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PARTIAL Δ-DIFFERENTIATION FOR MULTIVARIABLE FUNCTIONS ON n-DIMENSIONAL TIME SCALES

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Abstract. The general idea of this paper is to study a differential calculus for multivariable functions on time scales. Such a calculus can be used to develop a theory of partial dynamic equations on time scales.

1. Introduction

The unification and extension of continuous calculus, discrete calculus, q-calculus, and indeed arbitrary real-number calculus to time scale calculus was first accomplished by Hilger in his PhD thesis [9]. This theory is very important and useful in the mathematical modelling of several important dynamic processes. As a result the theory of dynamic systems on time scales is developed in ([1], [2], [4]–[8], [11]–[15]).

The present paper deals with the differential calculus for multivariable functions on time scales and intends to prepare an instrument for introducing and investigating partial dynamic equations on time scales.

There are a number of differences between the calculus one and of two variables. The calculus for functions of three or more variables differs only slightly from that of two variables. Bohner and Guseinov have published a paper about the partial differentiation on time scale. Here, authors introduced partial delta and nabla derivatives and the chain rule for multivariable functions on time scale and also the concept of the directional derivative [7].

The contents of this paper are as follows. In Section 2, we give a brief account of time scale calculus which will be used later. In Section 3, we introduce partial delta and nabla derivatives for multivariable functions on time scales and offer several new concepts related to differentiability. In Section 4, we give several useful mean value theorems for derivatives. In Section 5, we extend the chain rule for multivariable functions on time scales. Finally, in Section 6 we investigate some properties of directional derivative on time scales.

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2. Preliminaries from Time Scales Calculus

The following definitions and theorems will serve as a short primer on time scale calculus; they can be found in ([5], [6]). A time scale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$. Within that set, define the jump operators $\rho, \sigma : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\} \quad \text{and} \quad \sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\},$$

where $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$, where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, then we say that t is *right-scattered*. If $\sigma(t) = t$ and $t < \sup\mathbb{T}$, we say that t is *right-dense*. The backward jump operator, *left-scattered* and *left-dense* points are defined in similar way. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. The so-called graininess functions are $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, the delta derivative of f at t, denoted $f^{\Delta}(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right| \leqslant \varepsilon \left| \sigma(t) - s \right|$$

for all $s \in U$. For $\mathbb{T} = \mathbb{R}$, $f^{\Delta} = f'$, the usual derivative; for $\mathbb{T} = \mathbb{Z}$ the delta derivative is the forward difference operator, $f^{\Delta}(t) = f(t+1) - f(t)$.

THEOREM 1. *If $f, g : \mathbb{T} \to \mathbb{R}$ are $\Delta-$differentiable at $t \in \mathbb{T}^k$, then*

(i) *$f + g$ is $\Delta-$differentiable at t and*

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) *For any constant c, cf is $\Delta-$differentiable at t and*

$$(cf)^{\Delta}(t) = cf^{\Delta}(t).$$

(iii) *$f.g$ is $\Delta-$differentiable at t and*

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) \\ &= g^{\Delta}(t)f(t) + g(\sigma(t))f^{\Delta}(t).\end{aligned}$$

(iv) *If $g(t).g(\sigma(t)) \neq 0$ then $\dfrac{f}{g}$ is $\Delta-$differentiable at t and*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t).}{g(t).g(\sigma(t))}.$$

THEOREM 2. *Let $\mathbb{T}$ be a time scale and $\nu : \mathbb{T} \to \mathbb{R}$ be a strictly increasing function such that $\overline{\mathbb{T}} = \nu(\mathbb{T})$ is also a time scale. by $\overline{\sigma}$ we denote the jump function on $\overline{\mathbb{T}}$, and by $\overline{\Delta}$ we denote the derivative on $\overline{\mathbb{T}}$. Then*

$$\nu \circ \sigma = \overline{\sigma} \circ \nu.$$

THEOREM 3. (Chain Rule) *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\overline{\mathbb{T}} = \nu(\mathbb{T})$ is a time scale. Let $w : \overline{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $w^{\overline{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then $(w \circ \nu)^{\Delta}$ exist at t and satisfies the chain rule*

$$(w \circ \nu)^{\Delta} = (w^{\overline{\Delta}} \circ \nu)\nu^{\Delta} \ at \ t.$$

Many other information concerning time scales and dynamic equations on time scales can be found in the books ([5], [6]).

3. Partial Differentiation on time Scales

Let $n \in N$ be fixed and for each $i \in \{1,2,...,n\}$, $\mathbb{T}_i$ denote a time scale. Let us set

$$\wedge^n = \mathbb{T}_1 \times \mathbb{T}_2 \times ... \times \mathbb{T}_n = \{(l_1,l_2,...,l_n) : l_i \in \mathbb{T}_i \text{ for all } i \in \{1,2,...,n\}\}.$$

We call $\wedge^n$ an n-dimensional time scale. the set $\wedge^n$ is a complete metric space with the metric d defined by

$$d(t,s) = \left(\sum_{i=1}^{n} |t_i - s_i|^2\right)^{\frac{1}{2}}, \forall t,s \in \wedge^n.$$

Let $f : \wedge^n \to \mathbb{R}$ be a function. The partial delta derivative of f with respect to $t_i \in \mathbb{T}_i^k$ is defined as the limit

$$\lim_{\substack{s_i \to t_i \\ s_i \neq \sigma_i(t_i)}} \frac{f(t_1,t_2,...,t_{i-1},\sigma_i(t_i),t_{i+1},...,t_n) - f(t_1,t_2,...,t_{i-1},s_i,t_{i+1},...,t_n)}{\sigma_i(t_i) - s_i} = \frac{\partial f(t)}{\Delta_i t_i}.$$

Higher order partial delta derivatives are similarly defined.

