Journal of
Mathematical
Inequalities
Volume 3, Number 3 (2009), 305–313

THE HIERARCHY OF CONVEXITY AND SOME CLASSIC INEQUALITIES

GHEORGHE TOADER

Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. In what follows, a hierarchy of m-convexity is considered: we define m-starshaped functions, m-superadditive functions, Jensen m-convex functions, weak Jensen m-convex functions, Jensen m-superadditive functions, and weak m-superadditive functions. Some inclusions between such classes of functions are established. We also analyze the validity of the Hermite-Hadamard inequality, and of the Chebyshev-Andersson inequality for m-convex functions.

1. Introduction

Let us consider the sets of continuous, convex, starshaped, and superadditive functions on $[a,b]$ given by:

$$C[a,b] = \{f : [a,b] \longrightarrow \mathbb{R}, f \quad \text{continuous}\},$$

$$K[a,b] = \{f \in C[a,b]; f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y), \forall x,y \in [a,b], t \in [0,1]\},$$

$$S^*[a,b] = \left\{f \in C[a,b]; \frac{f(x)-f(a)}{x-a} \leqslant \frac{f(y)-f(a)}{y-a},\ a<x<y \leqslant b\right\},$$

and

$$S[a,b] = \{f \in C[a,b]; f(x)+f(y) \leqslant f(x+y-a)+f(a), \forall x,y,x+y-a \in [a,b]\},$$

respectively. For $a=0$ we denote by $C(b), K(b), S^*(b)$, and $S(b)$ respectively, the corresponding set of functions, restricted also under the condition $f(0)=0$. A. M. Bruckner and E. Ostrow have proven in [1] the strict inclusions:

$$K(b) \subset S^*(b) \subset S(b).$$

These inclusions, extended with some results of preservation of the above properties by the arithmetic integral mean, are collectively referred to in [6] as the *hierarchy of convexity*. Simple proofs and generalizations of the results of [1] may be found in [8].

Mathematics subject classification (2000): 26A51, 26D15.
Keywords and phrases: Hermite-Hadamard inequality, Chebyshev-Andersson inequality, hierarchy of m-convexity.

Paper JMI-03-30

Let us remark that we can also define a superadditive function by

$$f(x)+f(y) \leqslant f(x+y-a)+f(a), \forall x,y \in [a,b],$$

thus assuming $f \in C[a,2b-a]$. This is the preferred layout for superadditive functions in what follows.

In [9], one of the many generalizations on the convexity of functions – called m-convexity-was introduced. The set of m-convex functions is defined by:

$$K_m[a,b] = \{f \in C[a,b]; f(tx+m(1-t)y) \leqslant tf(x)+m(1-t)f(y),$$

$$\forall x,y \in [a,b], t \in [0,1]\},\ m \in [0,1].$$

If $a=0$ and $f(0) \leqslant 0$, we also obtain a hierarchy of convexity:

$$K[a,b] \subset K_m[a,b] \subset K_n[a,b] \subset S^*[a,b], \text{ for } 1 > m > n > 0.$$

A much larger generalization of convexity was given in [12]: the function $f:[a,b] \to \mathbb{R}$ is called (g,h,λ,μ)-convex if

$$g(f(tx+(1-t)\lambda(y))) \leqslant h(t)g(f(x))+[1-h(t)]\mu(f(y)), \forall x,y \in [a,b], \forall t \in [0,1].$$

It is shown that more interesting results can be obtained for $h(t)=t^\alpha$, with $\alpha \in [0,1]$. This case was combined with the m-convexity in [5] giving the (α,m)-convexity. In the next paragraph we define a hierarchy of (α,m)-convexity. Taking $\alpha=1$, we obtain a more fruitful hierarchy of m-convexity. Finally we study the Fejér inequality (generalization of the Hermite-Hadamard inequality) and the Chebyshev-Andersson inequality for m-convex functions.

