1}+\frac{|z_2|^2}{a_2}-\frac{|z_1+z_2|^2}{a_1+a_2}=\frac{|a_1z_2-a_2z_1|^2}{a_1a_2(a_1+a_2)} \tag{3}$$

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true for any $z_1, z_2 \in \mathbb{C}$ and any $a_1, a_2 \in \mathbb{R}\backslash\{0\}$ such that $a_1 + a_2 \neq 0$ (see [6], page 315).

In the following, we note $\mathbb{N} = \{1, 2, \ldots\}$

2. Main results

In this section we start with the generalization of identity (3).

THEOREM 3. *If $n \in \mathbb{N}$, $n \geqslant 2$, $z_1, z_2, \ldots, z_n \in \mathbb{C}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}\backslash\{0\}$, such that $a_1 + a_2 + \ldots + a_n \neq 0$, then*

$$\begin{aligned}&\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} + \ldots + \frac{|z_n|^2}{a_n} - \frac{|z_1 + z_2 + \ldots + z_n|^2}{a_1 + a_2 + \ldots + a_n} \\ &\qquad = \frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{1 \leqslant i < j \leqslant n} \frac{|a_i z_j - a_j z_i|^2}{a_i a_j}.\end{aligned} \tag{4}$$

Proof. We consider $z_k = x_k + i y_k$, $x_k, y_k \in \mathbb{R}$, $k \in \{1, 2, \ldots, n\}$. Then the member from the left from the (4) is equivalent with

$$\begin{aligned}&\frac{x_1^2+y_1^2}{a_1} + \frac{x_2^2+y_2^2}{a_2} + \ldots + \frac{x_n^2+y_n^2}{a_n} - \frac{(x_1+x_2+\ldots+x_n)^2 + (y_1+y_2+\ldots+y_n)^2}{a_1+a_2+\ldots+a_n} \\
&\quad = \frac{1}{a_1 a_2 \cdot \ldots \cdot a_n (a_1 + a_2 + \ldots + a_n)} \\
&\quad \cdot \Big((a_2^2 a_3 a_4 \cdot \ldots \cdot a_n + a_2 a_3^2 a_4 \cdot \ldots \cdot a_n + \ldots + a_2 a_3 \cdot \ldots \cdot a_{n-1} a_n^2)(x_1^2 + y_1^2) \\
&\quad + (a_1^2 a_3 a_4 \cdot \ldots \cdot a_n + a_1 a_3^2 a_4 \cdot \ldots \cdot a_n + \ldots + a_1 a_3 \cdot \ldots \cdot a_{n-1} a_n^2)(x_2^2 + y_2^2) + \ldots \\
&\quad + (a_1^2 a_2 a_3 \cdot \ldots \cdot a_{n-1} + a_1 a_2^2 a_3 \cdot \ldots \cdot a_{n-1} + \ldots + a_1 a_2 \cdot \ldots \cdot a_{n-2} a_{n-1}^2)(x_n^2 + y_n^2) \\
&\quad - 2 a_1 a_2 \cdot \ldots \cdot a_n \big((x_1 x_2 + x_1 x_3 + \ldots + x_1 x_n + \ldots + x_{n-1} x_n) \\
&\quad + (y_1 y_2 + y_1 y_3 + \ldots + y_1 y_n + \ldots + y_{n-1} y_n) \big) \Big) \\
&\quad = \frac{1}{a_1 a_2 \cdot \ldots \cdot a_n (a_1 + a_2 + \ldots + a_n)} \big(a_3 a_4 \ldots a_n ((a_1 x_2 - a_2 x_1)^2 + (a_1 y_2 - a_2 y_1)^2) \\
&\quad + a_2 a_4 a_5 \cdot \ldots \cdot a_n \big((a_1 x_3 - a_3 x_1)^2 + (a_1 y_3 - a_3 y_1)^2 \big) + \ldots \\
&\quad + a_1 a_2 \cdot \ldots \cdot a_{n-2} \big((a_{n-1} x_n - a_n x_{n-1})^2 + (a_{n-1} y_n - a_n y_{n-1})^2 \big) \big) \\
&\quad = \frac{1}{a_1 a_2 \cdot \ldots \cdot a_n (a_1 + a_2 + \ldots + a_n)} \big(a_3 a_4 \cdot \ldots \cdot a_n |a_1 z_2 - a_2 z_1|^2 \\
&\quad + a_2 a_4 a_5 \cdot \ldots \cdot a_n |a_1 z_3 - a_3 z_1|^2 + \ldots + a_1 a_2 \cdot \ldots \cdot a_{n-2} |a_{n-1} z_n - a_n z_{n-1}|^2 \big),
\end{aligned}$$

