Journal of
Mathematical
Inequalities
Volume 3, Number 2 (2009), 267–276

SOME CLASSES OF ANALYTIC FUNCTIONS RELATED WITH FUNCTIONS OF BOUNDED RADIUS ROTATION WITH RESPECT TO SYMMETRICAL POINTS

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(*Communicated by J. Pečarić*)

Abstract. In this paper, we introduce a class $R_k^s(\gamma)$ of analytic functions of bounded radius rotation with respect to symmetrical points and study some of its basic properties. Using this concept, two other classes $T_k^s(\delta,\gamma)$, $K_k^s(\delta,\gamma)$ are also defined. We study coefficient results, arc-length and radius problems for these classes.

1. Introduction

Let $\mathscr{A}$ be the class of analytic functions f defined on the unit disc $E=\{z:|z|<1\}$, normalized by $f(0)=f'(0)-1=0$ and of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \quad (z\in E). \tag{1.1}$$

Let $S,K,S^\star$ and C denote the subclasses of $\mathscr{A}$ which are univalent, close-to-convex, starlike and convex in E respectively. Let $P_k(\gamma)$ be the class of functions $p(z)$ analytic in the unit disc E satisfying the properties $p(0)=1$ and, for $z=re^{i\theta}, \quad k\geqslant 2$,

$$\int_0^{2\pi}\left|\mathrm{Re}\,\frac{p(z)-\gamma}{(1-\gamma)}\right|d\theta\leqslant k\pi, \quad (0\leqslant\gamma<1). \tag{1.2}$$

This class has been introduced in [6]. We note that $P_k(0)\equiv P_k$, see [14] and $P_2(\gamma)\equiv P(\gamma)$ is the class of analytic function with positive real part greater than γ. With $k=2$, $\gamma=0$, we have the class P of functions with positive real part.

We can write (1.2) as

$$p(z)=\frac{1}{2}\int_0^{2\pi}\frac{1+(1-2\gamma)ze^{-it}}{1-ze^{-it}}d\mu(t), \tag{1.3}$$

where $\mu(t)$ is a function with bounded variation on $[0,2\pi]$ such that

$$\int_o^{2\pi}d\mu(t)=2 \quad\text{and}\quad \int_o^{2\pi}|d\mu(t)\leqslant k. \tag{1.4}$$

Mathematics subject classification (2000): 30C45, 30C50.

Keywords and phrases: starlike, bounded radius rotation, symmetrical points, arc-length, coefficient problems.

Paper JMI-03-27

Also, for $p \in P_k(\gamma)$, we can write from (1.2)

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad p_1,\ p_2 \in P_2(\gamma),\ z \in E. \tag{1.5}$$

It is known [5] that $P_k(\gamma)$ is a convex set. Also $p \in P_k(\gamma)$ is in $P_2(\gamma) \equiv P(\gamma)$ for $|z| < r_1$, where

$$r_1 = \frac{1}{2}\left[k - \sqrt{k^2 - 4}\right]. \tag{1.6}$$

The classes $V_k(\gamma)$ of functions of bounded boundary rotation of order γ and $R_k(\gamma)$ of functions of bounded radius rotation of order γ are closely related with $P_k(\gamma)$. A function $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, analytic in E, is in $V_k(\gamma)$ if and only if $\left\{\frac{(zf'(z))'}{f'(z)}\right\} \in P_k(\gamma)$. Also

$$f \in R_k(\gamma) \quad \Longleftrightarrow \quad \left\{\frac{zf'(z)}{f(z)}\right\} \in P_k(\gamma).$$

It is clear that

$$f \in V_k(\gamma) \quad \Longleftrightarrow \quad zf'(z) \in R_k(\gamma) \tag{1.7}$$

When $k = 2,\ \gamma = 0,\ V_2(0)$ coincides with the class C and $R_2(0) \equiv S^\star$. We now define the following.

DEFINITION 1.1. Let $f \in \mathscr{A}$ and be given by (1.1). Then f is said to be of bounded radius rotation of order γ with respect to symmetrical points if and only if, for $|z| = r < 1 \quad (r \to 1)$,

$$\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} \in P_k(\gamma), \quad \text{for} \quad z \in E.$$

We shall denote the class of such functions as $R_k^s(\gamma)$. We note that $R_2^s(0)$ is the class $S_s^\star$ of univalent functions starlike with respect to symmetrical points defined by Sakaguchi [7]. Also $R_k^s(\gamma) \equiv R_k(\gamma)$.