2. A hierarchy of (α,m)-convexity

The set of (α,m)-*convex functions* is defined by

$$K_{m,\alpha}[a,b] = \{f \in C[ma,2b-ma]; f(tx+m(1-t)y) \leqslant t^\alpha f(x)+m(1-t^\alpha)f(y),$$

$$\forall x,y \in [a,b], t \in [0,1]\},\ m,\alpha \in [0,1].$$

Note that for $t=0$ and $y=a$ we have the condition $f(ma) \leqslant mf(a)$ meaning that the function must be defined on $ma \leqslant a$. In fact, to assure that all the definitions and results that follow are valid we will assume that the functions are defined on $[ma,2b-ma]$. Assuming $\alpha \neq 0, m \neq 0$, we define the following sets of functions:

$$S^*_{m,\alpha}[a,b] = \left\{ f \in C[ma,2b-ma]; \frac{f(x)-mf(a)}{(x-ma)^\alpha} \geqslant \frac{f(z)-mf(a)}{(z-ma)^\alpha},\ a<z<x \leqslant b \right\},$$

called (α,m)-*starshaped functions*;

$$S_{m,\alpha}[a,b] = \{f \in C[ma,2b-ma]; [f(x)-mf(a)]\,(x-ma)^{1-\alpha} + [f(y)-mf(a)]\,(y-ma)^{1-\alpha}$$

$$\leqslant [f(x+y-ma)-mf(a)](x+y-2ma)^{1-\alpha}, \forall x,y \in [a,b]\},$$

called (α,m)-*superadditive functions*;

$$J^*_{m,\alpha}[a,b] = \{f \in C[ma,2b-ma]; f(2x-ma)-mf(a) \geqslant 2^\alpha [f(x)-mf(a)], \forall x \in [a,b]\},$$

called *Jensen* (α,m)-*starshaped functions*;

$$J_{m,\alpha}[a,b] = \left\{ f \in C[ma,2b-ma]; f\left(\frac{m^{\frac{1}{\alpha}}x+my}{1+m^{\frac{1}{\alpha}}}\right)\right.$$

$$\left. \leqslant \frac{mf(x)+m\left[\left(1+m^{\frac{1}{\alpha}}\right)^\alpha - m\right]f(y)}{\left(1+m^{\frac{1}{\alpha}}\right)^\alpha}, \forall x,y \in [a,b] \right\},$$

called (α,m)-*Jensen convex functions*;

$$H_{m,\alpha}[a,b] = \left\{f \in C[ma,2b-ma]; f(tx) \leqslant \left[m+(t-m)^\alpha(1-m)^{1-\alpha}\right]f(x),\right.$$
$$\left. a \leqslant x \leqslant b, m \leqslant t \leqslant 1\right\},$$

called (α,m)-*subhomogenous functions*;

$$H^*_{m,\alpha}[a,b] = \left\{ f \in C[ma,2b-ma]; f\left(\frac{m+m^{\frac{1}{\alpha}}}{1+m^{\frac{1}{\alpha}}}x\right)\right.$$

$$\left. \leqslant m\left[1+\frac{1-m}{\left(1+m^{\frac{1}{\alpha}}\right)^\alpha}\right]f(x),\ a \leqslant x \leqslant b \right\},$$

called *Jensen* (α,m)-*subhomogenous functions;*

$$wS_{m,\alpha}[a,b] = \left\{f \in C[ma,2b-ma]; [f(a+t)-mf(a)](a+t-ma)^{1-\alpha}\right.$$
$$+[f(b-t)-mf(a)]\cdot(b-t-ma)^{1-\alpha} \leqslant [f(b+(1-m)a)-mf(a)](a+b-2ma)^{1-\alpha},$$
$$\left.\forall t \in [0,(b-a)/2]\right\},$$

called *weak* (α,m)-*superadditive*; and

$$wJ_{m,\alpha}[a,b] = \left\{ f \in C[ma,2b-ma]; \frac{m}{\left(1+m^{\frac{1}{\alpha}}\right)^\alpha}\left\{f(a+t)+\left[\left(1+m^{\frac{1}{\alpha}}\right)^\alpha - m\right]f(b-t)\right\}\right.$$

$$\left. \geqslant f\left(\frac{m^{\frac{1}{\alpha}}(a+t)+m(b-t)}{1+m^{\frac{1}{\alpha}}}\right), \forall t \in [0,(b-a)/2] \right\},$$

called *weak* (α,m)-*Jensen convex.*

For these sets, we have the following main results.

