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CHARACTERIZATION OF THE NORM TRIANGLE EQUALITY IN PRE–HILBERT C^*–MODULES AND APPLICATIONS

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Dedicated to Professor Josip Pečarić on the occasion of his 60th birthday

Abstract. The characterization of the norm triangle equality in pre-Hilbert C^*-modules is given. The result is applied for describing the case of equality in some generalizations of the Dunkl-Williams inequality.

1. Introduction

Let V be a normed linear space. Then the triangle inequality

$$\|x+y\| \leqslant \|x\| + \|y\| \tag{1}$$

holds for all $x, y \in V$. The problem when the equality in (1) holds has been studied for different types of normed linear spaces by many authors (see, for example [1, 11, 3, 15, 2]).

In this paper we give a historical review of this investigation. More precisely, we present some known results on the characterization of the triangle equality in the case when V is the algebra of all bounded linear operators acting on a uniformly convex Banach space or a Hilbert space. (Recall that a normed linear space W is uniformly convex if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(x, y \in W,\ \|x\| = \|y\| = 1,\ \|x-y\| \geqslant \varepsilon) \Rightarrow \left\|\frac{x+y}{2}\right\| \leqslant 1-\delta.)$$

We proceed by extending these results in the context of an arbitrary C^*-algebra and finally in a pre-Hilbert C^*-module setting. It enables us to characterize the equality attainedness in a generalized Dunkl-Williams inequality and its reverse inequality for elements of a pre-Hilbert C^*-module.

Throughout this paper, we shall use X for denoting a uniformly convex Banach space and H for a Hilbert space. By $B(X)$ (resp. $B(H)$) we denote the algebra of all bounded linear operators on X (resp. H).

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2. Characterization of the norm triangle equality

We consider the following problem: *When does the equality in* (1) *hold*?

It is well known that in the case of a strictly convex normed linear space V, the equality $\|x+y\| = \|x\|+\|y\|$ holds for nonzero vectors $x,y \in V$ if and only if $\frac{x}{\|x\|} = \frac{y}{\|y\|}$. (This characterization follows directly from Lemma 2.2 below.) In particular, the triangle equality in every inner-product space is characterized in this way.

Several authors considered this problem in the setting of the algebra $V = B(X)$. The first significant result was obtained in 1991 by Abramovich et al. [1] who solved this problem when one of the operators $A,B \in B(X)$ is the identity operator on X.

In what follows $\sigma_{ap}(T)$ will stand for the approximate point spectrum of $T \in B(X)$ (i.e., the set of all $\lambda \in \mathbb{C}$ for which there is a sequence (x_n) of unit vectors in X such that $\lim_{n\to\infty} \|Tx_n - \lambda x_n\| = 0$). By S^- we denote the topological closure of a set S.

THEOREM 2.1. ([1]) *If $A \in B(X)$, and $I \in B(X)$ is the identity operator, then $\|A+I\| = \|A\|+1$ if and only if $\|A\| \in \sigma_{ap}(A)$.*

Another type of result in this direction was obtained in 1999 by Lin [11] who established the following theorem. To prove it, we need the following lemma, the proof of which can be found in [1, Lemma 2.1].

LEMMA 2.2. ([1]) *Let x and y be vectors in a normed linear space such that $\|x+y\| = \|x\|+\|y\|$, and let α and β be non-negative real numbers. Then $\|\alpha x + \beta y\| = \alpha\|x\| + \beta\|y\|$.*

THEOREM 2.3. ([11]) *Whenever $A,B \in B(X)$ satisfy the triangle equality $\|A+B\| = \|A\|+\|B\|$, then $0 \in \sigma_{ap}(\|B\|A - \|A\|B)$. Moreover, if X is a Hilbert space and either A or B is isometric, then the converse holds.*

Proof. Suppose that $\|A+B\| = \|A\|+\|B\|$ for some nonzero operators $A,B \in B(X)$. Let us denote $P = \frac{A}{\|A\|}$ and $Q = \frac{B}{\|B\|}$. Then by Lemma 2.2 we have $\|P+Q\| = 2$. So, there exists a sequence (x_n) of unit vectors in X such that $\lim_{n\to\infty} \|Px_n + Qx_n\| = 2$. From this we get $\lim_{n\to\infty} \|Px_n - Qx_n\| = 0$, since X is uniformly convex and $\|Px_n\| \leqslant 1$, $\|Qx_n\| \leqslant 1$ for all $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} \|(\|B\|A - \|A\|B)x_n\| = 0$, that is, $0 \in \sigma_{ap}(\|B\|A - \|A\|B)$.

