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ON SOME INEQUALITIES FOR CONVEX FUNCTIONS

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. In this paper we derive new inequalities for convex functions. The results presented here are an extension of the inequalities obtained by S. S. Dragomir, J. Pečarić and L.-E. Persson.

1. Introduction

Let C be a convex subset of the real linear space X and $f : C \to \mathbb{R}$ a convex function on C. If $x_i \in C$ and $p_i \in (0,1)$ with $\sum_{i=1}^{n} p_i = 1$, then the following well-known form of Jensen's discrete inequality holds:

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leqslant \sum_{i=1}^{n} p_i f(x_i). \tag{1.1}$$

In [2] S.S. Dragomir, J. Pečarić and L.E. Persson proved the following refinement of Jensen's inequality in the general setting of linear spaces

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geqslant \max\left\{ p_i f(x_i) + p_j f(x_j) - (p_i + p_j) f\left(\frac{p_i x_i + p_j x_j}{p_i + p_j}\right)\right\}. \tag{1.2}$$

In 2006 S.S. Dragomir ([1]) proved the following result:

$$\max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\}\left[\sum_{j=1}^{n} q_j f(x_j) - f\left(\sum_{j=1}^{n} q_j x_j\right)\right] \geqslant \sum_{j=1}^{n} p_j f(x_j) - f\left(\sum_{j=1}^{n} p_j x_j\right) \geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\}\left[\sum_{j=1}^{n} q_j f(x_j) - f\left(\sum_{j=1}^{n} q_j x_j\right)\right] \tag{1.3}$$

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provided $f: C \to \mathbb{R}$ is convex on the convex subset C of the linear space X and p_i, q_i, $i \in \{1,2,\dots,n\}$ are probability sequences with $q_i > 0$ for each $i \in \{1,2,\dots,n\}$.

In particular, from (1.3) the following result is obtained:

$$n \max_{1 \leqslant i \leqslant n} \{p_i\} \left[\frac{1}{n} \sum_{j=1}^{n} f(x_j) - f\left(\frac{1}{n} \sum_{j=1}^{n} x_j \right) \right] \geqslant \sum_{j=1}^{n} p_j f(x_j) - f\left(\sum_{j=1}^{n} p_j x_j \right)$$

$$\geqslant n \min_{1 \leqslant i \leqslant n} \{p_i\} \left[\frac{1}{n} \sum_{j=1}^{n} f(x_j) - f\left(\frac{1}{n} \sum_{j=1}^{n} x_j \right) \right]. \tag{1.4}$$

In this paper some new results in connection with the inequalities (1.2)-(1.4) are given.

2. Main results

Let $\mathscr{F}$ be a linear set of functions defined on the interval I, $I \subseteq \mathbb{R}$ and A be a linear positive functional defined on $\mathscr{F}$. We suppose that $C(I) \subset \mathscr{F}$ and for every $X \subset I$ the characteristic function of X denoted by h_X belongs to $\mathscr{F}$ and for every continuous function f, $h_X \cdot f$ belongs to $\mathscr{F}$, too. In the following we suppose that A is a normalized functional. This means that

$$A(e_0) = 1,$$

where we denote by e_i, $i \in \mathbb{N}$ the monomial function, $e_i : I \to \mathbb{R}$,

$$e_i(x) = x^i, \quad x \in I.$$

THEOREM 2.1. *Let X be a subset of I such that $A(h_X) > 0$, and f a convex function defined on I. Then for every linear positive normalized functional we have:*

$$A(f) - f(a_1) \geqslant A(f h_X) - A(h_X) f\left(\frac{A(e_1 h_X)}{A(h_X)} \right), \tag{2.1}$$

where $a_1 = A(e_1)$.

Proof. From the equality

$$f = f h_X + f h_{I-X}$$

we get

$$A(f) - f(a_1) = A(f h_X) + A(f h_{I-X}) - A(h_X) f\left(\frac{A(e_1 h_X)}{A(h_X)} \right) - f(a_1)$$

$$+ A(h_X) f\left(\frac{A(e_1 h_X)}{A(h_X)} \right). \tag{2.2}$$

Let $B: C(I) \to \mathbb{R}$ be the linear functional defined by

$$B(f) = A(fh_{I-X}) + A(h_X) f\left(\frac{A(e_1 h_X)}{A(h_X)}\right). \tag{2.3}$$

We note that B is a linear positive functional.

We have

$$B(e_0) = A(h_{I-X}) + A(h_X) = A(e_0) = 1$$

$$B(e_1) = A(e_1 h_{I-X}) + A(e_1 h_X) = A(e_1) = a_1.$$

So, B is a normalized functional and $B(e_1) = a_1$.