We define the class $V_k^s(\gamma)$ as follows.

DEFINITION 1.2.

$$f \in V_k^s(\gamma) \quad \Longleftrightarrow \quad zf' \in R_k^s(\gamma), \quad \text{in} \quad E.$$

2. Basic Properties of $R_k^s(\gamma)$

THEOREM 2.1. *Let $f \in \mathscr{A}$. Then a necessary and sufficient condition for f to belong to $R_k^s(\gamma)$ is that $\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\} \in P_k(\gamma)$ for $z \in E$.*

Proof. Its proof is immediate when we follow essentially the same method given in [7]. □

THEOREM 2.2. *Let* $f \in R_k^s(\gamma)$. *Then the odd function*

$$\psi(z) = \frac{1}{2}\left[f(z) - f(-z)\right] \tag{2.1}$$

belongs to $R_k(\gamma)$ *in* E.

Proof. Differentiating (2.1) logarithmically, we have

$$\begin{aligned} \frac{z\psi'(z)}{\psi(z)} &= \frac{zf'(z)}{f(z)-f(-z)} + \frac{-zf'(-z)}{f(-z)-f(z)} \\ &= \frac{1}{2}\left[p_1(z)+p_2(z)\right], \quad p_1, p_2 \in P_k(\gamma). \end{aligned}$$

Since $P_k(\gamma)$ is a convex set, we have $\frac{z\psi'(z)}{\psi(z)} \in P_k(\gamma)$ for $z \in E$ and hence $\psi \in R_k(\gamma)$ in E. □

We note that $f \in R_k^s(\gamma)$ is close-to-convex for $|z| < r_1$, where r_1 is given by (1.6).

REMARK 2.1. Since ψ, defined in Theorem 2.2, is in $R_k(\gamma)$ and is odd, we can write

$$\psi(z) = \frac{(s_1(z))^{(\frac{k}{4}+\frac{1}{2})(1-\gamma)}}{(s_2(z))^{(\frac{k}{4}-\frac{1}{2})(1-\gamma)}}, \tag{2.2}$$

where s_1 and s_2 are odd starlike functions, see [1,5].

From relation (2.1) and Remark 2.1, we can easily derive the following.

THEOREM 2.3. *Let* $f \in R_k^s(0) \equiv R_k^s$. *Then with* $z = re^{i\theta}$ *and* $\theta_1 < \theta_2$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > -(k-1)\pi.$$

This is a necessary condition for a function f *to belong to* R_k^s. *For* $k = 2$, R_2^s *is a proper subclass of* S *and for* $k > 2$, $f \in R_k^s$ *need not even be finite-valent, see* [2].

REMARK 2.2. Let $f \in R_k^s(\gamma)$, and be given by (1.1). It is known [5] that for $p \in P_k(\gamma)$ with $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, we have $|c_n| \leqslant k(1-\gamma)$ for all n. Using this together with the fact that $\psi(z) = \frac{1}{2}\left[f(z) - f(-z)\right]$ is an odd function, we easily obtain $|a_2| \leqslant \frac{k}{2}(1-\gamma)$. Since, for $f \in R_k^{\gamma} \subset R_k^s$, the function $\frac{w_0 f(z)}{w_0 - f(z)}$, $f(z) \neq w_0$ is univalent in E. For $k = 2$, we see that $f \in R_2^s(\gamma)$ maps E onto a domain that contains the schlicht disc $|w| < \frac{1}{3-\gamma}$.

3. The Classes $T_k^s(\gamma)$ and $K_k^s(\gamma)$

DEFINITION 3.1. Let $f \in \mathscr{A}$. Then $f \in T_k^s(\gamma, \delta)$, $0 \leqslant \gamma, \delta < 1$, $k \geqslant 2$, if and only if, there exists a $g \in R_k^s(\gamma)$ such that

$$\left\{\frac{2zf'(z)}{g(z)-g(-z)}\right\} \in P(\delta), \quad \text{for} \quad z \in E.$$

DEFINITION 3.2. Let $f \in \mathscr{A}$. Then $f \in K_k^s(\gamma,\delta)$, $0 \leqslant \gamma,\delta < 1$, $k \geqslant 2$, if and only if, there exists a $\phi \in R_2^s(\gamma)$ such that

$$\left\{\frac{2zf'(z)}{\phi(z)-\phi(-z)}\right\} \in P_k(\delta), \quad \text{for} \quad z \in E.$$

We note that the classes $T_k^s(\gamma)$ and $K_k^s(\gamma)$ have same class of functions as a special case when $k=2$.