Let us now suppose that X is a Hilbert space and A is an isometric operator. Let $0 \in \sigma_{ap}(\|B\|A - B)$. Then $\lim_{n\to\infty} \|(\|B\|A - B)x_n\| = 0$ for some sequence (x_n) of unit vectors in X. Since

$$\|(\|B\|A-B)x_n\| \geqslant \big\| \|B\|Ax_n\| - \|Bx_n\| = \|B\|\|Ax_n\| - \|Bx_n\| = \|B\| - \|Bx_n\| \geqslant 0$$

we have

$$0 = \lim_{n\to\infty} \|(\|B\|A-B)x_n\| \geqslant \|B\| - \lim_{n\to\infty} \|Bx_n\| \geqslant 0,$$

wherefrom $\lim_{n\to\infty}\|Bx_n\| = \|B\|$. Furthermore,

$$2\|B\| \geqslant \|(\|B\|A+B)x_n\| = \|(\|B\|A-B)x_n + 2Bx_n\| \geqslant 2\|Bx_n\| - \|(\|B\|A-B)x_n\|,$$

from which it follows that

$$2\|B\| \geqslant \lim_{n\to\infty} \|(\|B\|A+B)x_n\| \geqslant 2\|B\|,$$

that is, $\lim_{n\to\infty}\|(\|B\|A+B)x_n)\| = 2\|B\|$. We conclude that $\|\,\|B\|A+B\| = 2\|B\|$, wherefrom $\|A+B\| = 1+\|B\|$ by Lemma 2.2. $\square$

Note that the above theorem completely describes the triangle equality for Hilbert space operators under the condition that one of the operators is isometric.

In 2002 Barraa et al. [3] extended Lin's result to general Hilbert space operators by characterizing the triangle equality in terms of the classical numerical range. (Recall that the *classical numerical range* of $T \in B(H)$ is defined as $W(T) = \{(Tx,x) : x \in H, \|x\| = 1\}$.)

THEOREM 2.4. ([3]) *Let $A,B \in B(H)$. Then the equality $\|A+B\| = \|A\|+\|B\|$ holds if and only if $\|A\|\|B\| \in W(A^*B)^-$.*

Proof. Suppose $\|A+B\| = \|A\|+\|B\|$. There exists a sequence (x_n) of unit vectors in H such that $\lim_{n\to\infty}\|Ax_n+Bx_n\| = \|A\|+\|B\|$. Since

$$\|Ax_n+Bx_n\| \leqslant \|Ax_n\|+\|Bx_n\| \leqslant \|A\|+\|Bx_n\| \leqslant \|A\|+\|B\|$$

we have $\lim_{n\to\infty}(\|A\|+\|Bx_n\|) = \|A\|+\|B\|$, that is, $\lim_{n\to\infty}\|Bx_n\| = \|B\|$. Similarly, we obtain $\lim_{n\to\infty}\|Ax_n\| = \|A\|$. Therefore, from the identity

$$\|Ax_n+Bx_n\|^2 = \|Ax_n\|^2+\|Bx_n\|^2+2\mathrm{Re}\,(A^*Bx_n,x_n) \tag{2}$$

it follows that

$$\lim_{n\to\infty}\mathrm{Re}\,(A^*Bx_n,x_n) = \|A\|\|B\|.$$

Since

$$|(A^*Bx_n,x_n)| = ((\mathrm{Re}\,(A^*Bx_n,x_n))^2+(\mathrm{Im}\,(A^*Bx_n,x_n))^2)^{\frac{1}{2}},$$

and $|(A^*Bx_n,x_n)| \leqslant \|A\|\|B\|$, we deduce that

$$\lim_{n\to\infty}|(A^*Bx_n,x_n)| = \|A\|\|B\|.$$

Thus, $\lim_{n\to\infty}\mathrm{Im}\,(A^*Bx_n,x_n) = 0$, wherefrom $\lim_{n\to\infty}(A^*Bx_n,x_n) = \|A\|\|B\|$, i.e., $\|A\|\|B\| \in W(A^*B)^-$.

