ON SOME INEQUALITIES FOR CONVEX FUNCTIONS

IOAN GAVREA

Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. In this paper we derive new inequalities for convex functions. The results presented here are an extension of the inequalities obtained by S. S. Dragomir, J. Pečarić and L.-E. Persson.

1. Introduction

Let \( C \) be a convex subset of the real linear space \( X \) and \( f : C \to \mathbb{R} \) a convex function on \( C \). If \( x_i \in C \) and \( p_i \in (0, 1) \) with \( \sum_{i=1}^{n} p_i = 1 \), then the following well-known form of Jensen’s discrete inequality holds:

\[
f \left( \sum_{i=1}^{n} p_ix_i \right) \leq \sum_{i=1}^{n} p_i f(x_i). \tag{1.1}
\]

In [2] S.S. Dragomir, J. Pečarić and L.E. Persson proved the following refinement of Jensen’s inequality in the general setting of linear spaces

\[
\sum_{i=1}^{n} p_if(x_i) - f \left( \sum_{i=1}^{n} p_ix_i \right) \\
\geq \max \left\{ p_if(x_i) + p_j f(x_j) - (p_i + p_j)f \left( \frac{p_ix_i + p_jx_j}{p_i + p_j} \right) \right\}. \tag{1.2}
\]

In 2006 S.S. Dragomir ([1]) proved the following result:

\[
\max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[ \sum_{j=1}^{n} q_j f(x_j) - f \left( \sum_{j=1}^{n} q_jx_j \right) \right] \\
\geq \sum_{j=1}^{n} p_j f(x_j) - f \left( \sum_{j=1}^{n} p_jx_j \right) \\
\geq \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[ \sum_{j=1}^{n} q_j f(x_j) - f \left( \sum_{j=1}^{n} q_jx_j \right) \right]. \tag{1.3}
\]

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provided \( f : C \to \mathbb{R} \) is convex on the convex subset \( C \) of the linear space \( X \) and \( p_i, q_i, i \in \{1, 2, \ldots, n\} \) are probability sequences with \( q_i > 0 \) for each \( i \in \{1, 2, \ldots, n\} \).

In particular, from (1.3) the following result is obtained:

\[
\begin{align*}
& n \max_{1 \leq i \leq n} \{ p_i \} \left[ \frac{1}{n} \sum_{j=1}^{n} f(x_j) - f \left( \frac{1}{n} \sum_{j=1}^{n} x_j \right) \right] \geq \sum_{j=1}^{n} p_j f(x_j) - f \left( \sum_{j=1}^{n} p_j x_j \right) \\
& \geq n \min_{1 \leq i \leq n} \{ p_i \} \left[ \frac{1}{n} \sum_{j=1}^{n} f(x_j) - f \left( \frac{1}{n} \sum_{j=1}^{n} x_j \right) \right].
\end{align*}
\]

(1.4)

In this paper some new results in connection with the inequalities (1.2)-(1.4) are given.

2. Main results

Let \( \mathcal{F} \) be a linear set of functions defined on the interval \( I, I \subseteq \mathbb{R} \) and \( A \) be a linear positive functional defined on \( \mathcal{F} \). We suppose that \( C(I) \subseteq \mathcal{F} \) and for every \( X \subset I \) the characteristic function of \( X \) denoted by \( h_X \) belongs to \( \mathcal{F} \) and for every continuous function \( f, h_X \cdot f \) belongs to \( \mathcal{F} \), too. In the following we suppose that \( A \) is a normalized functional. This means that

\[ A(e_0) = 1, \]

where we denote by \( e_i, i \in \mathbb{N} \) the monomial function, \( e_i : I \to \mathbb{R}, \)

\[ e_i(x) = x^i, \quad x \in I. \]

THEOREM 2.1. Let \( X \) be a subset of \( I \) such that \( A(h_X) > 0 \), and \( f \) a convex function defined on \( I \). Then for every linear positive normalized functional we have:

\[
A(f) - f(a_1) \geq A(fh_X) - A(h_X)f \left( \frac{A(e_1 h_X)}{A(h_X)} \right), \tag{2.1}
\]

where \( a_1 = A(e_1) \).