Let $L(r,f)$ denote the length of the image of the circle $|z|=r$ under f and $M(r) = \max_\theta \left|f(re^{i\theta})\right|$. We prove the following.

THEOREM 3.1. *Let $f \in T_k^s(0,\gamma)$. Then, for $0<r<1$,*

$$L(r,f) \leqslant c(k)M(r)\log\frac{1}{1-r},$$

where $c(k)$ is a constant.

Proof. With $z = re^{i\theta}$,

$$\begin{aligned} L(r,f) &= \int_0^{2\pi} \left|zf'(z)\right| d\theta \\ &= \int_0^{2\pi} |\psi(z)h(z)|\, d\theta, \quad \psi(z) = \frac{1}{2}[g(z)-g(-z)] \in R_k(\gamma), \quad h \in P(0) \equiv P \\ &\leqslant \int_0^r \int_0^{2\pi} \left|\psi'(\rho e^{i\theta})h(\rho e^{i\theta})\right| d\theta d\rho + \int_0^r \int_0^{2\pi} \left|\psi(\rho e^{i\theta})h'(\rho e^{i\theta})\right| d\theta d\rho \\ &= J_1(r) + J_2(r). \end{aligned} \tag{3.1}$$

Now

$$\begin{aligned} J_1(r) &= \int_0^r \int_0^{2\pi} \left|f'(\rho e^{i\theta})H(\rho e^{i\theta})\right| d\theta d\rho, \quad H = \frac{z\psi'}{\psi} \in P_k(\gamma) \\ &\leqslant 2\pi \int_0^r \left[\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f'(\rho e^{i\theta})\right|^2 d\theta\right)^{\frac{1}{2}} \left(\frac{1}{2}\int_0^{2\pi}\left|H(\rho e^{i\theta})\right|^2 d\theta\right)^{\frac{1}{2}}\right] d\rho. \end{aligned}$$

Thus, with $f(z)$ given by (1.1), $H(z) = 1+\sum_{n=1}^\infty c_n z^n$, $|c_n| \leqslant k(1-\gamma)$ and $n \geqslant 1$, we have

$$\begin{aligned} J_1(r) &\leqslant 2\pi \int_0^r \left[\left(\sum_{n=1}^\infty n^2|a_n|^2\rho^{2n-2}\right)^{\frac{1}{2}} \left(\sum_{n=0}^\infty |c_n|^2\rho^{2n}\right)^{\frac{1}{2}}\right] d\rho \\ &\leqslant \sqrt{2}k(1-\gamma)\pi \left(\sum_{n=1}^\infty n(a_n|^2 r^{2n-1}\right)^{\frac{1}{2}} \left(\log\frac{1+r}{1-r}\right)^{\frac{1}{2}}. \end{aligned}$$

But $A(r) = \pi\sum_{n=1}^\infty n|a_n|^2 r^{2n}$ is the area of the image of $|z|<r$ by $w=f(z)$, and, since $A(r) \leqslant \pi M^2(r)$, we have

$$J_1(r) \leqslant \sqrt{2}k(1-\gamma)\pi M(r)\left(\frac{1}{r}\log\frac{1+r}{1-r}\right)^{\frac{1}{2}}, \quad (r \to 1). \tag{3.2}$$

Next we estimate $J_2(r)$.