Conversely, let us suppose that $\|A\|\|B\| \in W(A^*B)^-$. Then there is a sequence (x_n) of unit vectors in H such that $\lim_{n\to\infty}(A^*Bx_n,x_n) = \|A\|\|B\|$. Thus, $\lim_{n\to\infty}\mathrm{Re}\,(A^*Bx_n,x_n) = \|A\|\|B\|$. Since $|(A^*Bx_n,x_n)| \leqslant \|Ax_n\|\|B\| \leqslant \|A\|\|B\|$, we conclude that $\lim_{n\to\infty}\|Ax_n\| = \|A\|$. Similarly, we get $\lim_{n\to\infty}\|Bx_n\| = \|B\|$. Now, (2) implies $\lim_{n\to\infty}\|Ax_n+Bx_n\| = \|A\|+\|B\|$, wherefrom $\|A+B\| = \|A\|+\|B\|$. $\square$

REMARK 2.5. It is known that $\|A\| \in W(A)^-$ if and only if $\|A\| \in \sigma_{ap}(A)$ (see e.g. [8]). From this, one can easily deduce that $\|A\|\,\|B\| \in W(A^*B)^-$ if and only if $\|A\|\,\|B\| \in \sigma_{ap}(A^*B)$.

The concept of the classical numerical range can be naturally extended for elements of an arbitrary C^*-algebra. If $\mathscr{A}$ is a C^*-algebra, we define the *numerical range* of an arbitrary element $a \in \mathscr{A}$ as the set

$$V(a) = \{\varphi(a) : \varphi \text{ is a state of } \mathscr{A}\}.$$

This set generalizes the classical numerical range in the sense that the numerical range $V(T)$ of a Hilbert space operator T (considered as an element of a C^*-algebra $B(H)$) coincides with the closure of its classical numerical range $W(T)$ (see [18]). (For more details about numerical ranges the reader is referred to [4, 5].)

Thus, one can ask if the C^*-algebraic version of Theorem 2.4 holds. The answer is affirmative and was given by Nakamoto et al. in [15]. They proved the following theorem.

THEOREM 2.6. ([15]) *Let $\mathscr{A}$ be a C^*-algebra and $a,b \in \mathscr{A}$. Then the equality $\|a+b\| = \|a\| + \|b\|$ holds if and only if $\|a\|\|b\| \in V(a^*b)$.*

Arambašić et al. [2] observed that Theorem 2.6 can be further generalized in the framework of pre-Hilbert C^*-modules. Pre-Hilbert C^*-modules are a straightforward generalization of inner-product spaces where the scalar field $\mathbb{C}$ is replaced by a C^*-algebra. The basic theory of pre-Hilbert C^*-modules can be found in [10, 19].

The formal definition is as follows.

A *right pre-Hilbert C^*-module V over a C^*-algebra $\mathscr{A}$* (or a *right pre-Hilbert $\mathscr{A}$-module*) is a linear space which is a right $\mathscr{A}$-module (with a compatible scalar multiplication: $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for $x \in V,\ a \in \mathscr{A},\ \lambda \in \mathbb{C}$), equipped with an $\mathscr{A}$-valued inner-product $\langle\cdot,\cdot\rangle : V \times V \to \mathscr{A}$ that is sesquilinear, positive definite and respects the module action. In other words:

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x,y\rangle + \beta\langle x,z\rangle$ for $x,y,z \in V,\ \alpha,\beta \in \mathbb{C}$,
(ii) $\langle x, ya\rangle = \langle x,y\rangle a$ for $x,y \in V,\ a \in \mathscr{A}$,
(iii) $\langle x,y\rangle^* = \langle y,x\rangle$ for $x,y \in V$,
(iv) $\langle x,x\rangle \geqslant 0$ for $x \in V$; if $\langle x,x\rangle = 0$ then $x = 0$.

For a pre-Hilbert C^*-module V the Cauchy-Schwarz inequality holds:

$$\|\langle x,y\rangle\|^2 \leqslant \|\langle x,x\rangle\|\,\|\langle y,y\rangle\|, \quad (x,y \in V).$$

Consequently, $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$ defines a norm on V.