THEOREM 1. *The following inclusions*

$$K_{m,\alpha}[a,b] \subseteq S^*_{m,\alpha}[a,b] \subseteq S_{m,\alpha}[a,b] \subseteq J^*_{m,\alpha}[a,b], S_{m,\alpha}[a,b] \subseteq wS_{m,\alpha}[a,b],$$

$$H^*_{m,\alpha}[a,b] \supseteq H_{m,\alpha}[a,b] \supseteq K_{m,\alpha}[a,b] \subseteq J_{m,\alpha}[a,b] \subseteq H^*_{m,\alpha}[a,b]$$

and

$$J_{m,\alpha}[a,b] \subseteq wJ_{m,\alpha}[a,b]$$

hold.

Proof. a) Taking $f \in K_{m,\alpha}[a,b]$ and $y = a$ we obtain

$$f(xt + m(1-t)a) - mf(a) \leqslant t^\alpha [f(x) - mf(a)].$$

Denoting $xt + m(1-t)y = z$ we prove that $f \in S^*_{m,\alpha}[a,b]$.

b) Assuming that $f \in S^*_{m,\alpha}[a,b]$ we have

$$\begin{aligned} &[f(x+y-ma) - mf(a)](x+y-2ma)^{1-\alpha} \\ &\quad = \frac{f(x+y-ma) - mf(a)}{(x+y-2ma)^\alpha} \cdot (x+y-2ma) \\ &\quad = \frac{f(x+y-ma) - mf(a)}{(x+y-2ma)^\alpha}(x-ma) + \frac{f(x+y-ma) - mf(a)}{(x+y-2ma)^\alpha}(y-ma) \\ &\quad \geqslant \frac{f(x) - mf(a)}{(x-ma)^\alpha}(x-ma) + \frac{f(y) - mf(a)}{(y-ma)^\alpha}(y-ma), \end{aligned}$$

thus $f \in S_{m,\alpha}[a,b]$.

c) For $f \in S_{m,\alpha}[a,b]$ if we take $x = y$ we obtain

$$2[f(x) - mf(a)](x-ma)^{1-\alpha} \leqslant [f(2x-ma) - mf(a)](2x-2ma)^{1-\alpha},$$

implying that $f \in J^*_{m,\alpha}[a,b]$.

d) For $f \in S_{m,\alpha}[a,b]$ if we take $x = a-t, y = b-t$ we obtain $f \in wS_{m,\alpha}[a,b]$.

e) If $f \in K_{m,\alpha}[a,b]$ for $t = m^{1/\alpha}/\left(1+m^{1/\alpha}\right)$ we deduce that $f \in J_{m,\alpha}[a,b]$.

f) For $f \in J_{m,\alpha}[a,b]$ if we take $x = y$ we obtain that $f \in H^*_{m,\alpha}[a,b]$.

g) If $f \in K_{m,\alpha}[a,b]$ for $x = y$ we obtain

$$f(x(m+t(1-m))) \leqslant [t^\alpha + m(1-t^\alpha)] f(x)$$

and denoting $m + t(1-m) = s$ we deduce that $f \in H_{m,\alpha}[a,b]$.

h) If $f \in H_{m,\alpha}[a,b]$, for $t = (m+m^{1/\alpha})/(1+m^{1/\alpha})$ it follows that $f \in H^*_{m,\alpha}[a,b]$.

k) For $f \in J_{m,\alpha}[a,b]$ if we take $x = a+t, y = b-t$ we obtain that $f \in wJ_{m,\alpha}[a,b]$.

3. A hierarchy of m-convexity

For $\alpha = 1$ we obtain the following sets of functions:

$$S_m^*[a,b] = \left\{ f \in C[ma, 2b-ma]; \frac{f(x)-mf(a)}{x-ma} \geqslant \frac{f(z)-mf(a)}{z-ma},\ a \leqslant z < x \leqslant b \right\},$$

called m-starshaped functions;

$$S_m[a,b] = \{ f \in C[ma, 2b-ma]; f(x)+f(y) \leqslant f(x+y-ma)+mf(a), \forall x,y \in [a,b] \},$$

called m-superadditive functions;

$$J_m^*[a,b] = \{ f \in C[ma, 2b-ma]; f(2x-ma)-mf(a) \geqslant 2\left[f(x)-mf(a)\right],\ a \leqslant x \leqslant b \},$$