With h given by (1.3) and (1.4), $\gamma=0,\ k=2,$ we have

$$h'(z)=\frac{1}{\pi}\int_0^{2\pi}\frac{e^{-it}}{(1-ze^{-it})^2}d\mu(t).$$

Since

$$\operatorname{Re} h(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-\rho^2)}{|1-z^{-it}|^2}d\mu(t),$$

we have

$$\begin{aligned}J_2(r)&\leqslant 2\int_0^r\int_0^{2\pi}\left|\psi(\rho e^{i\theta})\operatorname{Re} h(\rho e^{i\theta})\right|d\theta\frac{d\rho}{1-\rho^2}\\&=2\int_0^r\left(\int_0^{2\pi}\operatorname{Re}\left[(\rho e^{i\theta})f'(\rho e^{i\theta})e^{-i\arg\psi}\right]d\theta\right)\frac{d\rho}{1-\rho^2}.\end{aligned}$$

Integration by parts gives

$$J_2(r)\leqslant 4\pi\int_0^r\frac{M(\rho)}{1-\rho^2}d\rho. \tag{3.3}$$

from (3.1), (3.2) and (3.3), we obtain the required result. □

We note that, following the techniques of Theorem 3.1, we can prove similar arc length problem for the class $K_k^s(0,\gamma)$.

THEOREM 3.2. *Let $f\in T_k^s(\delta,\gamma)$ and be given by (1.1). Then*

$$|a_n|\leqslant b(k,\delta,\gamma)n^{\left\{(\frac{k}{4}+\frac{1}{2})(1-\gamma)\right\}-1}\quad(n\geqslant 1),$$

where $b(k,\delta,\gamma)$ is a constant depending only on $k,\delta,$ and γ. The function $f_0\in T_k^s(\delta,\gamma)$ defined by

$$f_0'(z)=\frac{\left(1+z^2\right)^{(\frac{k}{4}+\frac{1}{2})(1-\gamma)}}{\left(1-z^2\right)^{(\frac{k}{4}-\frac{1}{2})(1-\gamma)}}\left\{(1-\delta)\frac{1-z}{1+z}+\delta\right\} \tag{3.4}$$

shows that the exponential $\left\{(\frac{k}{4}+\frac{1}{2})(1-\gamma)-1\right\}$ is best possible.

Proof. We set

$$F(z)=\left(zf'(z)\right)'=\psi(z)\left[H(z)h(z)+zH'(z)\right],$$

where $h=\frac{z\psi'}{\psi}\in P_k(\gamma),\ H\in P(\delta)$ and $2\psi(z)=[g(z)-g(-z)],\ g$ is as defined in Definition 3.1. Thus, for $n\geqslant 1,\ z=re^{i\theta}$, Cauchy's Theorem gives us

$$\begin{aligned}n^2|a_n|&=\frac{1}{2\pi r^n}\left|\int_0^{2\pi}F(z)e^{-in\theta}d\theta\right|\\&\leqslant\frac{1}{2\pi r^n}\int_0^{2\pi}|\psi(z)||H(z)h(z)+zH'(z)|d\theta\\&\leqslant\frac{1}{2\pi r^n}\left(\frac{2}{r}\right)^{(\frac{k}{4}-\frac{1}{2})(1-\gamma)}\left(\frac{r}{1-r^2}\right)^{(\frac{k}{4}+\frac{1}{2})(1-\gamma)}\int_0^{2\pi}\left|H(z)h(z)+zH'(z)\right|d\theta,\end{aligned} \tag{3.5}$$

where we have used (2.2) and the well-known distortion theorem for odd starlike functions.

Now

$$\int_0^{2\pi} \left|H(z)h(z)+zH'(z)\right| d\theta \leqslant \left(\int_0^{2\pi} |h(z)|^2 d\theta\right)^{\frac{1}{2}} \left(\int_0^{2\pi} |H(z)|^2 d\theta\right)^{\frac{1}{2}} + \int_0^{2\pi} |zH'(z)| d\theta$$
$$\leqslant \left(\frac{1+\{k^2(1-\gamma)^2-1\}r^2}{1-r^2}\right)^{\frac{1}{2}} \left(\frac{1+\{4(1-\delta)^2-1\}r^2}{1-r^2}\right)^{\frac{1}{2}} + \frac{2(1-\delta)}{1-r^2}, \tag{3.6}$$

by using a modified version of a Lemma proved in [5] for $h,\ H \in P_k,\ k \geqslant 2$.