A pre-Hilbert $\mathscr{A}$-module which is complete with respect to its norm is called a *Hilbert C^*-module over $\mathscr{A}$*, or a *Hilbert $\mathscr{A}$-module*.

Given a positive functional φ of a C^*-algebra $\mathscr{A}$, we have the following useful version of the Cauchy-Schwarz inequality:

$$|\varphi(\langle x,y\rangle)|^2 \leqslant \varphi(\langle x,x\rangle)\varphi(\langle y,y\rangle)$$

for arbitrary elements x,y of a pre-Hilbert C^*-module V.

REMARK 2.7. (a) The ordinary inner-product spaces are left pre-Hilbert $\mathbb{C}$-modules.

(b) Every C^*-algebra $\mathscr{A}$ is a Hilbert $\mathscr{A}$-module under the inner product defined by $\langle a,b\rangle = a^*b$. The corresponding norm is just the norm on $\mathscr{A}$ because of the C^*-condition.

The following result characterizes the triangle equality for elements of a pre-Hilbert C^*-module.

THEOREM 2.8. ([2]) *Let $\mathscr{A}$ be a C^*-algebra, V a pre-Hilbert $\mathscr{A}$-module and $x,y \in V$. Then the equality $\|x+y\| = \|x\|+\|y\|$ holds if and only if $\|x\|\|y\| \in V(\langle x,y\rangle)$.*

Proof. Suppose that $\|x+y\| = \|x\|+\|y\|$. There is a state φ on $\mathscr{A}$ such that $\varphi(\langle x+y,x+y\rangle) = \|\langle x+y,x+y\rangle\| = \|x+y\|^2$. Then we have

$$\begin{aligned}\|x+y\|^2 &= \varphi(\langle x+y,x+y\rangle)\\ &= \varphi(\langle x,x\rangle)+\varphi(\langle x,y\rangle)+\varphi(\langle y,x\rangle)+\varphi(\langle y,y\rangle)\\ &\leqslant \|x\|^2+2\|x\|\|y\|+\|y\|^2 = (\|x\|+\|y\|)^2 = \|x+y\|^2,\end{aligned}$$

from which it follows that $\varphi(\langle x,x\rangle) = \|x\|^2$, $\varphi(\langle y,y\rangle) = \|y\|^2$ and $\varphi(\langle x,y\rangle) = \|x\|\|y\|$. Hence, $\|x\|\|y\| \in V(\langle x,y\rangle)$.

Conversely, suppose $\|x\|\|y\| \in V(\langle x,y\rangle)$. Then there is a state φ on $\mathscr{A}$ such that $\varphi(\langle x,y\rangle) = \|x\|\|y\|$. From the Cauchy-Schwarz inequality we get

$$\begin{aligned}\|x\|\|y\| = |\varphi(\langle x,y\rangle)| &\leqslant \varphi(\langle x,x\rangle)^{\frac{1}{2}}\varphi(\langle y,y\rangle)^{\frac{1}{2}}\\ &\leqslant \|\langle x,x\rangle\|^{\frac{1}{2}}\|\langle y,y\rangle\|^{\frac{1}{2}} = \|x\|\|y\|\end{aligned}$$

and then $\varphi(\langle x,x\rangle) = \|x\|^2$ and $\varphi(\langle y,y\rangle) = \|y\|^2$. Now we have

$$\begin{aligned}\varphi(\langle x+y,x+y\rangle) &= \varphi(\langle x,x\rangle)+\varphi(\langle x,y\rangle)+\varphi(\langle y,x\rangle)+\varphi(\langle y,y\rangle)\\ &= \|x\|^2+2\|x\|\|y\|+\|y\|^2 = (\|x\|+\|y\|)^2.\end{aligned}$$

Since $\varphi(\langle x+y,x+y\rangle) \leqslant \|\langle x+y,x+y\rangle\| = \|x+y\|^2$, we get $\|x+y\| = \|x\|+\|y\|$. □

REMARK 2.9. (a) We have already noted that every inner-product space $(V,(\cdot,\cdot))$ is a pre-Hilbert $\mathbb{C}$-module. Since the only state on $\mathbb{C}$ is the identity operator, the preceding theorem tells us that the equality $\|x+y\| = \|x\|+\|y\|$ is valid for $x,y \in V$ precisely when $(x,y) = \|x\|\|y\|$. The second equality holds if and only if $x = \lambda y$ or $y = \lambda x$ for some constant $\lambda \geqslant 0$, which is a well-known characterization of the equality $\|x+y\| = \|x\|+\|y\|$ in inner-product spaces.