If f is a convex function on I, by Jensen's inequality we obtain:

$$B(f) \geqslant f(a_1)$$

or

$$A(fh_{I-X}) + A(h_X) f\left(\frac{A(e_1 h_X)}{A(h_X)}\right) - f(a_1) \geqslant 0. \tag{2.4}$$

From (2.2) and (2.4) we get inequality (2.1). □

REMARK 2.2. Let A be the linear positive normalized functional defined by:

$$A(f) = \sum_{i=1}^{n} p_i f(x_i)$$

where $p_i \in (0,1)$, $i = \overline{1,n}$ and $\sum_{i=1}^{n} p_i = 1$.

For a given convex function f, let k and s be the natural number, $k,s \in \{0,1,2,\ldots,n\}$ for which:

$$\max\left\{p_i f(x_i) + p_j f(x_j) - (p_i + p_j) f\left(\frac{p_i x_i + p_j x_j}{p_i + p_j}\right)\right\}$$
$$= p_k f(x_k) + p_s f(x_s) - (p_k + p_s) f\left(\frac{p_k x_k + p_s x_s}{p_k + p_s}\right).$$

Let us consider $X = \{x_k, x_s\}$.

Then

$$A(h_X) = p_k + p_s$$

and

$$A(e_1 h_X) = x_k p_k + p_s x_k. \tag{2.5}$$

From (2.1) and (2.5) we get (1.2).

Let I be an interval of the real axis $\mathbb{R}$ and $\{X_i\}_{i=1}^{n}$ a partition of the interval I.

THEOREM 2.2. *Let f be a continuous convex function defined on I and $p_i, q_i \in (0,1)$, $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$. Then for every partition $\{X_i\}_{i=1}^{n}$ of the interval I such that $A(h_{X_i}) > 0$, $i = 1, \ldots, n$ the following inequalities hold:*

$$\sum_{i=1}^{n} p_i \frac{A(fh_{X_i})}{A(h_{X_i})} - f\left(\sum_{i=1}^{n} p_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right)$$

$$\leqslant \max_{i=\overline{1,n}} \left\{\frac{p_i}{q_i}\right\} \left[\sum_{i=1}^{n} q_i \frac{A(fh_{X_i})}{A(h_{X_i})} - f\left(\sum_{i=1}^{n} q_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right)\right] \tag{2.6}$$

$$\sum_{i=1}^{n} p_i \frac{A(fh_{X_i})}{A(h_{X_i})} - f\left(\sum_{i=1}^{n} p_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right)$$

$$\geqslant \min \left\{\frac{p_i}{q_i}\right\} \left[\sum_{i=1}^{n} q_i \frac{A(fh_{X_i})}{A(h_{X_i})} - f\left(\sum_{i=1}^{n} q_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right)\right]. \tag{2.7}$$

Proof. Let k be a natural number, $k \in \{1, \ldots, n\}$ such that

$$\frac{p_k}{q_k} = \max_{i=\overline{1,n}} \left\{\frac{p_i}{q_i}\right\}.$$

Inequality (2.6) is equivalent with the following inequality:

$$\sum_{i=1}^{n} \left(q_i \frac{p_k}{q_k} - p_i\right) \frac{A(fh_{X_i})}{A(h_{X_i})} + f\left(\sum_{i=1}^{n} p_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right)$$

$$\geqslant \frac{p_k}{q_k} f\left(\sum_{i=1}^{n} q_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right). \tag{2.8}$$

The last inequality can be written in the following form:

$$\sum_{i=1}^{n} q_i \left(\frac{p_k}{q_k} - \frac{p_i}{q_i}\right) \left[\frac{A(fh_{X_i})}{A(h_{X_i})} - f\left(\frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right)\right]$$

$$+ \sum_{i=1}^{n} q_i \left(\frac{p_k}{q_k} - \frac{p_i}{q_i}\right) f\left(\frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right) + f\left(\sum_{i=1}^{n} p_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right)$$

$$\geqslant \frac{p_k}{q_k} f\left(\sum_{i=1}^{n} q_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right). \tag{2.9}$$

Since A is a linear positive functional we have

$$\frac{A(fh_{X_i})}{A(h_{X_i})} - f\left(\frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right) \geqslant 0, \quad i = 1, \ldots, n. \tag{2.10}$$

From (2.10) we obtain:

$$\sum_{i=1}^{n}\left(q_i\frac{p_k}{q_k}-p_i\right)\left[\frac{A(fh_{X_i})}{A(h_{X_i})}-f\left(\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)\right]\geqslant 0. \tag{2.11}$$

Now, inequality (2.9) follows from (2.11) and from Jensen's inequality:

$$\sum_{i=1}^{n+1}\lambda_i f(x_i)\geqslant f\left(\sum_{i=1}^{n+1}\lambda_i x_i\right)$$

where

$$\lambda_i=\left(q_i\frac{p_k}{q_k}-p_i\right)\frac{q_k}{p_k},\quad x_i=\frac{A(e_1h_{X_i})}{A(h_{X_i})},\quad i=1,\dots,n$$

$$\lambda_{n+1}=\frac{q_k}{p_k},\quad x_{n+1}=\sum_{i=1}^{n}p_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}.$$

Let us prove inequality (2.7).