From (3.5) and (3.6), we obtain

$$n^2|a_n| \leqslant b(k,\gamma,\delta)\left(\frac{1}{1-r}\right)^{\{(\frac{k}{4}+\frac{1}{2}(1-\gamma)\}+1}, \quad (r \to 1).$$

Taking $r = 1-\frac{1}{n}$, we have the required result. □

We note, as a special cases, that for $k=2,\ a_n = O(1)n^{-\gamma}$.

Using the similar techniques, we can prove the following coefficient result for the class $K_k^s(\delta,\gamma)$.

THEOREM 3.3. *Let $f \in K_k^s(\delta,\gamma)$ and be given by (1.1). Then*

$$|a_n| \leqslant B(k,\delta,\gamma)n^{2-\gamma}, \quad (n \geqslant 1)$$

and $B(k,\delta,\gamma)$ is a constant which depends only on k,δ and γ. The function $f_1 \in K_k^s(\delta,\gamma)$ and defined by

$$f_1'(z) = \frac{1}{(1-z^2)^{(1-\gamma)}}\left\{\left(\frac{k}{4}+\frac{1}{2}\right)\left[(1-\delta)\frac{1-z}{1+z}+\delta\right] - \left(\frac{k}{4}-\frac{1}{2}\right)\left[(1-\delta)\frac{1+z}{1-z}+\delta\right]\right\} \tag{3.7}$$

shows that the exponent $(2-\gamma)$ is best possible.

For our next result, we need the following lemmas.

LEMMA 3.1. *Let $g \in R_2^s(\gamma)$ and for $m = 1,2,3,\ldots$, let G be defined by*

$$G(z) = \frac{m+1}{2z^m}\int_0^z t^{m-1}\{g(t)-g(-t)\}dt. \tag{3.8}$$

Then G is starlike for $z \in E$.

Proof. Let

$$J(z) = \int_0^z t^{m-1}\frac{[g(t)-g(-t)]}{2}dt.$$

Now, since $\frac{g(z)-g(-z)}{2}$ is starlike in E, $J(z)$ is $(m+1)$-valently starlike in E. We can write (3.8) as

$$z^m G(z) = (m+1)J(z),$$

and differentiating logarithmically, we have

$$\frac{zG'(z)}{G(z)} = \frac{zJ'(z) - mJ(z)}{J(z)}.$$

Setting $N(z) = zJ'(z) - mJ(z)$ and $D(z) = J(z)$, we see that $N(0) = D(0) = 0$. Also

$$\frac{N'(z)}{D'(z)} = \frac{1}{2}\left[\frac{2zg'(z)}{g(z)-g(-z)} + \frac{2zg'(-z)}{g(z)-g(-z)}\right] = p(z), \quad p \in P(\gamma)$$

in E, since $P(\gamma)$ is a convex set. Therefore, using a result from Libera [3], $\frac{N(z)}{D(z)} \in P(\gamma)$ for $z \in E$. □

LEMMA 3.2. *Let N and D be analytic functions in E with $N(0) = D(0) = 0$, D maps E onto a many sheeted region which is starlike of order γ with respect to origin and let $\frac{N'}{D'} \in P(\delta)$. Then $\frac{N}{D} \in P_k(\delta)$. in E.*

Proof. Let

$$\frac{N(z)}{D(z)} = H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z),$$

where H is analytic in E, with $H(0) = 1$. Then

$$\begin{aligned}\frac{N'(z)}{D'(z)} &= H(z) + \frac{zH'(z)}{H_0(z)}, \quad \text{where} \quad H_0(z) = \frac{zD'(z)}{D(z)} \in P(\gamma) \quad \text{in} \quad E\\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left[h_1(z) + \frac{zh_1'(z)}{H_0(z)}\right] - \left(\frac{k}{4}-\frac{1}{2}\right)\left[h_2(z) + \frac{zh_2'(z)}{H_0(z)}\right].\end{aligned}$$

Since $\frac{N'(z)}{D'(z)} \in P_k(\gamma)$, it follows that

$$\left\{h_i(z) = \frac{zh'(z)}{H_0(z)}\right\} \in P(\delta), \quad H_o \in P(\gamma), \quad i = 1,2.$$

With $h_i(z) = (1-\delta)p_i(z) + \delta$, we have

$$\left[(1-\delta)p_i(z) + \frac{(1-\delta)zp_i'(z)}{H_0(z)}\right] \in P \quad \text{in} \quad E.$$