called Jensen m-starshaped functions;

$$J_m[a,b] = \left\{ f \in C[ma, 2b-ma]; f\left(\frac{m(x+y)}{1+m}\right) \leqslant \frac{m\left[f(x)+f(y)\right]}{1+m}, \forall x,y \in [a,b] \right\},$$

called m-Jensen convex functions;

$$H_m[a,b] = \{ f \in C[ma, 2b-ma]; f(tx) \leqslant tf(x),\ a \leqslant x \leqslant b, m \leqslant t \leqslant 1 \},$$

called m-subhomogenous functions;

$$H_m^*[a,b] = \left\{ f \in C[ma, 2b-ma]; f\left(\frac{2mx}{1+m}\right) \leqslant \frac{2m}{1+m} f(x),\ a \leqslant x \leqslant b \right\},$$

called Jensen m-subhomogenous functions;

$$wS_m[a,b] = \{ f \in C[ma, 2b-ma]; f(a+t)+f(b-t) \leqslant f(b+(1-m)a)+mf(a),$$
$$\forall t \in\ [0,(b-a)/2] \},$$

called *weak m-superadditive*; and

$$wJ_m[a,b] = \left\{ f \in C[ma, 2b-ma]; \frac{m\left[f(a+t)+f(b-t)\right]}{1+m} \right.$$
$$\left. \geqslant f\left(\frac{m(a+b)}{1+m}\right), \forall t \in [0,(b-a)/2] \right\},$$

called *weak m-Jensen convex.*

From the hierarchy of m-convexity we underline only some results.

THEOREM 2. *The following inclusions*

$$K_m[a,b] \subseteq S_m^*[a,b] \subseteq S_m[a,b] \subseteq wS_m[a,b]$$

and

$$H_m^*[a,b] \supseteq H_m[a,b] \supseteq K_m[a,b] \subseteq J_m[a,b] \subseteq wJ_m[a,b]$$

hold.

Moreover, in this simple case $\alpha = 1$ we can characterize the functions of $wS_m[a,b]$ and those of $wJ_m[a,b]$. For this we begin with the following:

LEMMA 3. *For every function* $f \in C[a,b]$ *we can determine two functions* $f_1 : [a(1-m),(b+(1-2m)a)/2] \longrightarrow \mathbb{R}$ *and* $f_2 : [0,(b+(1-2m)a)/2] \longrightarrow \mathbb{R}$ *such that:*

$$f(x) = \begin{cases} f_1(x-ma) & for \quad x \in \left[a, \frac{a+b}{2}\right] \\ f_1\left(\frac{b+(1-2m)a}{2}\right) + f_2\left(\frac{b+(1-2m)a}{2}\right) & \\ -f_2(b+(1-m)a-x) & for \quad x \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Proof. We can take:

$$f_1(t) = f(ma+t), \forall t \in [a(1-m),(b+(1-2m)a)/2]$$

and

$$f_2(t) = f((b+a)/2) + c - f(b+a(1-m)-t), \forall t \in [0,(b+(1-2m)a)/2],$$

where c is an arbitrary real number.

Using this lemma we can obtain the characterization and a method of construction of functions from $wS_m[a,b]$ and $wJ_m[a,b]$.

THEOREM 4. *The function* f *belongs to:*

a) $wS_m[a,b]$ *if and only if*

$$f_1(t+a(1-m)) - mf_1(a(1-m)) \leqslant f_2(t+a(1-m)) - f_2(0);$$

b) $wJ_m[a,b]$ *if and only if*

$$f_1(t+a(1-m)) + f_1\left(\frac{b+(1-2m)a}{2}\right) - \frac{1+m}{m} f_1\left(\frac{m(b-am)}{1+m}\right)$$
$$\geqslant f_2(t+a(1-m)) - f_2\left(\frac{b+(1-2m)a}{2}\right).$$

COROLLARY 1. *The function* f *belongs to* $wJ_m[a,b]$ *if*

$$f_1(t) = f_2(t), \forall t \in [a(1-m),(b+(1-2m)a)/2]$$

and

$$f_1\left(\frac{b+(1-2m)a}{2}\right) \geqslant \frac{1+m}{2m} f_1\left(\frac{m(b-am)}{1+m}\right).$$