DEFINITION 1. We say that a function $f : \wedge^n \to \mathbb{R}$ is completely delta differentiable at the point $t^0 \in \mathbb{T}_1^k \times \mathbb{T}_2^k \times ... \times \mathbb{T}_n^k$ if there exist numbers $A_1,...,A_n$ independent of $t = (t_1,...,t_n) \in \wedge^n$ (but, generally, dependent on $(t_1^0,...,t_n^0)$) such that all $t \in U_\delta(t^0)$,

$$f(t_1^0,t_2^0,...,t_n^0) - f(t_1,t_2,...,t_n) = \sum_{i=1}^{n} A_i(t_i^0 - t_i) + \sum_{i=1}^{n} \alpha_i(t_i^0 - t_i) \tag{3.1}$$

and, for $j \in \{1,...,n\}$ and all $t \in U_\delta(t^0)$,

$$\begin{aligned} &f(t_1^0,...,t_{j-1}^0,\sigma_j(t_j^0),t_{j+1}^0,...,t_n^0) - f(t_1,...,t_{i-1},t_i,t_{i+1},...,t_n) \\ &= A_j\left[\sigma_j(t_j^0) - t_j\right] + \sum_{\substack{i=1\\ i\neq j}}^{n} A_i\left[t_i^0 - t_i\right] + \beta_j\left[\sigma_j(t_j^0) - t_j\right] + \sum_{\substack{i=1\\ i\neq j}}^{n} \beta_i\left[t_i^0 - t_i\right], \end{aligned} \tag{3.2}$$

where δ is a sufficiently small positive number, $U_\delta(t^0)$ is the the $\delta-$neighborhood of t^0 in $\wedge^n$, $\alpha_i = \alpha_i(t^0,t)$ and $\beta_i = \beta_i(t^0,t)$ are defined on $U_\delta(t^0)$ such that they are equal to zero at $t = t^0$ and such that

$$\lim_{t\to t^0} \alpha_i(t^0,t) = 0 \quad \text{and} \quad \lim_{t\to t^0} \beta_i(t^0,t) = 0 \quad \text{for all } i \in \{1,...,n\}.$$

In the case $\mathbb{T}_1 = ... = \mathbb{T}_n = \mathbb{R}$, this definition coincides with the classical(total) differentiability of functions of $n-$ variables ([3], [10]).

It follows from Definition 1 that if the function $f : \wedge^n \to \mathbb{R}$ is completely delta differentiable at the point $t^0 \in \mathbb{T}_1^k \times \mathbb{T}_2^k \times ... \times \mathbb{T}_n^k$, then it is continuous at that point and has at t^0 the first order partial delta derivatives equal, respectively, to $A_1,...,A_n$:

$$\frac{\partial f(t^0)}{\Delta_1 t_1} = A_1,..., \frac{\partial f(t^0)}{\Delta_n t_n} = A_n.$$

The continuity of f at t^0 follows, in fact, from any one of (3.1) and (3.2) for some $j \in \{1,...,n\}$. Indeed, (3.1) obviously yields the continuity of f at t^0. Let now (3.2) hold for some $j \in \{1,...,n\}$. In the case $\sigma_j(t_j^0) = t_j^0$, (3.2) immediately gives the continuity of f at t^0. Consider the case $\sigma_j(t_j^0) > t_j^0$. Except of $f(t)$, each term in (3.2) has a limit as $t \to t^0$. Therefore $f(t)$ also has a limit as $t \to t^0$, and passing to the limit we get

$$f(t_1^0,...,t_{j-1}^0, \sigma_j(t_j^0), t_{j+1}^0,...,t_n^0) - \lim_{t \to t^0} f(t) = A_j[\sigma_j(t_j^0) - t_j^0].$$

Further, letting $t = t^0$ in (3.2), we obtain

$$f(t_1^0,...,t_{j-1}^0, \sigma_j(t_j^0), t_{j+1}^0,...,t_n^0) - f(t^0) = A_j[\sigma_j(t_j^0) - t_j^0].$$

Comparing the last two relations gives

$$\lim_{t \to t^0} f(t) = f(t^0)$$

so that the continuity of f at t^0 is shown. Next, setting in (3.2) $t_i = t_i^0$ for all $i \neq j$ and then dividing both sides by $\sigma_j(t_j^0) - t_j$ and passing to the limit as $t_j \to t_j^0$, we arrive at $\frac{\partial f(t^0)}{\nabla_j t_j} = A_j$. This also shows the uniqueness of the numbers $A_1,...,A_n$ presented in (3.1), (3.2). Note also that due to the continuity of f at t^0 we get from (3.2) in the case $\sigma_j(t_j^0) > t_j^0$ the formula

$$\frac{\partial f(t^0)}{\Delta_j t_j} = \frac{f(t_{j1}^0,...,t_{j-1}^0, \sigma_j(t_j^0), t_{j+1}^0,...,t_n^0) - f(t^0)}{\sigma_j(t_j^0) - t_j}.$$

DEFINITION 2. We say that a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \times ... \times \mathbb{T}_n \to \mathbb{R}$ is σ_j-completely delta differentiable at a point $t^0 = (t_1^0,...,t_n^0) \in \mathbb{T}_1^k \times \mathbb{T}_2^k \times ... \times \mathbb{T}_n^k$ if it is completely delta differentiable at that point in the sense of conditions (3.1), (3.2) and moreover, along with the numbers $A_1,...,A_n$ presented in (3.1) and (3.2) there exists also numbers $B_1,...,B_n$ independent of $t = (t_1,...,t_n) \in \mathbb{T}_1 \times \mathbb{T}_2 \times ... \times \mathbb{T}_n$ (but, generally, dependent on $(t_1^0,...,t_n^0)$) such that for $j \in \{1,...,n\}$

$$f(\sigma_1(t_1^0), \sigma_2(t_2^0), ..., \sigma_n(t_n^0)) - f(t_1,t_2,...,t_n)$$

$$= A_j \left[\sigma_j(t_j^0) - t_j\right] + \sum_{\substack{i=1 \\ i \neq j}}^{n} B_i \left[\sigma_i(t_i^0) - t_i\right] + \gamma_j \left[\sigma_j(t_j^0) - t_j\right] + \sum_{\substack{i=1 \\ i \neq j}}^{n} \gamma_i \left[\sigma_i(t_i^0) - t_i\right] \tag{3.3}$$

for all $t=(t_1,...,t_n)\in V^{\sigma_j}(t_1^0,...,t_n^0)$, where $V^{\sigma_j}(t_1^0,...,t_n^0)$ is a union of some neighborhoods of the points $(t_1^0,...,t_n^0)$ and $(\sigma_1(t_1^0),...,t_i^0...,\sigma_n(t_n^0))$, and the functions $\gamma_j=\gamma_j(t^0;t)$ and $\gamma_i=\gamma_i(t^0;t_i)$ are equal to zero for $(t_1,...,t_n)=(t_1^0,...,t_n^0)$ and

$$\lim_{t\to t^0}\gamma_j(t^0;t)=0 \quad\text{and}\quad \lim_{t_i\to t^0}\gamma_i(t^0;t_i)=0.$$

Note that in (3.3) the function γ_{ji} depends only on the variable t_i. Setting $t_1=\sigma_1(t_1),...,\ t_{i-1}=\sigma_{i-1}(t_{i-1}),\ t_i\neq\sigma_i(t_i),\ t_{i+1}=\sigma_{i+1}(t_{i+1}),...,t_n=\sigma_n(t_n)$ in (3.3) yields

$$B_i=\frac{\partial f(\sigma_1(t_1^0),\sigma_2(t_2^0),...,t_i,...,\sigma_n(t_n^0))}{\Delta_i t_i}.$$