We form the functional $\Psi(u,v)$ by taking $u = p_i(z)$, $v = zp_i'(z)$ with $u = u_1 + iu_2$, $v = v_1 + iv_2$, and use a well-known Lemma due to Miller [4] to conclude that $p_i \in P$, $i = 1,2$ and therefore $h_i \in P(\delta)$, $i = 1,2$ for $z \in E$. Consequently $H \in P_k(\delta)$ in E and the proof is complete. □

THEOREM 3.4. *Let $f \in K_k^s(\delta,\gamma)$. Then the function F defined by*

$$F(z) = \frac{m+1}{2z^m}\int_0^z t^{m-1}\,[f(t)-f(-t)]\,dt \tag{3.9}$$

also belongs to $K_k^s(\delta,\gamma)$ for $z \in E$ and $m = 1,2,3,\ldots$.

Proof. Since $f \in K_2^s(\delta,\gamma)$, there is a function $g \in R_2^s(\gamma)$ such that $\left\{\frac{2zf'(z)}{g(z)-g(-z)}\right\} \in P_k(\delta)$ in E. Now, by Lemma 3.1, G defined by (3.8) belongs to $R_k^s(\gamma)$ in E, and by definition it follows that there exists $G_1 \in V_2^s(\gamma)$ such that $G = zG_1'$ in E. Thus, from (3.9), we have with $g = zg_1'$

$$\begin{aligned}\frac{2F'(z)}{(G_1(z)-g_1(-z))'} &= \frac{z^m\,[f(z)-f(-z)] - m\int_0^z t^{m-1}\,[f(t)-f(-t)]\,dt}{z^m\,[g_1(z)-g_1(-z)] - m\int_0^z t^{m-1}\,[g_1(t)-g_1(-t)]\,dt}\\ &= \frac{N(z)}{D(z)}, \quad \text{say.}\end{aligned}$$

We note that $N(0) = D(0) = 0$ and for $g_1 \in V_2^s(\gamma)$,

$$\frac{(zD'(z))'}{D'(z)} = m + \frac{[z[g_1(z)-g_1(-z)]']'}{[g_1(z)-g_1(-z)]'} \in P(\gamma_1) \subset P(\gamma) \quad \text{in} \quad E.$$

This implies g_1 is convex and hence starlike in E. Since

$$\frac{N'(z)}{D'(z)} = \frac{1}{2}\left[\frac{2zf'(z)}{(g_1(z)-g_1(-z))'} + \frac{2zf'(-z)}{(g_1(z)-g_1(-z))'}\right] \in P_k(\delta), \quad \text{for} \quad z \in E,$$

we use Lemma 3.2 to have $\frac{N(z)}{D(z)} \in P_k(\delta)$ in E. This completes the proof. □

THEOREM 3.5. *Let $f \in K_k^s(0,0) \equiv K_k^s$ and let*

$$F_1(z) = \frac{1}{1+m} z^{1-m}\,[z^m f(z)]', \quad m = 1,2,\ldots. \tag{3.10}$$

Then $F_1 \in K_k^s$ for

$$|z| < r_1 = \frac{1+m}{2+\sqrt{3+m^2}}. \tag{3.11}$$

Proof. Let

$$F_1(z) = \frac{1}{1+m}\left[mf(z) + zf'(z)\right]. \tag{3.12}$$

Since, $f \in K_k^s$, there exists $g \in R_2^s(0) \equiv R_2^s$ such that

$$\left\{\frac{2zf'(z)}{g(z)-g(-z)}\right\} \in P_k, \quad z \in E.$$

Therefore, from (3.12), we can write

$$\frac{2zF_1'(z)}{g(z)-g(-z)} = \frac{1}{1+m}\left[\frac{2mzf'(z)}{g(z)-g(-z)} + \frac{2z(zf'(z))'}{g(z)-g(-z)}\right]$$
$$= \frac{1}{1+m}\left[mp(z)+zp'(z)+p(z)h(z)\right],$$

where $p \in P_k$, $h(z) = \frac{z\psi'(z)}{\psi(z)} \in P.$ with $\psi = \frac{1}{2}\left[g(z)-g(-z)\right]$. Since $p \in P_k$, we use (1.5) with $\gamma = 0$ to have

$$\frac{2zF_1'(z)}{g(z)-g(-z)} = \left(\frac{k}{4}+\frac{1}{2}\right)\left\{\frac{1}{1+m}\left[mp_1(z)+zp_1'(z)+p_1(z)h(z)\right]\right\}$$
$$-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{\frac{1}{1+m}\left[mp_2(z)+zp_2'(z)+p_2(z)h(z)\right]\right\}, \quad p_1,p_2,h \in P.$$

