THE HIERARCHY OF CONVEXITY AND SOME CLASSIC INEQUALITIES

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. In what follows, a hierarchy of \( m \)-convexity is considered: we define \( m \)-starshaped functions, \( m \)-superadditive functions, Jensen \( m \)-convex functions, weak Jensen \( m \)-convex functions, Jensen \( m \)-superadditive functions, and weak \( m \)-superadditive functions. Some inclusions between such classes of functions are established. We also analyze the validity of the Hermite-Hadamard inequality, and of the Chebyshev-Andersson inequality for \( m \)-convex functions.

1. Introduction

Let us consider the sets of continuous, convex, starshaped, and superadditive functions on \([a, b]\) given by:

\[
C[a, b] = \{ f : [a, b] \rightarrow \mathbb{R}, f \text{ continuous} \},
\]

\[
K[a, b] = \{ f \in C[a, b]; f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y), \forall x, y \in [a, b], t \in [0, 1] \},
\]

\[
S^{*}[a, b] = \left\{ f \in C[a, b]; \frac{f(x) - f(a)}{x - a} \leq \frac{f(y) - f(a)}{y - a}, a < x < y \leq b \right\},
\]

and

\[
S[a, b] = \{ f \in C[a, b]; f(x) + f(y) \leq f(x + y - a) + f(a), \forall x, y, x + y - a \in [a, b] \},
\]

respectively. For \( a = 0 \) we denote by \( C(b), K(b), S^{*}(b), \) and \( S(b) \) respectively, the corresponding set of functions, restricted also under the condition \( f(0) = 0 \). A. M. Bruckner and E. Ostrow have proven in [1] the strict inclusions:

\[
K(b) \subset S^{*}(b) \subset S(b).
\]

These inclusions, extended with some results of preservation of the above properties by the arithmetic integral mean, are collectively referred to in [6] as the hierarchy of convexity. Simple proofs and generalizations of the results of [1] may be found in [8].


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Let us remark that we can also define a superadditive function by

\[ f(x) + f(y) \leq f(x + y) + f(a), \forall x, y \in [a, b], \]

thus assuming \( f \in C[a, 2b - a] \). This is the preferred layout for superadditive functions in what follows.

In [9], one of the many generalizations on the convexity of functions – called \( \alpha \)-convexity – was introduced. The set of \( \alpha \)-convex functions is defined by:

\[
K_m[a, b] = \{ f \in C[a, b]; f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y), \forall x, y \in [a, b], t \in [0, 1] \}, m \in [0, 1].
\]

If \( a = 0 \) and \( f(0) \leq 0 \), we also obtain a hierarchy of convexity:

\[
K[a, b] \subset K_m[a, b] \subset K_n[a, b] \subset S^*[a, b], \text{ for } 1 > m > n > 0.
\]

A much larger generalization of convexity was given in [12]: the function \( f : [a, b] \to \mathbb{R} \) is called \((g, h, \lambda, \mu)\)-convex if

\[
g(f(tx + (1 - t)y)) \leq h(t)g(f(x)) + [1 - h(t)]\mu(f(y)), \forall x, y \in [a, b], \forall t \in [0, 1].
\]

It is shown that more interesting results can be obtained for \( h(t) = t^\alpha \), with \( \alpha \in [0, 1] \). This case was combined with the \( m \)-convexity in [5] giving the \((\alpha, m)\)-convexity. In the next paragraph we define a hierarchy of \((\alpha, m)\)-convexity. Taking \( \alpha = 1 \), we obtain a more fruitful hierarchy of \( m \)-convexity. Finally we study the Fejér inequality (generalization of the Hermite-Hadamard inequality) and the Chebyshev-Andersson inequality for \( m \)-convex functions.

\[2. \text{ A hierarchy of } (\alpha, m)\text{-convexity}\]

The set of \((\alpha, m)\)-convex functions is defined by

\[
K_{m, \alpha}[a, b] = \{ f \in C[ma, 2b - ma]; f(tx + m(1 - t)y) \leq t^\alpha f(x) + m(1 - t^\alpha)f(y), \forall x, y \in [a, b], t \in [0, 1] \}, m, \alpha \in [0, 1].
\]