COROLLARY 2. *The function* f *belongs to* $wS_m[a,b]$ *if*

$$f_1(t) = f_2(t), \forall t \in [a(1-m), (b+(1-2m)a)/2]$$

and

$$f_2(0) \leqslant mf_1(a(1-m)).$$

COROLLARY 3. *The function* f *belongs to* $wS_m[a,b] \cap wJ_m[a,b]$ *if*

$$f_1(t) = f_2(t), \forall t \in [a(1-m), (b+(1-2m)a)/2]$$

$$f_2(0) \leqslant mf_1(a(1-m))$$

and

$$f_1\left(\frac{b+(1-2m)a}{2}\right) \geqslant \frac{1+m}{2m} f_1\left(\frac{m(b-am)}{1+m}\right).$$

REMARK 5. For $m = 1$ these results were proven in [11].

4. Fejér's inequality

Let $L(\cdot,a,b) : C[a,b] \longrightarrow \mathbb{R}$ be an *isotonic linear functional*, that is, for $t,s \in \mathbb{R}$, $f,g \in C[a,b]$:

$$L(f;a,b) \geqslant 0 \quad if \quad f \geqslant 0$$

$$L(tf+sg;a,b) = tL(f;a,b) + sL(g;a,b).$$

If $f \in C[a,b]$ we denote by f_- the function defined by:

$$f_-(x) = f(a+b-x) \quad for \quad x \in [a,b].$$

DEFINITION 6. The functional $L(\cdot,a,b)$ is *symmetric* if:

$$L(f_-;a,b) = L(f;a,b), \ \forall f \in C[a,b].$$

THEOREM 7. *If* $L(\cdot;a,b)$ *is a symmetric isotonic linear functional, such that* $L(1;a,b) = 1$*, then:*

$$L(f;a,b) \leqslant [f(b+(1-m)a) + mf(a)]/2, \ \forall f \in wS_m[a,b]$$

and

$$L(f;a,b) \geqslant \frac{m+1}{2m} f\left(\frac{m(a+b)}{1+m}\right), \ \forall f \in wJ_m[a,b].$$

Proof. Indeed in the first case we have

$$f(a+t)+f(b-t)=f(x)+f_-(x) \leqslant f(b+(1-m)a)+mf(a), \forall x \in [a,b]$$

while in the second:

$$f(x)+f_-(x) \geqslant \frac{m+1}{m} f\left(\frac{m(a+b)}{1+m}\right), \forall x \in [a,b].$$

We need only to apply the functional $L(\cdot;a,b)$.

COROLLARY 4. *If $L(\cdot;a,b)$ is a symmetric isotonic linear functional, such that $L(1;a,b) = 1$, then:*

$$\frac{m+1}{2m} f\left(\frac{m(a+b)}{1+m}\right) \leqslant L(f;a,b) \leqslant [f(b+(1-m)a)+mf(a)]/2,$$
$$\forall f \in wS_m[a,b] \cap wJ_m[a,b].$$

REMARK 8. If $g \in C[a,b]$ is symmetric with respect to $\frac{a+b}{2}$, the functional defined by:

$$L(f;a,b) = \int_a^b f(x)g(x)dx \Big/ \int_a^b g(x)dx$$

is a symmetric isotonic linear functional. As $K_m[a,b] \subset wS_m[a,b] \cap wJ_m[a,b]$ we obtained a generalization of the result of L. Fejér from [3], thus also of the Hermite-Hadamard inequality. The generalization is effective even for $m = 1$ as was pointed out in [11]. Other generalizations of the Hermite-Hadamard inequality for m-convex functions were given in [2], [7], and [4].

5. Chebyshev-Andersson's inequality

In [10] we have shown that Chebyshev-Andersson's inequality is not only valid for convex functions but also for starshaped functions. A general result of this type was also proven in [12]. Let us now consider the case of (α, m)-starshaped functions. Denote by e the function defined by $e(x) = x$ and by c the constant function with value c.