from where, the relation (4) results.

COROLLARY 1. *If* $n\in\mathbb{N}$, $n\geqslant 2$, $x_1,x_2,\ldots,x_n\in\mathbb{R}$ *and* $a_1,a_2,\ldots,a_n\in\mathbb{R}\setminus\{0\}$, *such that* $a_1+a_2+\ldots+a_n\neq 0$, *then*

$$\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}+\ldots+\frac{x_n^2}{a_n}-\frac{(x_1+x_2+\ldots+x_n)^2}{a_1+a_2+\ldots+a_n} = \frac{1}{a_1+a_2+\ldots+a_n}\sum_{1\leqslant i<j\leqslant n}\frac{(a_ix_j-a_jx_i)^2}{a_ia_j}. \tag{5}$$

Proof. In the identity (4) we consider $z_k=x_k\in\mathbb{R}$, $k\in\{1,2,\ldots,n\}$.

COROLLARY 2. *If* $n\in\mathbb{N}$, $x\geqslant 2$, $x_k,y_k\in\mathbb{R}$, $k\in\{1,2,\ldots,n\}$, *then*

$$(x_1^2+x_2^2+\ldots+x_n^2)(y_1^2+y_2^2+\ldots+y_n^2) - (x_1y_1+x_2y_2+\ldots+x_ny_n)^2=\sum_{1\leqslant i<j\leqslant n}(x_iy_j-x_jy_i)^2. \tag{6}$$

Proof. Changing a_k by y_k^2 and x_k by x_ky_k, $k\in\{1,2,\ldots,n\}$ in identity (5), we obtain identity (6).

REMARK 1. *The identity from Corollary 2 is called Lagrange's identity.*

COROLLARY 3. *If* $z_1,z_2,\ldots,z_n\in\mathbb{C}$ *and* $a_1,a_2,\ldots,a_n\in(0,\infty)$, *then*

$$\frac{|z_1|^2}{a_1}+\frac{|z_2|^2}{a_2}+\ldots+\frac{|z_n|^2}{a_n}\geqslant\frac{|z_1+z_2+\ldots+z_n|^2}{a_1+a_2+\ldots+a_n}, \tag{7}$$

with equality if and only if $a_iz_j=a_jz_i$ *for any* $i,j\in\{1,2,\ldots,n\}$.

Proof. The inequality (7) results immediately from Theorem 3.

THEOREM 4. *If* $n\in\mathbb{N}$, $n\geqslant 2$, $z_1,z_2,\ldots,z_n\in\mathbb{C}$ *and* $a_1,a_2,\ldots,a_n\in\mathbb{R}\setminus\{0\}$ *such that* $a_1+a_2+\ldots+a_n>0$, *then*

$$\frac{|z_1|^2}{a_1}+\frac{|z_2|^2}{a_2}+\ldots+\frac{|z_n|^2}{a_n}\geqslant\frac{1}{a_1+a_2+\ldots+a_n}\sum_{1\leqslant i<j\leqslant n}\frac{|a_iz_j-a_jz_i|^2}{a_ia_j}, \tag{8}$$

with equality if and only if $z_1+z_2+\ldots+z_n=0$.