Note that for \( t = 0 \) and \( y = a \) we have the condition \( f(ma) \leq mf(a) \) meaning that the function must be defined on \( ma \leq a \). In fact, to assure that all the definitions and results that follow are valid we will assume that the functions are defined on \([ma, 2b - ma]\). Assuming \( \alpha \neq 0, m \neq 0 \), we define the following sets of functions:

\[
S^*_{m, \alpha}[a, b] = \left\{ f \in C[ma, 2b - ma]; \frac{f(x) - mf(a)}{(x - ma)^\alpha} \geq \frac{f(z) - mf(a)}{(z - ma)^\alpha}, a < z < x \leq b \right\},
\]

called \((\alpha, m)\)-starshaped functions;

\[
S_{m, \alpha}[a, b] = \{ f \in C[ma, 2b - ma]; [f(x) - mf(a)](x - ma)^{1 - \alpha} + [f(y) - mf(a)](y - ma)^{1 - \alpha}
\]
called $(\alpha,m)$-superadditive functions;

$$J_{m,\alpha}[a,b] = \{ f \in C[ma,2b-ma]; f(2x-ma)-mf(a) \geq 2^\alpha [f(x)-mf(a)], \forall x \in [a,b] \},$$
called Jensen $(\alpha,m)$-starshaped functions;

$$J_{m,\alpha}[a,b] = \left\{ f \in C[ma,2b-ma]; f \left( \frac{m \frac{1}{\alpha} x + my}{1 + m \frac{1}{\alpha}} \right) \right\}, \forall x, y \in [a,b]$$
called $(\alpha,m)$-Jensen convex functions;

$$H_{m,\alpha}[a,b] = \{ f \in C[ma,2b-ma]; f(tx) \leq [m + (t-m)^\alpha(1-m)^{1-\alpha}] f(x),$$
$$a \leq x \leq b, m \leq t \leq 1 \},$$
called $(\alpha,m)$-subhomogenous functions;

$$H_{m,\alpha}^*[a,b] = \left\{ f \in C[ma,2b-ma]; f \left( \frac{m + m \frac{1}{\alpha}}{1 + m \frac{1}{\alpha}} x \right) \right\}$$
$$\leq m \left[ 1 + \frac{1-m}{(1+m \frac{1}{\alpha})} \right] f(x), a \leq x \leq b \},$$
called Jensen $(\alpha,m)$-subhomogenous functions;

$$wS_{m,\alpha}[a,b] = \{ f \in C[ma,2b-ma]; f(a+t) - mf(a) \] (a+t - ma)^{1-\alpha}$$
$$+ [f(b-t) - mf(a)] \] (b-t - ma)^{1-\alpha} \leq [f(b + (1-m)a) - mf(a)] \] (a+b-2ma)^{1-\alpha},$$
$$\forall t \in [0,(b-a)/2] \},$$
called weak $(\alpha,m)$-superadditive; and

$$wJ_{m,\alpha}[a,b] = \left\{ f \in C[ma,2b-ma]; \frac{m}{(1+m \frac{1}{\alpha})^\alpha} \left\{ f(a+t)+\left[ (1+m \frac{1}{\alpha})^\alpha -m \right] f(b-t) \right\} \right\}$$
$$\geq f \left( \frac{m \frac{1}{\alpha} (a+t)+m(b-t)}{1 + m \frac{1}{\alpha}} \right), \forall t \in [0,(b-a)/2] \},$$
called weak $(\alpha,m)$-Jensen convex.

For these sets, we have the following main results.
Theorem 1. The following inclusions

\[ K_{m,\alpha}[a, b] \subseteq S^*_m[a, b] \subseteq S_m[a, b] \subseteq J^*_m[a, b], S_m[a, b] \subseteq wS_m[a, b], \]

\[ H^*_m[a, b] \supseteq H_m[a, b] \supseteq K_m[a, b] \subseteq J_m[a, b] \subseteq H_m[a, b] \]

and

\[ J_m[a, b] \subseteq wJ_m[a, b] \]

hold.