(b) It is evident from Remark 2.7 (b) that Theorem 2.8 also generalizes Nakamoto and Takahasi's result (i.e., Theorem 2.6) in the context of pre-Hilbert C^*-modules.

Characterization of the triangle equality for an arbitrary number of finitely many elements of a pre Hilbert C^*-module was also obtained in [2].

THEOREM 2.10. ([2]) *Let $\mathscr{A}$ be a C^*-algebra, V a pre-Hilbert $\mathscr{A}$-module and $x_1,\ldots,x_n$ nonzero elements of V. Then the equality $\|x_1+\cdots+x_n\| = \|x_1\|+\cdots+\|x_n\|$ holds if and only if there is a state φ of $\mathscr{A}$ such that $\varphi(\langle x_i,x_n\rangle) = \|x_i\|\|x_n\|$ for $i = 1,\ldots,n-1$.*

Proof. Indeed, from $\|x_1+\cdots+x_n\| = \|x_1\|+\cdots+\|x_n\|$ we obtain that

$$\|x_1\|+\cdots+\|x_n\| = \|x_1+\cdots+x_n\| \leqslant \|x_1+\cdots+x_{n-1}\|+\|x_n\|$$

from which it follows that $\|x_1+\cdots+x_{n-1}\| = \|x_1\|+\cdots+\|x_{n-1}\|$ and so $\|(x_1+\cdots+x_{n-1})+x_n\| = \|x_1+\cdots+x_{n-1}\|+\|x_n\|$. By Theorem 2.8 there is a state φ on $\mathscr{A}$ such that

$$\varphi(\langle x_1+\cdots+x_{n-1},x_n\rangle) = \|x_1+\cdots+x_{n-1}\|\|x_n\| = (\|x_1\|+\cdots+\|x_{n-1}\|)\|x_n\|.$$

Then $\varphi(\langle x_1,x_n\rangle)+\cdots+\varphi(\langle x_{n-1},x_n\rangle) = \|x_1\|\|x_n\|+\cdots+\|x_{n-1}\|\|x_n\|$ which implies $\varphi(\langle x_i,x_n\rangle) = \|x_i\|\|x_n\|$ for $i = 1,\ldots,n-1$. The converse is proved in a similar way. □

3. Some applications

Maligranda [12] obtained the following refinement of the triangle inequality and its reverse inequality.

THEOREM 3.1. ([12]) *For any nonzero elements x and y in a normed linear space V we have*

$$\|x+y\| \leqslant \|x\|+\|y\| - \left(2 - \left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\min\{\|x\|,\|y\|\} \tag{3}$$

and

$$\|x+y\| \geqslant \|x\|+\|y\| - \left(2 - \left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\max\{\|x\|,\|y\|\}. \tag{4}$$

Proof. Without loss of generality we may assume that $\|x\| \leqslant \|y\|$. Then, by the triangle inequality

$$\begin{aligned}\|x+y\| &= \left\|\tfrac{\|x\|}{\|x\|}x+\tfrac{\|x\|}{\|y\|}y+\left(1-\tfrac{\|x\|}{\|y\|}\right)y\right\| \\ &\leqslant \|x\|\left\|\tfrac{x}{\|x\|}+\tfrac{y}{\|y\|}\right\|+\|y\|-\|x\| \\ &= \|y\|+\left(\left\|\tfrac{x}{\|x\|}+\tfrac{y}{\|y\|}\right\|-1\right)\|x\| \\ &= \|x\|+\|y\|+\left(\left\|\tfrac{x}{\|x\|}+\tfrac{y}{\|y\|}\right\|-2\right)\|x\|,\end{aligned}$$