Let s be a natural number, $s\in\{1,2,\dots,n\}$ for which we have

$$\frac{p_s}{q_s}=\min_{i=\overline{1,n}}\left\{\frac{p_i}{q_i}\right\}.$$

Inequality (2.7) is equivalent with the inequality:

$$\sum_{i=1}^{n}\left(p_i-\frac{p_s}{q_s}q_i\right)\frac{A(fh_{X_i})}{A(h_{X_i})}+\frac{p_s}{q_s}f\left(\sum_{i=1}^{n}q_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)$$
$$\geqslant f\left(\sum_{i=1}^{n}p_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right) \tag{2.12}$$

or

$$\begin{aligned}\sum_{i=1}^{n}\left(p_i-\frac{p_s}{q_s}q_i\right)&\left[\frac{A(fh_{X_i})}{A(h_{X_i})}-f\left(\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)\right]\\&+\sum_{i=1}^{n}\left(p_i-\frac{p_s}{q_s}q_i\right)f\left(\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)+\frac{p_s}{q_s}f\left(\sum_{i=1}^{n}q_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)\\ \geqslant f\left(\sum_{i=1}^{n}p_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)&.\end{aligned} \tag{2.13}$$

We note that:

$$\sum_{i=1}^{n}\left(p_i-\frac{p_s}{q_s}q_i\right)+\frac{p_s}{q_s}=1$$

and

$$\sum_{i=1}^{n}\left(p_i-\frac{p_s}{q_s}q_i\right)\frac{A(e_1h_{X_i})}{A(h_{X_i})}+\frac{p_s}{q_s}\sum_{i=1}^{n}q_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}=\sum_{i=1}^{n}p_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}.$$

Jensen's inequality leads to the inequality:

$$\sum_{i=1}^{n}\left(p_i-\frac{p_s}{q_s}q_i\right)f\left(\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)+\frac{p_s}{q_s}f\left(\sum_{i=1}^{n}q_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)$$
$$\geqslant f\left(\sum_{i=1}^{n}p_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right). \tag{2.14}$$

From (2.14), and using the fact that

$$\frac{A(fh_{X_i})}{A(h_{X_i})}-f\left(\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)\geqslant 0$$

we obtain inequality (2.13).

The proof of the theorem is finished. □

COROLLARY 2.3. *Let $(X_i)_{i=1}^n$ be a partition of the interval I and A be a linear positive normalized functional, such that $A(h_{X_i})>0$, for every $i\in\{1,\dots,n\}$. If $q_i\in(0,1)$, $i=\overline{1,n}$ and $\sum_{i=1}^{n}q_i=1$, then for every convex function f, $f\in C(I)$ we have:*

$$\begin{aligned}A(f)-f(a_1)&\leqslant\max_{i=\overline{1,n}}\left\{\frac{A(h_{X_i})}{q_i}\right\}\left[\sum_{i=1}^{n}q_i\frac{A(fh_{X_i})}{A(h_{X_i})}-f\left(\sum_{i=1}^{n}q_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)\right],\\ A(f)-f(a_1)&\geqslant\min_{i=\overline{1,n}}\left\{\frac{A(h_{X_i})}{q_i}\right\}\left[\sum_{i=1}^{n}q_i\frac{A(fh_{X_i})}{A(h_{X_i})}-f\left(\sum_{i=1}^{n}q_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)\right].\end{aligned}\tag{2.15}$$

Proof. Let p_i $(i=\overline{1,n})$ be positive numbers defined by:

$$p_i=A(h_{X_i}),\quad i=\overline{1,n}.$$

We note that

$$\sum_{i=1}^{n}p_i=A\left(\sum_{i=1}^{n}h_{X_i}\right)=A(e_0)=1.$$

Now, the inequalities from Corollary 2.3 follow by Theorem 2.2. □

COROLLARY 2.4. *Let $(X_i)_{i=1}^n$ be a partition of the interval I and A be a linear positive normalized functional such that $A(h_{X_i})>0$ for every $i\in\{1,2,\dots,n\}$. Then for every convex function f the following inequalities are true:*

$$\begin{aligned}A(f)-f(a_1)&\leqslant n\max\{A(h_{X_i})\}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{A(fh_{X_i})}{A(h_{X_i})}-f\left(\frac{1}{n}\sum_{i=1}^{n}\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)\right],\\ A(f)-f(a_1)&\geqslant n\min\{A(h_{X_i})\}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{A(fh_{X_i})}{A(h_{X_i})}-f\left(\frac{1}{n}\sum_{i=1}^{n}\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)\right].\end{aligned}\tag{2.16}$$

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