Proof. a) Taking \( f \in K_m[a, b] \) and \( y = a \) we obtain

\[ f(xt + m(1 - t)a) - mf(a) \leq t^\alpha [f(x) - mf(a)]. \]

Denoting \( xt + m(1 - t)y = z \) we prove that \( f \in S^*_m[a, b] \).

b) Assuming that \( f \in S^*_m[a, b] \) we have

\[ [f(x + y - ma) - mf(a)](x + y - 2ma)^{1 - \alpha} \]
\[ = \frac{f(x + y - ma) - mf(a)}{(x + y - 2ma)^\alpha} \cdot (x + y - 2ma) \]
\[ = \frac{f(x + y - ma) - mf(a)}{(x + y - 2ma)^\alpha} (x - ma) + \frac{f(x + y - ma) - mf(a)}{(x + y - 2ma)^\alpha} (y - ma) \]
\[ \geq \frac{f(x) - mf(a)}{(x - ma)^\alpha} (x - ma) + \frac{f(y) - mf(a)}{(y - ma)^\alpha} (y - ma), \]

thus \( f \in S_m[a, b] \).

c) For \( f \in S_m[a, b] \) if we take \( x = y \) we obtain

\[ 2 [f(x) - mf(a)](x - ma)^{1 - \alpha} \leq [f(2x - ma) - mf(a)](2x - 2ma)^{1 - \alpha}, \]

implying that \( f \in J^*_m[a, b] \).

d) For \( f \in S_m[a, b] \) if we take \( x = a - t, y = b - t \) we obtain \( f \in wS_m[a, b] \).

e) If \( f \in K_m[a, b] \) for \( t = m^{1/\alpha} / (1 + m^{1/\alpha}) \) we deduce that \( f \in J_m[a, b] \).

f) For \( f \in J_m[a, b] \) if we take \( x = y \) we obtain that \( f \in H^*_m[a, b] \).

g) If \( f \in K_m[a, b] \) for \( x = y \) we obtain

\[ f(x(m + t(1 - m)) \leq [t^\alpha + m(1 - t^\alpha)] f(x) \]

and denoting \( m + t(1 - m) = s \) we deduce that \( f \in H_m[a, b] \).

h) If \( f \in H_m[a, b] \), for \( t = (m + m^{1/\alpha}) / (1 + m^{1/\alpha}) \) it follows that \( f \in H^*_m[a, b] \).

k) For \( f \in J_m[a, b] \) if we take \( x = a + t, y = b - t \) we obtain that \( f \in wJ_m[a, b] \).
3. A hierarchy of $m$-convexity

For $\alpha = 1$ we obtain the following sets of functions:

$$S_m[a, b] = \left\{ f \in C[ma, 2b - ma]; \frac{f(x) - mf(a)}{x - ma} \geq \frac{f(z) - mf(a)}{z - ma}, a \leq z < x \leq b \right\},$$
called $m$-starshaped functions;

$$S_m[a, b] = \left\{ f \in C[ma, 2b - ma]; f(x) + f(y) \leq f(x + y - ma) + mf(a), \forall x, y \in [a, b] \right\},$$
called $m$-superadditive functions;

$$J_m[a, b] = \left\{ f \in C[ma, 2b - ma]; f(2x - ma) - mf(a) \geq 2[f(x) - mf(a)], a \leq x \leq b \right\},$$
called Jensen $m$-starshaped functions;

$$J_m[a, b] = \left\{ f \in C[ma, 2b - ma]; f \left( \frac{m(x + y)}{1 + m} \right) \leq \frac{m[f(x) + f(y)]}{1 + m}, \forall x, y \in [a, b] \right\},$$
called $m$-Jensen convex functions;

$$H_m[a, b] = \left\{ f \in C[ma, 2b - ma]; f(tx) \leq tf(x), a \leq x \leq b, m \leq t \leq 1 \right\},$$
called $m$-subhomogenous functions;

$$H_m'[a, b] = \left\{ f \in C[ma, 2b - ma]; f \left( \frac{2mx}{1 + m} \right) \leq \frac{2m}{1 + m}f(x), a \leq x \leq b \right\},$$
called Jensen $m$-subhomogenous functions;

$$wS_m[a, b] = \left\{ f \in C[ma, 2b - ma]; f(a + t) + f(b - t) \leq f(b + (1 - m)a) + mf(a), \forall t \in [0, (b - a)/2] \right\},$$
called weak $m$-superadditive; and

$$wJ_m[a, b] = \left\{ f \in C[ma, 2b - ma]; \frac{m[f(a + t) + f(b - t)]}{1 + m} \geq f \left( \frac{m(a + b)}{1 + m} \right), \forall t \in [0, (b - a)/2] \right\},$$
called weak $m$-Jensen convex.

From the hierarchy of $m$-convexity we underline only some results.