Proof. From the equality

\[ f = fh_X + fh_{I-X} \]

we get

\[
A(f) - f(a_1) = A(fh_X) + A(fh_{I-X}) - A(h_X)f \left( \frac{A(e_1 h_X)}{A(h_X)} \right) - f(a_1) + A(h_X)f \left( \frac{A(e_1 h_X)}{A(h_X)} \right). \tag{2.2}
\]
Let $B : C(I) \to \mathbb{R}$ be the linear functional defined by

$$B(f) = A(fh_{I-X}) + A(h_X)f\left(\frac{A(e_1h_X)}{A(h_X)}\right). \quad (2.3)$$

We note that $B$ is a linear positive functional. We have

$$B(e_0) = A(h_{I-X}) + A(h_X) = A(e_0) = 1$$
$$B(e_1) = A(e_1h_{I-X}) + A(e_1h_X) = A(e_1) = a_1.$$ 

So, $B$ is a normalized functional and $B(e_1) = a_1$. If $f$ is a convex function on $I$, by Jensen’s inequality we obtain:

$$B(f) \geq f(a_1)$$

or

$$A(fh_{I-X}) + A(h_X)f\left(\frac{A(e_1h_X)}{A(h_X)}\right) - f(a_1) \geq 0. \quad (2.4)$$

From (2.2) and (2.4) we get inequality (2.1). \qed

**Remark 2.2.** Let $A$ be the linear positive normalized functional defined by:

$$A(f) = \sum_{i=1}^{n} p_i f(x_i)$$

where $p_i \in (0, 1), i = 1, n$ and $\sum_{i=1}^{n} p_i = 1$.

For a given convex function $f$, let $k$ and $s$ be the natural number, $k, s \in \{0, 1, 2, \ldots, n\}$ for which:

$$\max \left\{ p_if(x_i) + p_jf(x_j) - (p_i + p_j)f\left(\frac{p_ix_i + p_jx_j}{p_i + p_j}\right) \right\}$$

$$= p_kf(x_k) + p_sf(x_s) - (p_k + p_s)f\left(\frac{p_kx_k + p sx_s}{p_k + p s}\right).$$

Let us consider $X = \{x_k, x_s\}$. Then

$$A(h_X) = p_k + p_s$$

and

$$A(e_1h_X) = x_k p_k + p_s x_s. \quad (2.5)$$

From (2.1) and (2.5) we get (1.2).

Let $I$ be an interval of the real axis $\mathbb{R}$ and $\{X_i\}_{i=1}^{n}$ a partition of the interval $I$. 

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THEOREM 2.2. Let $f$ be a continuous convex function defined on $I$ and $p_i, q_i \in (0, 1)$, $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$. Then for every partition $\{X_i\}_{i=1}^{n}$ of the interval $I$ such that $A(h_{X_i}) > 0$, $i = 1, \ldots, n$ the following inequalities hold:

$$\sum_{i=1}^{n} p_i \frac{A(fh_{X_i})}{A(h_{X_i})} - f \left( \sum_{i=1}^{n} p_i \frac{A(e_{1}h_{X_i})}{A(h_{X_i})} \right)$$

$$\leq \max_{i=1,n} \left\{ \frac{p_i}{q_i} \right\} \left[ \sum_{i=1}^{n} q_i \frac{A(fh_{X_i})}{A(h_{X_i})} - f \left( \sum_{i=1}^{n} q_i \frac{A(e_{1}h_{X_i})}{A(h_{X_i})} \right) \right]$$

(2.6)

$$\sum_{i=1}^{n} p_i \frac{A(fh_{X_i})}{A(h_{X_i})} - f \left( \sum_{i=1}^{n} p_i \frac{A(e_{1}h_{X_i})}{A(h_{X_i})} \right)$$

$$\geq \min \left\{ \frac{p_i}{q_i} \right\} \left[ \sum_{i=1}^{n} q_i \frac{A(fh_{X_i})}{A(h_{X_i})} - f \left( \sum_{i=1}^{n} q_i \frac{A(e_{1}h_{X_i})}{A(h_{X_i})} \right) \right].$$

(2.7)