For $j=1$, Definition 2 becomes the following: A function $f:\mathbb{T}_1\times\mathbb{T}_2\times...\times\mathbb{T}_n\to\mathbb{R}$ is σ_1-completely delta differentiable at a point $t^0=(t_1^0,...,t_n^0)\in\mathbb{T}_1^k\times\mathbb{T}_2^k\times...\times\mathbb{T}_n^k$ if it is completely delta differentiable at that point in the sense of conditions (3.1), (3.2) and moreover, along with the numbers $A_1,...,A_n$ presented in (3.1) and (3.2) there exists also numbers $B_1,...,B_n$ independent of $t=(t_1,...,t_n)\in\mathbb{T}_1\times\mathbb{T}_2\times...\times\mathbb{T}_n$ (but, generally, dependent on $(t_1^0,...,t_n^0)$) such that

$$\begin{aligned}&f(\sigma_1(t_1^0),\sigma_2(t_2^0),...,\sigma_n(t_n^0))-f(t_1,t_2,...,t_n)\\&=A_1\left[\sigma_1(t_1^0)-t_1\right]+\sum_{i=2}^{n}B_i\left[\sigma_i(t_i^0)-t_i\right]+\gamma_1\left[\sigma_1(t_1^0)-t_1\right]+\sum_{i=2}^{n}\gamma_i\left[\sigma_i(t_i^0)-t_i\right]\end{aligned}\tag{3.4}$$

for all $t=(t_1,...,t_n)\in V^{\sigma_1}(t_1^0,...,t_n^0)$. Here if we take $n=2$, then we have $i=2$ because of $j=1$. Therefore, equality (3.4) reduces:

$$\begin{aligned}f(\sigma_1(t_1^0),\sigma_2(t_2^0))-f(t_1,t_2)&=A_1\left[\sigma_1(t_1^0)-t_1\right]+B_2\left[\sigma_2(t_2^0)-t_2\right]\\&\quad+\gamma_1\left[\sigma_1(t_1^0)-t_1\right]+\gamma_2\left[\sigma_2(t_2^0)-t_2\right]\end{aligned}\tag{3.5}$$

where $A_1=\dfrac{\partial f(t_1^0,t_2^0)}{\Delta_1 t_1}$ and $B_2=\dfrac{\partial f(\sigma_1(t_1^0),t_2^0)}{\Delta_2 t_2}$.

Also, for $j=2$ the equality (3.3) in Definition 2 becomes:

$$\begin{aligned}&f(\sigma_1(t_1^0),\sigma_2(t_2^0),...,\sigma_n(t_n^0))-f(t_1,t_2,...,t_n)\\&=A_2\left[\sigma_2(t_2^0)-t_2\right]+\sum_{\substack{i=1\\i\neq2}}^{n}B_i\left[\sigma_i(t_i^0)-t_i\right]+\gamma_2\left[\sigma_2(t_2^0)-t_2\right]+\sum_{\substack{i=1\\i\neq2}}^{n}\gamma_i\left[\sigma_i(t_i^0)-t_i\right].\end{aligned}\tag{3.6}$$

Here if we take again $n=2$, then we have $i=1$ because of $j=2$. Therefore, equality (3.4) reduces to the following:

$$\begin{aligned}f(\sigma_1(t_1^0),\sigma_2(t_2^0))-f(t_1,t_2)&=B_1\left[\sigma_1(t_1^0)-t_1\right]+A_2\left[\sigma_2(t_2^0)-t_2\right]\\&\quad+\gamma_1\left[\sigma_1(t_1^0)-t_1\right]+\gamma_2\left[\sigma_2(t_2^0)-t_2\right].\end{aligned}\tag{3.7}$$

where $B_1 = \dfrac{\partial f(t_1^0, \sigma_2(t_2^0))}{\Delta_1 t_1}$ and $A_2 = \dfrac{\partial f(t_1^0, t_2^0)}{\Delta_2 t_2}$. Consequently, for $n = 2$ equality (3.4) reduces equalities (3.5) and (3.7) which are given in Definition 2.3 and 2.4, respectively by Bohner et. al. [7].

4. Mean Value Theorems

First we present mean value results in the single and two variables case ([7], [8]).

THEOREM 4. (Mean Value Theorem) *Suppose that f is a continuous function on $[a,b]$ and has a delta derivative at each point of $[a,b)$. Then there exist $\xi, \xi' \in [a,b)$ such that*

$$f^{\Delta}(\xi')(b-a) \leqslant f(b) - f(a) \leqslant f^{\Delta}(\xi)(b-a).$$

THEOREM 5. *Let a and b be two arbitrary points in $\mathbb{T}$ and let us set $\alpha = \min\{a,b\}$ and $\beta = \max\{a,b\}$. Let, further, f be a continuous function on $[\alpha,\beta]$ that has a delta derivative at each point of $[\alpha,\beta)$. Then there exist $\xi, \xi' \in [a,b)$ such that*

$$f^{\Delta}(\xi')(b-a) \leqslant f(b) - f(a) \leqslant f^{\Delta}(\xi)(b-a).$$

THEOREM 6. (Mean Value Theorem) *Let (a_1,a_2) and (b_1,b_2) be any two points in $\mathbb{T}_1 \times \mathbb{T}_2$ and let us set*

$$\alpha_i = \min\{a_i,b_i\} \quad \text{and} \quad \beta_i = \max\{a_i,b_i\} \ \text{for} \ i \in \{1,2\}.$$

Let, further, $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be a continuous function on $[\alpha_1,\beta_1] \times [\alpha_2,\beta_2] \subset \mathbb{T}_1 \times \mathbb{T}_2$ that has first order partial derivatives $\frac{\partial f(t,a_2)}{\Delta_1 t}$ for each $t \in [\alpha_1,\beta_1)$ and $\frac{\partial f(b_1,s)}{\Delta_1 s}$ for each $s \in [\alpha_2,\beta_2)$. Then there exist $\xi, \xi' \in [\alpha_1,\beta_1)$ and $\eta, \eta' \in [\alpha_2,\beta_2)$ such that

$$\begin{aligned}\frac{\partial f(\xi',a_2)}{\Delta_1 t}(a_1-b_1) + \frac{\partial f(b_1,\eta')}{\Delta_2 s}(a_2-b_2) &\leqslant f(a_1,a_2) - f(b_1,b_2) \\ &\leqslant \frac{\partial f(\xi,a_2)}{\Delta_1 t}(a_1-b_1) + \frac{\partial f(b_1,\eta)}{\Delta_2 s}(a_2-b_2).\end{aligned}$$