**Theorem 2.** The following inclusions

$$K_m[a, b] \subseteq S_m[a, b] \subseteq S_m[a, b] \subseteq wS_m[a, b]$$

and

$$H_m'[a, b] \supseteq H_m[a, b] \supseteq K_m[a, b] \subseteq J_m[a, b] \subseteq wJ_m[a, b]$$

hold.
Moreover, in this simple case $\alpha = 1$ we can characterize the functions of $wS_m[a, b]$ and those of $wJ_m[a, b]$. For this we begin with the following:

**Lemma 3.** For every function $f \in C[a,b]$ we can determine two functions $f_1 : [a(1-m), (b + (1 - 2m)a)/2] \to \mathbb{R}$ and $f_2 : [0, (b + (1 - 2m)a)/2] \to \mathbb{R}$ such that:

$$f(x) = \begin{cases} f_1(x - ma) & \text{for } x \in [a, \frac{a+b}{2}] \\ f_1 \left( \frac{b+(1-2m)a}{2} \right) + f_2 \left( \frac{b+(1-2m)a}{2} \right) & \text{for } x \in \left[\frac{a+b}{2}, b\right] \\ -f_2(b + (1-m)a - x) & \text{for } x \in [a(1-m), a(1-m) + b/2]. \end{cases}$$

**Proof.** We can take:

$$f_1(t) = f(ma + t), \forall t \in [a(1-m), (b + (1 - 2m)a)/2]$$

and

$$f_2(t) = f((b + a)/2) + c - f(b + a(1-m) - t), \forall t \in [0, (b + (1 - 2m)a)/2],$$

where $c$ is an arbitrary real number.

Using this lemma we can obtain the characterization and a method of construction of functions from $wS_m[a, b]$ and $wJ_m[a, b]$.

**Theorem 4.** The function $f$ belongs to:

a) $wS_m[a, b]$ if and only if

$$f_1(t + a(1-m)) - mf_1(a(1-m)) \leq f_2(t + a(1-m)) - f_2(0);$$

b) $wJ_m[a, b]$ if and only if

$$f_1(t + a(1-m)) + f_1 \left( \frac{b+(1-2m)a}{2} \right) - \frac{1+m}{m} f_1 \left( \frac{m(b-am)}{1+m} \right) \geq f_2(t + a(1-m)) - f_2 \left( \frac{b+(1-2m)a}{2} \right).$$

**Corollary 1.** The function $f$ belongs to $wJ_m[a, b]$ if

$$f_1(t) = f_2(t), \forall t \in [a(1-m), (b + (1 - 2m)a)/2]$$

and

$$f_1 \left( \frac{b+(1-2m)a}{2} \right) \geq \frac{1+m}{2m} f_1 \left( \frac{m(b-am)}{1+m} \right).$$
COROLLARY 2. The function $f$ belongs to $wS_m[a,b]$ if
\[
f_1(t) = f_2(t), \forall t \in [a(1-m), (b+(1-2m)a)/2] \]
and
\[
f_2(0) \leq mf_1(a(1-m)).
\]

COROLLARY 3. The function $f$ belongs to $wS_m[a,b] \cap wJ_m[a,b]$ if
\[
f_1(t) = f_2(t), \forall t \in [a(1-m), (b+(1-2m)a)/2] \]
and
\[
f_2(0) \leq mf_1(a(1-m)), \quad f_1\left(\frac{b+(1-2m)a}{2}\right) \geq \frac{1+m}{2m} f_1\left(\frac{m(b-am)}{1+m}\right).
\]

REMARK 5. For $m = 1$ these results were proven in [11].

4. Fejér’s inequality

Let $L(\cdot,a,b) : C[a,b] \rightarrow \mathbb{R}$ be an isotonic linear functional, that is, for $t,s \in \mathbb{R}, f,g \in C[a,b]$:
\[
L(f; a,b) \geq 0 \quad \text{if} \quad f \geq 0
\]
\[
L(tf + sg; a,b) = tL(f; a,b) + sL(g; a,b).
\]

If $f \in C[a,b]$ we denote by $f_-$ the function defined by:
\[
f_-(x) = f(a + b - x) \quad \text{for} \quad x \in [a,b].
\]

DEFINITION 6. The functional $L(\cdot,a,b)$ is symmetric if:
\[
L(f_-; a,b) = L(f; a,b), \forall f \in C[a,b].
\]

THEOREM 7. If $L(\cdot,a,b)$ is a symmetric isotonic linear functional, such that $L(1; a,b) = 1$, then:
\[
L(f; a,b) \leq [f(b + (1-m)a) + mf(a)] / 2, \forall f \in wS_m[a,b]
\]
and
\[
L(f; a,b) \geq \frac{m+1}{2m} f\left(\frac{m(a+b)}{1+m}\right), \forall f \in wJ_m[a,b].
\]
Proof. Indeed in the first case we have
\[ f(a+t) + f(b-t) = f(x) + f_{-}(x) \leq f(b + (1-m)a) + mf(a), \forall x \in [a,b] \]
while in the second:
\[ f(x) + f_{-}(x) \geq \frac{m+1}{m} f \left( \frac{m(a+b)}{1+m} \right), \forall x \in [a,b]. \]
We need only to apply the functional \( L(\cdot; a, b) \).