Proof. Let $k$ be a natural number, $k \in \{1, \ldots, n\}$ such that

$$\frac{p_k}{q_k} = \max_{i=1,n} \left\{ \frac{p_i}{q_i} \right\}.$$

Inequality (2.6) is equivalent with the following inequality:

$$\sum_{i=1}^{n} \left( q_i \frac{p_k}{q_k} - p_i \right) \frac{A(fh_{X_i})}{A(h_{X_i})} + f \left( \sum_{i=1}^{n} p_i \frac{A(e_{1}h_{X_i})}{A(h_{X_i})} \right)$$

$$\geq \frac{p_k}{q_k} f \left( \sum_{i=1}^{n} q_i \frac{A(e_{1}h_{X_i})}{A(h_{X_i})} \right).$$

(2.8)

The last inequality can be written in the following form:

$$\sum_{i=1}^{n} q_i \left( \frac{p_k}{q_k} - p_i \right) \left[ \frac{A(fh_{X_i})}{A(h_{X_i})} - f \left( \frac{A(e_{1}h_{X_i})}{A(h_{X_i})} \right) \right]$$

$$+ \sum_{i=1}^{n} q_i \left( \frac{p_k}{q_k} - p_i \right) f \left( \frac{A(e_{1}h_{X_i})}{A(h_{X_i})} \right) + f \left( \sum_{i=1}^{n} p_i \frac{A(e_{1}h_{X_i})}{A(h_{X_i})} \right)$$

$$\geq \frac{p_k}{q_k} f \left( \sum_{i=1}^{n} q_i \frac{A(e_{1}h_{X_i})}{A(h_{X_i})} \right).$$

(2.9)

Since $A$ is a linear positive functional we have

$$\frac{A(fh_{X_i})}{A(h_{X_i})} - f \left( \frac{A(e_{1}h_{X_i})}{A(h_{X_i})} \right) \geq 0, \quad i = 1, \ldots, n.$$

(2.10)
From (2.10) we obtain:

\[
\sum_{i=1}^{n} \left( q_i \frac{p_k}{q_k} - p_i \right) \left[ \frac{A(fh_X_i)}{A(h_X_i)} - f \left( \frac{A(e_1h_X_i)}{A(h_X_i)} \right) \right] \geq 0. \tag{2.11}
\]

Now, inequality (2.9) follows from (2.11) and from Jensen’s inequality:

\[
\sum_{i=1}^{n+1} \lambda_i f(x_i) \geq f \left( \sum_{i=1}^{n+1} \lambda_i x_i \right)
\]

where

\[
\lambda_i = \left( q_i \frac{p_k}{q_k} - p_i \right) \frac{q_k}{p_k}, \quad x_i = \frac{A(e_1h_X_i)}{A(h_X_i)} , \quad i = 1, \ldots, n
\]

\[
\lambda_{n+1} = \frac{q_k}{p_k}, \quad x_{n+1} = \sum_{i=1}^{n} p_i \frac{A(e_1h_X_i)}{A(h_X_i)}.
\]

Let us prove inequality (2.7).

Let \( s \) be a natural number, \( s \in \{1, 2, \ldots, n\} \) for which we have

\[
\frac{p_s}{q_s} = \min_{i=1,n} \left\{ \frac{p_i}{q_i} \right\}.
\]

Inequality (2.7) is equivalent with the inequality:

\[
\sum_{i=1}^{n} \left( p_i - \frac{p_s}{q_s} q_i \right) \frac{A(fh_X_i)}{A(h_X_i)} + \frac{p_s}{q_s} f \left( \sum_{i=1}^{n} q_i \frac{A(e_1h_X_i)}{A(h_X_i)} \right)
\]

\[
\geq f \left( \sum_{i=1}^{n} p_i \frac{A(e_1h_X_i)}{A(h_X_i)} \right) \tag{2.12}
\]

or

\[
\sum_{i=1}^{n} \left( p_i - \frac{p_s}{q_s} q_i \right) \left[ \frac{A(fh_X_i)}{A(h_X_i)} - f \left( \frac{A(e_1h_X_i)}{A(h_X_i)} \right) \right]
\]

\[
+ \sum_{i=1}^{n} \left( p_i - \frac{p_s}{q_s} q_i \right) f \left( \frac{A(e_1h_X_i)}{A(h_X_i)} \right) + \frac{p_s}{q_s} f \left( \sum_{i=1}^{n} q_i \frac{A(e_1h_X_i)}{A(h_X_i)} \right)
\]

\[
\geq f \left( \sum_{i=1}^{n} p_i \frac{A(e_1h_X_i)}{A(h_X_i)} \right). \tag{2.13}
\]