Also, if f has first order partial derivatives $\frac{\partial f(t,b_2)}{\Delta_1 t}$ for each $t \in [\alpha_1,\beta_1)$ and $\frac{\partial f(a_1,s)}{\Delta_1 s}$ for each $s \in [\alpha_2,\beta_2)$, then there exist $\tau, \tau' \in [\alpha_1,\beta_1)$ and $\theta, \theta' \in [\alpha_2,\beta_2)$ such that

$$\begin{aligned}\frac{\partial f(\tau',b_2)}{\Delta_1 t}(a_1-b_1) + \frac{\partial f(a_1,\theta')}{\Delta_2 s}(a_2-b_2) &\leqslant f(a_1,a_2) - f(b_1,b_2) \\ &\leqslant \frac{\partial f(\tau,b_2)}{\Delta_1 t}(a_1-b_1) + \frac{\partial f(a_1,\theta)}{\Delta_2 s}(a_2-b_2).\end{aligned}$$

Passing now to the n-variable case, we consider functions $f : \mathbb{T}_1 \times ... \times \mathbb{T}_n \to \mathbb{R}$ of the variables $(t_1,...,t_n) \in \mathbb{T}_1 \times ... \times \mathbb{T}_n$.

THEOREM 7. (Mean Value Theorem) *Let* $(a_1,a_2,...,a_n)$ *and* $(b_1,b_2,...,b_n)$ *be any two points in* $\mathbb{T}_1\times...\times\mathbb{T}_n$ *and let us set*

$$\alpha_i=\min\{a_i,\ b_i\}\quad \text{and}\quad \beta_i=\max\{a_i,\ b_i\}\ \text{for}\ i\in\{1,2,...,n\}.$$

Let, further, f *be a continuous function on* $[\alpha_1,\beta_1]\times...\times[\alpha_n,\beta_n]\subset\mathbb{T}_1\times...\times\mathbb{T}_n$ *that has first order partial derivatives* $\frac{\partial f(b_1,b_2,...b_{i-1},t_i,a_{i+1},...,a_n)}{\Delta_i t_i}$ *for each* $t_i\in[\alpha_i,\beta_i)$ $i=1,...,n$*. Then there exist* $\xi_i,\xi_i'\in[\alpha_i,\beta_i)$ $i=1,...,n$ *such that*

$$\begin{aligned}
&\frac{\partial f(\xi_1',a_2,...,a_n)}{\Delta_1 t_1}(a_1-b_1)+\frac{\partial f(b_1,\xi_2',a_3,...,a_n)}{\Delta_2 t_2}(a_2-b_2)+...\\
&\quad+\frac{\partial f(b_1,b_2,...b_{i-1},\xi_i',a_{i+1}...,a_n)}{\Delta_i t_i}(a_i-b_i)+...+\frac{\partial f(b_1,b_2,...,\xi_n')}{\Delta_n t_n}(a_n-b_n)\\
&\leqslant f(a_1,...,a_n)-f(b_1,...,b_n)\\
&\leqslant\frac{\partial f(\xi_1,a_2,...,a_n)}{\Delta_1 t}(a_1-b_1)+\frac{\partial f(b_1,\xi_2,a_3,...,a_n)}{\Delta_2 t_2}(a_2-b_2)+...\\
&\quad+\frac{\partial f(b_1,b_2,...b_{i-1},\xi_i,a_{i+1},...,a_n)}{\Delta_i t_i}(a_i-b_i)+...+\frac{\partial f(b_1,b_2,...,\xi_n)}{\Delta_n t_n}(a_n-b_n).
\end{aligned}\tag{4.1}$$

Also, if f *has first order partial derivatives* $\frac{\partial f(a_1,a_2,...a_{i-1},t_i,b_{i+1},...,b_n)}{\Delta_i t_i}$ *for each* $t_i\in[\alpha_i,\beta_i)$ $i=1,...,n$*. Then there exist* $\eta_i,\eta_i'\in[\alpha_i,\beta_i)$ $i=1,...,n$ *such that*

$$\begin{aligned}
&\frac{\partial f(\eta_1',b_2,...,b_n)}{\Delta_1 t}(a_1-b_1)+\frac{\partial f(a_1,\eta_2',b_3,...,b_n)}{\Delta_2 t_2}(a_2-b_2)+...\\
&\quad+\frac{\partial f(a_1,a_2,...,a_{i-1},\eta_i',b_{i+1},...,b_n)}{\Delta_i t_i}(a_i-b_i)+...+\frac{\partial f(a_1,a_2,...,\eta_n')}{\Delta_n t_n}(a_n-b_n)\\
&\leqslant f(a_1,...,a_n)-f(b_1,...,b_n)\\
&\leqslant\frac{\partial f(\eta_1,b_2,...,b_n)}{\Delta_1 t}(a_1-b_1)+\frac{\partial f(a_1,\eta_2,b_3,...,b_n)}{\Delta_2 t_2}(a_2-b_2)...\\
&\quad+\frac{\partial f(a_1,a_2,...,a_{i-1},\eta_i,b_{i+1}...,b_n)}{\Delta_i t_i}(a_i-b_i)+...+\frac{\partial f(a_1,a_2,...,\eta_n)}{\Delta_n t_n}(a_n-b_n).
\end{aligned}\tag{4.2}$$

Proof. To prove (4.1) we consider the difference

$$\begin{aligned}
f(a_1,...,a_n)-f(b_1,...,b_n)&=[f(a_1,...,a_n)-f(b_1,a_2,...,a_n)]\\
&\quad+[f(b_1,a_2,...,a_n)-f(b_1,b_2,a_3,...,a_n)]\\
&\quad+[f(b_1,b_2,a_3,...,a_n)-f(b_1,b_2,b_3,a_4,...,a_n)]\\
&\qquad\vdots\\
&\quad+[f(b_1,...,b_i,a_{i+1},...,a_n)-f(b_1,...,b_i,b_{i+1},a_{i+2},...,a_n)]\\
&\qquad\vdots\\
&\quad+[f(b_1,...,b_{n-1},a_n)-f(b_1,...,b_n)].
\end{aligned}\tag{4.3}$$