Proof. It results from Theorem 3.

COROLLARY 4. *If* $n\in\mathbb{N}$, $n\geqslant 2$ *and* $z_1,z_2,\ldots,z_n\in\mathbb{C}$, *then*

$$|z_1|^2+|z_2|^2+\ldots+|z_n|^2\geqslant\frac{1}{n}\sum_{1\leqslant i<j\leqslant n}|z_j-z_i|^2. \tag{9}$$

Proof. The inequality (9) results from Theorem 4 if we take $a_1 = a_2 = \ldots = a_n = 1$.

THEOREM 5. *If $n \in \mathbb{N}$, $n \geqslant 2$, $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and $a_1, a_2, \ldots, a_n \in (0, \infty)$, then*

$$\begin{aligned} &\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \ldots + \frac{x_n^2}{a_n} - \frac{(x_1 + x_2 + \ldots + x_n)^2}{a_1 + a_2 + \ldots + a_n} \\ &\qquad \geqslant A_{k,l} + \frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{\substack{1 \leqslant i < j \leqslant n \\ i,j \notin \{k,l\}}} \frac{(a_i x_j - a_j x_i)^2}{a_i a_j}, \end{aligned} \tag{10}$$

where

$$A_{k,l} = \max_{1 \leqslant i < j \leqslant n} \frac{(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)} = \frac{(a_k x_l - a_l x_k)^2}{a_k a_l (a_k + a_l)}, \ 1 \leqslant k < l \leqslant n.$$

Proof. We have that

$$\begin{aligned} T &= \sum_{\substack{m=1 \\ m \notin \{k,l\}}}^{n} \left(\frac{(a_m x_l - a_l x_m)^2}{a_m a_l} + \frac{(a_m x_k - a_k x_m)^2}{a_m a_k} \right) \\ &= \sum_{\substack{m=1 \\ m \notin \{k,l\}}}^{m} \left(\frac{\left(a_k x_l - \frac{a_l a_k}{a_m} x_m\right)}{\frac{a_l a_k^2}{a_m}} + \frac{\left(\frac{a_l a_k}{a_m} x_m - a_l x_k\right)^2}{\frac{a_k a_l^2}{a_m}} \right) \end{aligned}$$

and applying the inequality (1) for $n = 2$, we obtain that

$$T \geqslant \sum_{\substack{m=1 \\ m \notin \{k,l\}}}^{n} \frac{(a_k x_l - a_l x_k)^2}{\frac{a_k a_l (a_k + a_l)}{a_m}} = \sum_{\substack{m=1 \\ m \notin \{k,l\}}}^{n} \frac{a_m (a_k x_l - a_l x_k)^2}{a_k a_l (a_k + a_l)}.$$

Taking the inequality above into account, we have that

$$\begin{aligned} &\frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{1 \leqslant i < j \leqslant n} \frac{(a_i x_j - a_j x_i)^2}{a_i a_j} \\ &\quad = \frac{1}{a_1 + a_2 + \ldots + a_n} \left(\frac{(a_k x_l - a_l x_k)^2}{a_k a_l} + \sum_{\substack{m=1 \\ m \notin \{k,l\}}}^{n} \left(\frac{(a_m x_l - a_l x_m)^2}{a_m a_l} + \frac{(a_m x_k - a_k x_m)^2}{a_m x_k} \right) \right) \\ &\qquad + \frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{\substack{1 \leqslant i < j \leqslant n \\ i,j \notin \{k,l\}}} \frac{(a_i x_j - a_j x_i)^2}{a_i a_j} \\ &\quad \geqslant \frac{1}{a_1 + a_2 + \ldots + a_n} \left(\frac{(a_k x_l - a_l x_k)^2}{a_k a_l} + \sum_{\substack{m=1 \\ m \notin \{k,l\}}}^{n} \frac{a_m (a_k x_l - a_l x_k}{a_k a_l (a_k + a_l)} \right) \\ &\qquad + \frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{\substack{1 \leqslant i < j \leqslant n \\ i,j \notin \{k,l\}}} \frac{(a_i x_j - a_j x_i)^2}{a_i a_j}, \end{aligned}$$

from where and taking identity (5) into account, the inequality (10) results.