THEOREM 9. *If A and B are isotonic linear functionals, $f \in S^*_{m,\alpha}[a,b]$ and $g \in S^*_{n,\beta}[a,b]$ then the following inequality holds:*

$$\begin{aligned} &A\left((e-ma)^{\alpha}(e-na)^{\beta}\right)B\left((f-mf(a))(g-ng(a))\right) \\ &\quad + B\left((e-ma)^{\alpha}(e-na)^{\beta}\right)A\left((f-mf(a))(g-ng(a))\right) \\ &\geqslant A\left((e-ma)^{\alpha}(g-ng(a))\right)B\left((e-na)^{\beta}(f-mf(a))\right) \\ &\quad + B\left((e-ma)^{\alpha}(g-ng(a))\right)A\left((e-na)^{\beta}(f-mf(a))\right). \end{aligned}$$

Proof. We have

$$\left[\frac{f(x)-mf(a)}{(x-ma)^{\alpha}}-\frac{f(z)-mf(a)}{(z-ma)^{\alpha}}\right](x-ma)^{\alpha}(z-ma)^{\alpha}$$
$$\cdot\left[\frac{g(x)-ng(a)}{(x-na)^{\beta}}-\frac{g(z)-ng(a)}{(z-na)^{\beta}}\right](x-na)^{\beta}(z-na)^{\beta}\geqslant 0,$$

or

$$\begin{aligned}&(z-ma)^{\alpha}(z-na)^{\beta}[f(x)-mf(a)][g(x)-ng(a)]\\&\quad-(z-ma)^{\alpha}[g(z)-ng(a)](x-na)^{\beta}[f(x)-mf(a)]\\&\quad-(z-na)^{\beta}[f(z)-mf(a)](x-ma)^{\alpha}[g(x)-ng(a)]\\&\quad+(x-ma)^{\alpha}(x-na)^{\beta}[f(z)-mf(a)][g(z)-ng(a)]\geqslant 0.\end{aligned}$$

If we now take the value of A for the functions of x and then the value of B for the functions of z, we obtain the announced inequality.

REMARK 10. Taking $A=B$ and/or $m=n,\alpha=\beta$, we deduce some consequences of the Chebyshev-Andersson type inequalities.

REFERENCES

[1] A. M. BRUCKNER, E. OSTROW, *Some function classes related to the class of convex functions*, Pacific J. Math., **12** (1962), 1203–1215.

[2] S. S. DRAGOMIR, G. TOADER, *Some inequalities for m-convex functions*, Studia Univ. Babeş-Bolyai, **38**, 1 (1993), 21–28.

[3] L. FEJÉR, *On Fourier like sequences* (Hungarian), Matem. Term. Ertesitö, **204** (1906), 369–390.

[4] M. KLARIČIĆ BAKULA, J. PEČARIĆ, M. RIBIČIĆ, *Companion inequalities to Jensen's inequalities for m-convex and (α,m)-convex functions*, J. Inequ. Pure Appl. Math., **7**, 5 (2006), Article 194.

[5] V. MIHEŞAN, *A generalization of the convexity*, Itinerant Sem. Funct. Equat. Approx. Conv., Cluj-Napoca, Romania, 1993.

[6] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer Verlag, Berlin, 1970.

[7] P. T. MOCANU, I. ŞERB, G. TOADER, *Real star-convex functions*, Studia Univ. Babeş-Bolyai, **42**, 3 (1997), 65–80.

[8] G. TOADER, *On the hierarchy of convexity of functions*, Anal. Numér. Théor. Approx., **15**, 2 (1986), 167–172.

[9] G. TOADER, *On a generalization of the convexity*, Mathematica, **30**, 53 (1988), 1 83–87.

[10] G. TOADER, *On Chebyshev's inequality for functionals*, Acta Techn. Napoc., Appl. Math. Mech., **35** (1992), 77–80.

[11] G. TOADER, *Superadditivity and Hermite-Hadamard's inequality,* Studia Univ. Babeş-Bolyai, **39**, 2 (1994), 27–32.

[12] G. TOADER, S. TOADER, *Chebyshev-Andersson's inequality*, Ineq. Th. Appl., Y. J. Cho, J. K. Kim, S. S. Dragomir, eds., Vol. **3** (2003), 181–188.

(Received October 18, 2008)

Gheorghe Toader
Department of Mathematics
Technical University
Cluj-Napoca
Romania
e-mail: gheorghe.toader@math.utcluj.ro