By Theorem 5 there exist $\xi_i, \xi_i' \in [\alpha_i, \beta_i)$ $i = 1, ..., n$ such that

$$\frac{\partial f(\xi_1', a_2, ..., a_n)}{\Delta_1 t_1}(a_1 - b_1) \leqslant f(a_1, ..., a_n) - f(b_1, a_2, ..., a_n) \leqslant \frac{\partial f(\xi_1, a_2, ..., a_n)}{\Delta_1 t_1}(a_1 - b_1)$$

$$\frac{\partial f(b_1, \xi_2', a_3, ..., a_n)}{\Delta_2 t_2}(a_2 - b_2) \leqslant f(a_1, ..., a_n) - f(b_1, a_2, ..., a_n) \leqslant \frac{\partial f(b_1, \xi_2, a_3, ..., a_n)}{\Delta_2 t_2}(a_2 - b_2)$$

$$\vdots$$

$$\frac{\partial f(b_1, b_2, ... b_{i-1}, \xi_i', a_{i+1}, ..., a_n)}{\Delta_i t_i}(a_i - b_i) \leqslant f(b_1, ..., b_i, a_{i+1}, ..., a_n) - f(b_1, ..., b_i, b_{i+1}, a_{i+2}, ..., a_n)$$

$$\leqslant \frac{\partial f(b_1, b_2, ... b_{i-1}, \xi_i, a_{i+1}, ..., a_n)}{\Delta_i t_i}(a_i - b_i)$$

$$\vdots$$

$$\frac{\partial f(b_1, b_2, ..., \xi_n')}{\Delta_n t_n}(a_n - b_n) \leqslant f(b_1, ..., b_{n-1}, a_n) - f(b_1, ..., b_n) \leqslant \frac{\partial f(b_1, b_2, ..., \xi_n)}{\Delta_n t_n}(a_n - b_n).$$

Adding these inequalities side by side and taking into account (4.3), we obtain (4.1). Using the relation

$$\begin{aligned} f(a_1, ..., a_n) - f(b_1, ..., b_n) = & [f(a_1, b_2 ..., b_n) - f(b_1, b_2, ..., b_n)] \\ & + [f(a_1, a_2, b_3, ..., b_n) - f(a_1, b_2, ..., b_n)] \\ & + [f(a_1, a_2, a_3, b_4, ..., a_n) - f(a_1, a_2, b_3, ..., b_n)] \\ & \quad \vdots \\ & + [f(a_1, ..., a_i, b_{i+1}, ..., b_n) - f(a_1, ..., a_{i-1}, b_i, ..., b_n)] \\ & \quad \vdots \\ & + [f(a_1, ..., a_{n-1}, a_n) - f(a_1, ..., a_{n-1}, b_n)] \end{aligned}$$

the inequalities (4.2) can be proved similarly. □

REMARK 1. If we take $n = 2$ in Theorem 7, then it reduces Theorem 4.2 proved by Bohner et. al. [7]. So, our results are generalizations of the corresponding results of Bohner et. al. and Guseinov et. al. ([7], [8]).

5. The Chain Rule

The chain rule for one-variable and two-variable functions on time scales have been investigated in ([1], [5], [7]). To get an extension to $n-$ variable functions on time scales we start with a time scale $\mathbb{T}$. Denote its forward jump operator by σ_i and its delta differentiation operator by Δ_i for $i = 1, ..., n$. Let, further, $n-$functions

$$\varphi_i : \mathbb{T} \to \mathbb{R} \quad \text{for } i = 1, ..., n$$

be given. Let us set

$$\varphi_i(\mathbb{T}) = \mathbb{T}_i \quad \text{for } i = 1, \ldots, n.$$

We will assume that $\mathbb{T}_1, \ldots, \mathbb{T}_n$ are time scales. Denote by $\sigma_1, \Delta_1, \ldots, \sigma_n, \Delta_n$ the forward jump operators and delta operators for $\mathbb{T}_1, \ldots, \mathbb{T}_n,$ respectively. Take a point $\xi^0 \in \mathbb{T}^k$ and put

$$t_i^0 = \varphi_i(\xi^0) \quad \text{for } i = 1, \ldots, n.$$

We will also assume that

$$\varphi_i(\sigma(\xi^0)) = \sigma_i(\varphi_i(\xi^0)) \quad \text{for } i = 1, \ldots, n \tag{5.1}$$

Under the above assumptions let a function $f : \mathbb{T}_1 \times \ldots \times \mathbb{T}_n \to \mathbb{R}$ be given.

THEOREM 8. *Let the function f be σ_j-completely delta differentiable at the point $(t_1^0, \ldots, t_n^0)$. If the function φ_i $(i = 1, \ldots, n)$ has delta derivatives at the point ξ^0, then the composite function*

$$F(\xi) = f(\varphi_1(\xi), \ldots, \varphi_n(\xi)) \quad \textit{for } \xi \in \mathbb{T} \tag{5.2}$$

has a delta derivative at that point which is given by the formula

$$F^\Delta(\xi^0) = \frac{\partial f(t_1^0, \ldots, t_n^0)}{\Delta_j t_j} \varphi_j^\Delta(\xi^0) + \sum_{\substack{i=1 \\ i \neq j}}^n \frac{\partial f(\sigma_1(t_1^0), \ldots, t_i^0, \ldots, \sigma_n(t_n^0))}{\Delta_i t_i} \varphi_i^\Delta(\xi^0) \tag{5.3}$$

for each $j \in \{1, \ldots, n\}$.

Proof. Using (5.1) and (3.3) with

$$A_j = \frac{\partial f(t_1^0, \ldots, t_n^0)}{\Delta_j t_j} \text{ and } B_i = \frac{\partial f(\sigma_1(t_1^0), \ldots, t_i^0, \ldots, \sigma_n(t_n^0))}{\Delta_i t_i},$$

we obtain

$$\begin{aligned}
F(\sigma(\xi^0)) - F(\xi) &= f(\varphi_1(\sigma(\xi^0)), \ldots, \varphi_n(\sigma(\xi^0))) - f(\varphi_1(\xi), \ldots, \varphi_n(\xi)) \\
&= f(\sigma_1(\varphi_1(\xi^0)), \ldots, \sigma_n(\varphi_n(\xi^0))) - f(\varphi_1(\xi), \ldots, \varphi_n(\xi)) \\
&= \frac{\partial f(\sigma_1(\varphi_1(\xi^0)), \ldots, \sigma_n(\varphi_n(\xi^0)))}{\Delta_j t_j} \left[\sigma_j(\varphi_j(\xi^0)) - \varphi_j(\xi)\right] \\
&\quad + \sum_{\substack{i=1 \\ i \neq j}}^n \frac{\partial f(\sigma_1(\varphi_1(\xi^0)), \ldots, \varphi_i(\xi^0), \ldots, \sigma_n(\varphi_n(\xi^0)))}{\Delta_i t_i} \\
&\quad \times \left[\sigma_i(\varphi_i(\xi^0)) - \varphi_i(\xi)\right] \\
&\quad + \gamma_j \left[\sigma_j(\varphi_j(\xi^0)) - \varphi_j(\xi)\right] + \sum_{\substack{i=1 \\ i \neq j}}^n \gamma_i \left[\sigma_i(\varphi_i(\xi^0)) - \varphi_i(\xi)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial f(t_1^0,...,t_n^0)}{\Delta_j t_j}\left[\varphi_j(\sigma(\xi^0)) - \varphi_j(\xi)\right] \\
&\quad + \sum_{\substack{i=1 \\ i\neq j}}^{n} \frac{\partial f(\sigma_1(t_1^0),...,t_i^0,...,\sigma_n(t_n^0))}{\Delta_i t_i}\left[\varphi_i(\sigma(\xi^0)) - \varphi_i(\xi)\right] \\
&\quad + \gamma_{jj}\left[\varphi_j(\sigma(\xi^0)) - \varphi_j(\xi)\right] + \sum_{\substack{i=1 \\ i\neq j}}^{n} \gamma_i\left[\varphi_i(\sigma(\xi^0)) - \varphi_i(\xi)\right].
\end{aligned}
$$