REMARK 2. From Theorem 5 the inequalities from (1) and (2) results.

THEOREM 6. *If* $n \in \mathbb{N},\ n \geqslant 2,\ x_i, y_i \in \mathbb{R},\ i \in \{1,2,\ldots,3\}$, *then*

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2 \geqslant \left(\sum_{i=1}^{n} y_i^2\right)\frac{(x_k y_l - x_l y_k)^2}{y_k^2 + y_l^2} + \sum_{\substack{1 \leqslant i < j \leqslant n \\ i,j \notin \{k,l\}}} (x_i y_j - x_j y_i)^2 \tag{11}$$

if $y_i \neq 0$ *for any* $i \in \{1,2,\ldots,n\}$,

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2 \geqslant \left(\sum_{i=1}^{n} x_i^2\right)\frac{(x_p y_q - x_q y_p)^2}{x_p^2 + x_q^2} + \sum_{\substack{1 \leqslant i < j \leqslant n \\ i,j \notin \{p,q\}}} (x_i y_j - x_j y_i)^2 \tag{12}$$

if $x_i \neq 0$ *for any* $i \in \{1,2,\ldots,\}$, *where* $k,l,p,q \in \{1,2,\ldots,n\},\ k \neq l,\ p \neq q$ *such that* $\max_{1 \leqslant i < j \leqslant n} \frac{(x_i y_j - x_j y_i)^2}{y_i^2 + y_j^2}$ *is obtained for* $i = k$ *and* $j = l$, $\max_{1 \leqslant i < j \leqslant n} \dfrac{(x_i y_j - x_j y_i)^2}{x_i^2 + x_j^2}$ *is obtained for* $i = p$ *and* $j = q$.

Proof. In Theorem 5 we change a_i by y_i^2 and x_i with $x_i y_i$, $i \in \{1,2,\ldots,n\}$ and we obtain the inequality (11), respectively x_i by y_i, $i \in \{1,2,\ldots,n\}$ in inequality (11) and then we obtain inequality (12).

COROLLARY 5. *If* $n \in \mathbb{N},\ n \geqslant 2,\ x_i, y_i \in \mathbb{R}\backslash\{0\},\ i \in \{1,2,\ldots,n\}$, *then the inequalities*

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2 \geqslant \left(\sum_{i=1}^{n} y_i^2\right)\frac{(x_i y_j - x_j y_i)^2}{y_i^2 + y_j^2} \tag{13}$$

and

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2 \geqslant \left(\sum_{i=1}^{n} x_i\right)^2\frac{(x_i y_j - x_j y_i)^2}{x_i^2 + y_j^2} \tag{14}$$

hold for any $i, j \in \{1,2,\ldots,n\}$.

Proof. It results from Theorem 6.

REMARK 3. Using the inequalities from Corollary 5, we obtain some inequalities from paper [5].

REFERENCES

[1] G. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge, 1934.
[2] E. F. BECHENCBACH AND R. BELLMAN, *Inequalities*, Springer, Berlin, Göttingen and Heidelberg, 1961.
[3] R. BELLMAN, *Notes on Matrix Theory* - IV (An Inequality Due to Bergström), Amer. Math. Montly, Vol. **62** (1955), 172–173
[4] H. BERGSTRÖM, *A triangle inequality for matrices*, Den Elfte Skandinaviske Matematikerkongress, 1949, Trodheim, Johan Grundt Tanums Forlag, Oslo, 1952, 264–267.
[5] D. MĂRGHIDANU, J. L. DÍAZ-BARRERO AND S. RĂDULESCU, *New refinements of some classical inequalities*, to appear in Math. Ineq.& Appl.
[6] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

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