COROLLARY 4. If \( L(\cdot; a, b) \) is a symmetric isotonic linear functional, such that \( L(1; a, b) = 1 \), then:
\[ \frac{m+1}{2m} f \left( \frac{m(a+b)}{1+m} \right) \leq L(f; a, b) \leq \frac{[f(b + (1-m)a) + mf(a)]}{2}, \]
\[ \forall f \in wS_m[a, b] \cap wJ_m[a, b]. \]

REMARK 8. If \( g \in C[a, b] \) is symmetric with respect to \( \frac{a+b}{2} \), the functional defined by:
\[ L(f; a, b) = \int_a^b f(x)g(x)dx / \int_a^b g(x)dx \]
is a symmetric isotonic linear functional. As \( K_m[a, b] \subset wS_m[a, b] \cap wJ_m[a, b] \) we obtained a generalization of the result of L. Fejér from [3], thus also of the Hermite-Hadamard inequality. The generalization is effective even for \( m = 1 \) as was pointed out in [11]. Other generalizations of the Hermite-Hadamard inequality for \( m \)-convex functions were given in [2], [7], and [4].

5. Chebyshev-Andersson’s inequality

In [10] we have shown that Chebyshev-Andersson’s inequality is not only valid for convex functions but also for starshaped functions. A general result of this type was also proven in [12]. Let us now consider the case of \((\alpha, m)\)-starshaped functions. Denote by \( e \) the function defined by \( e(x) = x \) and by \( c \) the constant function with value \( c \).

THEOREM 9. If \( A \) and \( B \) are isotonic linear functionals, \( f \in S^*_{m, \alpha}[a, b] \) and \( g \in S^*_{n, \beta}[a, b] \) then the following inequality holds:
\[ A \left( (e - ma)^{\alpha} (e - na)^{\beta} \right) B \left( (f - mf(a)) (g - ng(a)) \right) \]
\[ + B \left( (e - ma)^{\alpha} (e - na)^{\beta} \right) A \left( (f - mf(a)) (g - ng(a)) \right) \]
\[ \geq A \left( (e - ma)^{\alpha} (g - ng(a)) \right) B \left( e - na \right)^{\beta} (f - mf(a)) \]
\[ + B \left( (e - ma)^{\alpha} (g - ng(a)) \right) A \left( e - na \right)^{\beta} (f - mf(a)) \].
Proof. We have

\[
\left[ \frac{f(x) - mf(a)}{(x - ma)^{\alpha}} - \frac{f(z) - mf(a)}{(z - ma)^{\alpha}} \right] (x - ma)^{\alpha} (z - ma)^{\alpha}
\]

\[
\cdot \left[ \frac{g(x) - ng(a)}{(x - na)^{\beta}} - \frac{g(z) - ng(a)}{(z - na)^{\beta}} \right] (x - na)^{\beta} (z - na)^{\beta} \geq 0,
\]

or

\[
(z - ma)^{\alpha} (z - na)^{\beta} [f(x) - mf(a)] [g(x) - ng(a)]
\]

\[
- (z - ma)^{\alpha} [g(z) - ng(a)] (x - na)^{\beta} [f(x) - mf(a)]
\]

\[
- (z - na)^{\beta} [f(z) - mf(a)] (x - ma)^{\alpha} [g(x) - ng(a)]
\]

\[
+ (x - ma)^{\alpha} (x - na)^{\beta} [f(z) - mf(a)] [g(z) - ng(a)] \geq 0.
\]

If we now take the value of \( A \) for the functions of \( x \) and then the value of \( B \) for the functions of \( z \), we obtain the announced inequality.

**Remark 10.** Taking \( A = B \) and/or \( m = n, \alpha = \beta \), we deduce some consequences of the Chebyshev-Andersson type inequalities.

**References**


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