Dividing both sides of this equality by $\sigma(\xi^0)-\xi$ and passing to the limit as $\xi \to \xi^0$, we get the formula (5.3) because $\xi \to \xi^0$ implies $\gamma_j \to 0$ and $\gamma_i \to 0$ for $i = 1,2,..,n$. □

REMARK 2. Let $n = 2$. Then, in the case $j = 1$ and $i = 2$ equality (5.3) in our Theorem 8 reduces to equality (7.3) in Theorem 7.1. which is proved by Bohner et. al. [7]. Further, in the case $j = 2$ and $i = 1$, equality (5.3) in our Theorem 8 reduces to equality in Theorem 7.2. which is proved by Bohner et. al. [7]. Hence our results in Theorem 8 are generalizations of the corresponding results of Bohner et. al. ([1], [5], [7]).

REMARK 3. One or all of the functions $\varphi_1,...,\varphi_n$ may be constant. In that case one or all of $\mathbb{T}_1,...,\mathbb{T}_n$ will be a single point time scale. For a single point time scale $\mathbb{T}_i = \{t_i\}$ we assume that $\sigma_i(t_i) = t_i$ and for each function $g : \mathbb{T}_i \to \mathbb{R}$ we assume that $g^{\Delta_i}(t_i) = 0$.

Let now n-time scales $\mathbb{T}_{(1)},...,\mathbb{T}_{(n)}$ be given. Denote their forward jump operators and delta differentiation operators by $\sigma_{(1)},\Delta_{(1)},...,\sigma_{(n)},\Delta_{(n)}$, respectively. Let, also, $m-$ functions

$$\varphi_i : \mathbb{T}_{(1)} \times ... \times \mathbb{T}_{(n)} \to \mathbb{R}, \quad i = 1,...,m$$

of $n-$ variables $(\xi_1,...,\xi_n) \in \mathbb{T}_{(1)} \times ... \times \mathbb{T}_{(n)}$, and a fixed point $(\xi_1^0,...,\xi_n^0) \in \mathbb{T}_{(1)}^k \times ... \times \mathbb{T}_{(n)}^k$ be given. Let us set

$$\mathbb{T}_i = \mathbb{T}_i(\xi_1^0,...,\xi_{i-1}^0,\xi_{i+1}^0,...,\xi_n^0) = \varphi_i(\xi_1^0,...,\xi_{i-1}^0,\mathbb{T}_{(i)},\xi_{i+1}^0,...,\xi_n^0)$$

and

$$t_i^0 = \varphi_i(\xi_1^0,...,\xi_n^0), \quad \text{for } i = 1,...,m.$$

We will assume that $\mathbb{T}_1,...,\mathbb{T}_m$ are time scales. Denote their forward jump operators and delta differentiation operators by $\sigma_1,\Delta_1,...,\sigma_n,\Delta_n$, respectively. We will assume for each $k \in \{1,...,n\}$

$$\varphi_i(\xi_1^0,...,\xi_{k-1}^0,\sigma_{(k)}(\xi_i^0),\xi_{k+1}^0,...,\xi_n^0) = \sigma_i(\varphi_i(\xi_1^0,...,\xi_{k-1}^0,\xi_k^0,\xi_{k+1}^0,...,\xi_n^0)) \tag{5.4}$$

for $i = 1,...,m$. Under the above conditions let a function $f : \mathbb{T}_1 \times ... \times \mathbb{T}_m \to \mathbb{R}$ of $m-$ variables $(t_1,...,t_m) \in \mathbb{T}_1 \times ... \times \mathbb{T}_m$ be given.

If we take $n=2$, then $k\in\{1,2\}$. So, in the case $k=1$ equality (5.4) becomes the following form:

$$\varphi_i(\sigma_{(1)}(\xi_1^0),\xi_2^0)=\sigma_i(\varphi_i(\xi_1^0,\xi_2^0)),\quad i=1,...,m.$$

Here, for $m=2$ we have

$$\begin{aligned}\varphi_1(\sigma_{(1)}(\xi_1^0),\xi_2^0) &= \sigma_1(\varphi_1(\xi_1^0,\xi_2^0))\\ \varphi_2(\sigma_{(1)}(\xi_1^0),\xi_2^0) &= \sigma_2(\varphi_2(\xi_1^0,\xi_2^0)).\end{aligned}\tag{5.5}$$

On the other hand, in the case $k=2$ equality (5.4) becomes the following form:

$$\varphi_i(\xi_1^0,\sigma_{(2)}(\xi_2^0))=\sigma_i(\varphi_i(\xi_1^0,\xi_2^0)),\quad i=1,...,m.$$

Again, for $m=2$ we have

$$\begin{aligned}\varphi_1(\xi_1^0,\sigma_{(2)}(\xi_2^0)) &= \sigma_1(\varphi_1(\xi_1^0,\xi_2^0))\\ \varphi_2(\xi_1^0,\sigma_{(2)}(\xi_2^0)) &= \sigma_2(\varphi_2(\xi_1^0,\xi_2^0)).\end{aligned}\tag{5.6}$$

So, for $n=2,\ m=2$ the equality (5.4) reduces equality (5.5) and (5.6) which is given by Bohner et. al. [7].