Now

$$\operatorname{Re}\left\{\frac{1}{1+m}\left[mp_i(z)+zp_i'(z)+p_i(z)h(z)\right]\right\} \geqslant \frac{\operatorname{Re} p_i(z)}{1+m}\left[m+\frac{1-r}{1+r}-\frac{2r}{1-r^2}\right]$$
$$= \frac{\operatorname{Re} p_i(z)}{1+m}\left[\frac{(1-m)r^2-4r+(1+m)}{1-r^2}\right],$$

and the right hand side is positive for $|z| < r_1$ and consequently $F_1 \in K_k^s$ for $|z| < r_1$, where r_1 is given by (3.11). This completes the proof. □

THEOREM 3.6. *Let* $f \in T_k^s(0,0) \equiv T_k^s$ *and let* F_1 *be defined by* (3.10). *Then* $F_1 \in T_k^s$ *for* $|z| < r_1$, *where* r_1 *is given by* (3.11).

Proof. Since $f \in T_k^s$, there exists $g \in R_k^s(0) = R_k^s$ such that $\left\{\frac{2zf'(z)}{g(z)-g(-z)}\right\} = p \in P$, $z \in E$. Now, from (3.12), we have

$$\frac{2zF_1'(z)}{g(z)-g(-z)} = \frac{1}{1+m}\left[mp(z)+zp'(z)+p(z)h(z)\right],$$

where $p \in P$ and $h = \frac{z\psi'(z)}{\psi(z)} \in P_k$ with $\psi(z) = \frac{1}{2}\left[g(z)-g(-z)\right]$. We use (1.5) to have

$$\frac{2zF_1'(z)}{g(z)-g(-z)} = \left(\frac{k}{4}+\frac{1}{2}\right)\left[\frac{1}{1+m}\left\{mp(z)+zp'(z)+p(z)h_1(z)\right\}\right]$$
$$-\left(\frac{k}{4}+\frac{1}{2}\right)\left[\frac{1}{1+m}\left\{mp(z)+zp'(z)+p(z)h_2(z)\right\}\right],$$
$$h_1,h_2 \in P \quad \text{for} \quad z \in E.$$

We note that

$$\operatorname{Re}\left[\frac{1}{1+m}\left\{mp(z)+zp'(z)+p(z)h_i(z)\right\}\right] \geqslant \frac{\operatorname{Re} p(z)}{1+m}\left[m+\frac{1-r}{1+r}-\frac{2r}{1-r^2}\right], \quad i=1,2$$

and the right hand side is positive for $|z| < r_1$, where r_1 is given by (3.11), $\square$

Acknowledgement. The authors would like to express their gratitude to the referee for his/her constructive comments and to Dr. M. Junaid Zaidi, Rector, CIIT, for providing excellent research facilities.

REFERENCES

[1] D. A. BRANNAN, *On functions of bounded boundary rotation 1*, Proc. Edingburgh Math. Soc., **16** (1968/69), 339–347.
[2] A. W. GOODMAN, *On close-to-convex functions of higher order*, Ann. Univ. Sci. Buda Eotous Sect. Math., **25** (1972), 17–30.
[3] R. J. LIBERA, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., **16** (1965), 755–758.
[4] S. S. MILLER, *Differential inequalities and Caratheodory functions*, Bull. Amer. Math. Soc., **81** (1975), 79–81.
[5] K. INAYAT NOOR, *On subclasses of close-to-convex functions of higher order*, Inter. J. Math. Math. Sci., **15** (1992), 279–290.
[6] B. PINCHUK, *Functions with bounded boundary rotation*, Isr. J. Math., **10** (1971), 7–16.
[7] K. SAKAGUCHI, ON A CERTAIN UNIVALENT MAPPING, J. Math. Soc. Japan, **11** (1959), 72–75.

(Received May 17, 2007)

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