We note that:

\[
\sum_{i=1}^{n} \left( p_i - \frac{p_s}{q_s} q_i \right) + \frac{p_s}{q_s} = 1
\]

and

\[
\sum_{i=1}^{n} \left( p_i - \frac{p_s}{q_s} q_i \right) \frac{A(e_1h_X_i)}{A(h_X_i)} + \frac{p_s}{q_s} \sum_{i=1}^{n} q_i \frac{A(e_1h_X_i)}{A(h_X_i)} = \sum_{i=1}^{n} p_i \frac{A(e_1h_X_i)}{A(h_X_i)}.
\]
Jensen’s inequality leads to the inequality:
\[
\sum_{i=1}^{n} \left( p_i - \frac{p_s}{q_s} q_i \right) f \left( \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right) + \frac{p_s}{q_s} f \left( \sum_{i=1}^{n} q_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right)\]
\[
\geq f \left( \sum_{i=1}^{n} p_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right).
\] (2.14)

From (2.14), and using the fact that
\[
\frac{A(f h_{X_i})}{A(h_{X_i})} - f \left( \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right) \geq 0
\]
we obtain inequality (2.13).

The proof of the theorem is finished. □

**COROLLARY 2.3.** Let \((X_i)_{i=1}^{n}\) be a partition of the interval \(I\) and \(A\) be a linear positive normalized functional, such that \(A(h_{X_i}) > 0\), for every \(i \in \{1, \ldots, n\}\). If \(q_i \in (0, 1)\), \(i = 1, n\) and \(\sum_{i=1}^{n} q_i = 1\), then for every convex function \(f\), \(f \in C(I)\) we have:
\[
A(f) - f(a_1) \leq \max_{i=1, \ldots, n} \left\{ \frac{A(h_{X_i})}{q_i} \right\} \left[ \sum_{i=1}^{n} q_i \frac{A(f h_{X_i})}{A(h_{X_i})} - f \left( \sum_{i=1}^{n} q_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right) \right],
\] (2.15)

\[
A(f) - f(a_1) \geq \min_{i=1, \ldots, n} \left\{ \frac{A(h_{X_i})}{q_i} \right\} \left[ \sum_{i=1}^{n} q_i \frac{A(f h_{X_i})}{A(h_{X_i})} - f \left( \sum_{i=1}^{n} q_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right) \right].
\]

**Proof.** Let \(p_i\) \((i = 1, n)\) be positive numbers defined by:
\[
p_i = A(h_{X_i}), \quad i = 1, n.
\]
We note that
\[
\sum_{i=1}^{n} p_i = A \left( \sum_{i=1}^{n} h_{X_i} \right) = A(e_0) = 1.
\]

Now, the inequalities from Corollary 2.3 follow by Theorem 2.2. □

**COROLLARY 2.4.** Let \((X_i)_{i=1}^{n}\) be a partition of the interval \(I\) and \(A\) be a linear positive normalized functional such that \(A(h_{X_i}) > 0\) for every \(i \in \{1, 2, \ldots, n\}\). Then for every convex function \(f\) the following inequalities are true:
\[
A(f) - f(a_1) \leq n \max \{A(h_{X_i})\} \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{A(f h_{X_i})}{A(h_{X_i})} - f \left( \frac{1}{n} \sum_{i=1}^{n} \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right) \right],
\] (2.16)

\[
A(f) - f(a_1) \geq n \min \{A(h_{X_i})\} \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{A(f h_{X_i})}{A(h_{X_i})} - f \left( \frac{1}{n} \sum_{i=1}^{n} \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right) \right].
\]
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Ioan Gavrea
Technical University of Cluj-Napoca
Department of Mathematics
Romania

e-mail: Ioan.Gavrea@math.utcluj.ro