THEOREM 9. *Let the function f be σ_j-completely delta differentiable at the point $(t_1^0,...,t_n^0)$. If the function φ_i $(i=1,...,n)$ has first order partial delta derivatives at the point $\xi^0=(\xi_1^0,...,\xi_n^0)$, then the composite function*

$$F(\xi^0)=f(\varphi_1(\xi^0),...,\varphi_n(\xi^0))\ \ \textit{for}\ \ \xi^0=(\xi_1^0,...,\xi_n^0)\in\mathbb{T}_{(1)}\times...\times\mathbb{T}_{(n)}\tag{5.7}$$

has a delta derivative at that point which is expresses by the formula

$$\begin{aligned}\frac{\partial F(\xi_1^0,...,\xi_n^0)}{\Delta_{(k)}\xi_k} &= \frac{\partial f(t_1^0,...,t_m^0)}{\Delta_j t_j}\frac{\partial\varphi_j(\xi_1^0,...,\xi_n^0)}{\Delta_{(k)}\xi_k}\\ &\quad+\sum_{\substack{i=1\\ i\neq j}}^m\frac{\partial f(\sigma_1(t_1^0),...,t_i^0,...,\sigma_m(t_m^0))}{\Delta_i t_i}\frac{\partial\varphi_i(\xi_1^0,...,\xi_n^0)}{\Delta_{(k)}\xi_k}\end{aligned}\tag{5.8}$$

for each $k\in\{1,...,n\}$.

Proof. For the sake of simplicity, we take $\xi'=(\xi_1^0,...,\xi_{k-1}^0,\xi_k,\xi_{k+1}^0,...,\xi_n^0)$ and $\xi''=(\xi_1^0,...,\xi_{k-1}^0,\xi_k^0,\xi_{k+1}^0,...,\xi_n^0)$. Using (5.4) and (3.3) with $A_j=\dfrac{\partial f(t_1^0,...,t_m^0)}{\Delta_j t_j}$ and $B_i=\dfrac{\partial f(\sigma_1(t_1^0),...,t_i^0,...,\sigma_m(t_m^0))}{\Delta_i t_i}$, we obtain

$$\begin{aligned}&F(\xi_1^0,...,\xi_{k-1}^0,\sigma_{(k)}(\xi_k^0),\xi_{k+1}^0,...,\xi_n^0)-F(\xi')\\ &\quad=f(\varphi_1(\xi_1^0,...,\xi_{k-1}^0,\sigma_{(k)}(\xi_k^0),\xi_{k+1}^0,...,\xi_n^0),...,\varphi_m(\xi_1^0,...,\xi_{k-1}^0,\sigma_{(k)}(\xi_k^0),\xi_{k+1}^0,...,\xi_n^0))\end{aligned}$$

$$
\begin{aligned}
&-f(\varphi_1(\xi'),\dots,\varphi_m(\xi')) \\
&= f(\sigma_1(\varphi_1(\xi'')),\dots,\sigma_m(\varphi_m(\xi''))) - f(\varphi_1(\xi'),\dots,\varphi_n(\xi') \\
&= \frac{\partial f(\sigma_1(\varphi_1(\xi^0)),\dots,\sigma_m(\varphi_m(\xi^0)))}{\Delta_j t_j}\left[\sigma_j(\varphi_j(\xi^0)) - \varphi_j(\xi')\right] \\
&\quad + \sum_{\substack{i=1\\ i\neq j}}^{m} \frac{\partial f(\sigma_1(\varphi_1(\xi^0)),\dots,\varphi_i(\xi^0),\dots,\sigma_m(\varphi_m(\xi^0)))}{\Delta_i t_i}\left[\sigma_i(\varphi_i(\xi^0)) - \varphi_i(\xi')\right] \\
&\quad + \gamma_j\left[\sigma_j(\varphi_j(\xi^0)) - \varphi_j(\xi')\right] + \sum_{\substack{i=1\\ i\neq j}}^{m} \gamma_i\left[\sigma_i(\varphi_i(\xi^0)) - \varphi_i(\xi')\right] \\
&= \frac{\partial f(t_1^0,\dots,t_m^0)}{\Delta_j t_j}\left[\varphi_j(\xi_1^0,\dots,\xi_{k-1}^0,\sigma_{(k)}(\xi_k^0),\xi_{k+1}^0,\dots,\xi_n^0) - \varphi_j(\xi')\right] \\
&\quad + \sum_{\substack{i=1\\ i\neq j}}^{m} \frac{\partial f(\sigma_1(t_1^0),\dots,t_i^0,\dots,\sigma_m(t_m^0))}{\Delta_i t_i}\left[\varphi_i(\xi_1^0,\dots,\xi_{k-1}^0,\sigma_{(k)}(\xi_k^0),\xi_{k+1}^0,\dots,\xi_n^0) - \varphi_i(\xi')\right] \\
&\quad + \gamma_j\left[\varphi_j(\xi_1^0,\dots,\xi_{k-1}^0,\sigma_{(k)}(\xi_k^0),\xi_{k+1}^0,\dots,\xi_n^0) - \varphi_j(\xi')\right] \\
&\quad + \sum_{\substack{i=1\\ i\neq j}}^{m} \gamma_i\left[\varphi_i(\xi_1^0,\dots,\xi_{k-1}^0,\sigma_{(k)}(\xi_k^0),\xi_{k+1}^0,\dots,\xi_n^0) - \varphi_i(\xi')\right].
\end{aligned}
$$

On dividing both sides of this equality by $\sigma_{(k)}(\xi_k^0) - \xi_k$ and passing to the limit as $\xi \to \xi^0$, we get the formula (5.7) because $\xi \to \xi^0$ implies $\gamma_j \to 0$ and $\gamma_i \to 0$ for $i = 1,\dots,m$. $\square$

REMARK 4. Let $n = 2$. Then we have

$$
\begin{aligned}
\frac{\partial F(\xi_1^0,\xi_2^0)}{\Delta_{(1)}\xi_1} &= \frac{\partial f(t_1^0,\dots,t_m^0)}{\Delta_j t_j}\frac{\partial \varphi_j(\xi_1^0,\xi_2^0)}{\Delta_{(1)}\xi_1} \\
&\quad + \sum_{\substack{i=1\\ i\neq j}}^{m} \frac{\partial f(\sigma_1(t_1^0),\dots,t_i^0,\dots,\sigma_m(t_m^0))}{\Delta_i t_i}\frac{\partial \varphi_i(\xi_1^0,\xi_2^0)}{\Delta_{(1)}\xi_1} \quad \text{for} \quad k=1
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial F(\xi_1^0,\xi_2^0)}{\Delta_{(2)}\xi_2} &= \frac{\partial f(t_1^0,\dots,t_m^0)}{\Delta_j t_j}\frac{\partial \varphi_j(\xi_1^0,\xi_2^0)}{\Delta_{(2)}\xi_2} \\
&\quad + \sum_{\substack{i=1\\ i\neq j}}^{m} \frac{\partial f(\sigma_1(t_1^0),\dots,t_i^0,\dots,\sigma_m(t_m^0))}{\Delta_i t_i}\frac{\partial \varphi_i(\xi_1^0,\xi_2^0)}{\Delta_{(2)}\xi_2} \quad \text{for} \quad k=2.
\end{aligned}
$$

Here, let $m=2$. Then for $j=1, i=2$ we have the following equalities:

$$\frac{\partial F(\xi_1^0,\xi_2^0)}{\Delta_{(1)}\xi_1}=\frac{\partial f(t_1^0,t_2^0)}{\Delta_1 t_1}\frac{\partial \varphi_j(\xi_1^0,\xi_2^0)}{\Delta_{(1)}\xi_1}+\frac{\partial f(\sigma_1(t_1^0),t_2^0)}{\Delta_2 t_2}\frac{\partial \varphi_i(\xi_1^0,\xi_2^0)}{\Delta_{(1)}\xi_1} \quad \text{for} \quad k=1 \tag{5.9}$$

and

$$\frac{\partial F(\xi_1^0,\xi_2^0)}{\Delta_{(2)}\xi_2}=\frac{\partial f(t_1^0,t_2^0)}{\Delta_1 t_1}\frac{\partial \varphi_j(\xi_1^0,\xi_2^0)}{\Delta_{(2)}\xi_2}+\frac{\partial f(\sigma_1(t_1^0),t_2^0)}{\Delta_2 t_2}\frac{\partial \varphi_i(\xi_1^0,\xi_2^0)}{\Delta_{(2)}\xi_2} \quad \text{for} \quad k=2. \tag{5.10}$$

Similarly way, for $j=2, i=1$ we have

$$\frac{\partial F(\xi_1^0,\xi_2^0)}{\Delta_{(1)}\xi_1}=\frac{\partial f(\sigma_1(t_1^0),t_2^0)}{\Delta_1 t_1}\frac{\partial \varphi_j(\xi_1^0,\xi_2^0)}{\Delta_{(1)}\xi_1}+\frac{\partial f(t_1^0,t_2^0)}{\Delta_2 t_2}\frac{\partial \varphi_i(\xi_1^0,\xi_2^0)}{\Delta_{(1)}\xi_1} \quad \text{for} \quad k=1 \tag{5.11}$$

and

$$\frac{\partial F(\xi_1^0,\xi_2^0)}{\Delta_{(2)}\xi_2}=\frac{\partial f(\sigma_1(t_1^0),t_2^0)}{\Delta_1 t_1}\frac{\partial \varphi_j(\xi_1^0,\xi_2^0)}{\Delta_{(2)}\xi_2}+\frac{\partial f(t_1^0,t_2^0)}{\Delta_2 t_2}\frac{\partial \varphi_i(\xi_1^0,\xi_2^0)}{\Delta_{(2)}\xi_2} \quad \text{for} \quad k=2. \tag{5.12}$$

Therefore, for $n=2$, $m=2$ the equality (5.7) reduces equality (5.9)–(5.10) in Theorem 7.4 and (5.11)–(5.12) in Theorem 7.5 which are given by Bohner et. al. [7].

6. The Directional Derivative

Let $\mathbb{T}$ be a time scale with the forward jump operator σ and the delta operator Δ. We will assume that $0\in\mathbb{T}$. Further, let $\omega=(\omega_1,...,\omega_n)\in\mathbb{R}^n$ be a unit vector and let $(t_1^0,...,t_n^0)$ be a fixed point in $\mathbb{R}^n$. Let us set

$$\mathbb{T}_i=\{t_i=t_i^0+\xi\omega_i:\ \xi\in\mathbb{T}\},\ \ i=1,...,n.$$

Then $\mathbb{T}_1,...,\mathbb{T}_n$ are time scales and $t_i^0\in\mathbb{T}_i$ for $i=1,...,n$. Denote the forward jump operators of $\mathbb{T}_i$ by σ_i, the delta operators by Δ_i for $i=1,...,n$.

DEFINITION 3. Let a function $f:\mathbb{T}_1\times...\times\mathbb{T}_n\to\mathbb{R}$ be given. The directional delta derivative of the function f at the point $(t_1^0,...,t_n^0)$ in the direction of the vector ω (along ω) is defined as the number

$$\frac{\partial f(t_1^0,...,t_n^0)}{\Delta\omega}=F^{\Delta}(0), \tag{6.1}$$

provided it exists, where

$$F(\xi)=f(t_1^0+\xi\omega_1,...,t_n^0+\xi\omega_n)\ \ \text{for}\ \ \xi\in\mathbb{T}. \tag{6.2}$$

THEOREM 10. *Suppose that the function* f *is* σ_j-*completely delta differentiable at the point* $(t_1^0,...,t_n^0)$*. Then the directional delta derivative of* f *at* $(t_1^0,...,t_n^0)$ *in the direction of the vector* ω *exists and is expressed by the formula*

$$\frac{\partial f(t_1^0,...,t_n^0)}{\Delta\omega} = \frac{\partial f(t_1^0,...,t_n^0)}{\Delta_j t_j}\omega_j + \sum_{\substack{i=1\\ i\neq j}}^{n} \frac{\partial f(\sigma_1(t_1^0),...,t_i^0,...,\sigma_n(t_n^0))}{\Delta_i t_i}\omega_i \tag{6.3}$$

for each $j \in \{1,...,n\}$.

Proof. The proof is obtained from the Definitions 3 and 10 by applying Theorem 8. □

REMARK 5. For $\omega_j = 1$ and $\omega_i = 0$, (6.3) coincides with $\frac{\partial f(t_1^0,...,t_n^0)}{\Delta_j t_j}$, while for $\omega_j = 0$ and $\omega_i = 1$ it coincides with $\frac{\partial f(t_1^0,...,t_n^0)}{\Delta_i t_i}$ because then $\mathbb{T}_i = \{t_i^0\}$ and hence $\sigma_i(t_i^0) = t_i^0$ (see Remark 3).

REMARK 6. Let $n = 2$. Then for $j = 1$, $i = 2$ we have

$$\frac{\partial f(t_1^0,t_2^0)}{\Delta\omega} = \frac{\partial f(t_1^0,t_2^0)}{\Delta_1 t_1}\omega_1 + \frac{\partial f(\sigma_1(t_1^0),t_2^0)}{\Delta_2 t_2}\omega_2 \tag{6.4}$$

and

$$\frac{\partial f(t_1^0,t_2^0)}{\Delta\omega} = \frac{\partial f(t_1^0,\sigma_2(t_2^0))}{\Delta_1 t_1}\omega_1 + \frac{\partial f(t_1^0,t_2^0)}{\Delta_2 t_2}\omega_2. \tag{6.5}$$

Therefore, for $n = 2$ equality (6.3) yields (6.4) and (6.5) which are proved by Bohner et. al. [7].

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