which establishes the estimate (3). Similarly, the computation

$$\begin{aligned}\|x+y\| &= \left\|\tfrac{\|y\|}{\|y\|}y+\tfrac{\|y\|}{\|x\|}x+\left(1-\tfrac{\|y\|}{\|x\|}\right)x\right\| \\ &\geqslant \|y\|\left\|\tfrac{y}{\|y\|}+\tfrac{x}{\|x\|}\right\|-|\,\|x\|-\|y\|\,| \\ &= \|y\|\left\|\tfrac{y}{\|y\|}+\tfrac{x}{\|x\|}\right\|-\|y\|+\|x\| \\ &= \|x\|+\|y\|-\left(2-\left\|\tfrac{x}{\|x\|}+\tfrac{y}{\|y\|}\right\|\right)\|y\|\end{aligned}$$

gives the inequality (4). $\square$

Observe that the inequalities (3) and (4) can be rewritten as the estimates for the *angular distance* $\alpha[x,y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$ introduced by Clarkson in [6]. Namely, it is evident that (4) is equivalent to

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{\|x-y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}}, \tag{5}$$

while (3) is equivalent to

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geqslant \frac{\|x-y\| - |\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}}. \tag{6}$$

(For another proof of (6) see also [14].)

The inequality (5) provides a refinement of the Massera-Schäffer inequality [13] which states that

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{2\|x-y\|}{\max\{\|x\|, \|y\|\}} \tag{7}$$

for any two nonzero elements x and y in a normed linear space V. Also, (7) sharpens the Dunkl-Williams inequality [7]:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{4\|x-y\|}{\|x\| + \|y\|} \quad (x, y \in V \setminus \{0\}). \tag{8}$$

Generalizations of (3) and (4) for $n \in \mathbb{N}$ elements of a normed linear space were given by Kato et al. [9] who proved the following theorem.

THEOREM 3.2. ([9]) *Let V be a normed linear space and $x_1, \dots, x_n$ nonzero elements of V. Then we have*

$$\left\| \sum_{j=1}^{n} x_j \right\| \leqslant \sum_{j=1}^{n} \|x_j\| - \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{j \in \{1,\dots,n\}} \|x_j\| \tag{9}$$

and

$$\left\| \sum_{j=1}^{n} x_j \right\| \geqslant \sum_{j=1}^{n} \|x_j\| - \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \max_{j \in \{1,\dots,n\}} \|x_j\|. \tag{10}$$

The above result was improved in [16]:

THEOREM 3.3. ([16]) *Let V be a normed linear space and $x_1, \dots, x_n$ nonzero elements of V. Then we have*

$$\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \leqslant \min_{i \in \{1,\dots,n\}} \left\{ \frac{1}{\|x_i\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} |\|x_j\| - \|x_i\|| \right) \right\} \tag{11}$$

and

$$\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \geqslant \max_{i \in \{1,\dots,n\}} \left\{ \frac{1}{\|x_i\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} |\|x_j\| - \|x_i\|| \right) \right\}. \tag{12}$$

REMARK 3.4. When $n=2$ the inequalities (5) and (6) immediately follow by putting $x_1 := x$ and $x_2 := -y$ in (11) and (12), respectively.

Moreover, it can be easily verified that (11) implies (10), while (12) implies (9).

Finally, we present the characterizations of the case of equality in (11) and (12) for elements of pre-Hilbert C^*-modules. These characterizations were obtained in [17] by using Theorem 2.10.

THEOREM 3.5. ([17]) *Let V be a pre-Hilbert $\mathscr{A}$-module and $x_1,\dots,x_n$ nonzero elements of V such that $\sum_{j=1}^n x_j \neq 0$. Then equality holds in* (11) *if and only if $\|x_1\| = \cdots = \|x_n\|$ or there exist $i \in \{1,\dots,n\}$ and a state φ of $\mathscr{A}$ satisfying*

$$\operatorname{sgn}(\|x_i\| - \|x_k\|) \sum_{j=1}^n \varphi(\langle x_j, x_k\rangle) = \left\| \sum_{j=1}^n x_j \right\| \|x_k\|$$

for all $k \in \{1,\dots,n\}$ such that $\|x_k\| \neq \|x_i\|$.

THEOREM 3.6. ([17]) *Let V be a pre-Hilbert $\mathscr{A}$-module and $x_1,\dots,x_n$ nonzero elements of V such that $\sum_{j=1}^n x_j = 0$. Then equality holds in* (11) *if and only if $\|x_1\| = \cdots = \|x_n\|$ or there exist $i,k \in \{1,\dots,n\}$ satisfying $\|x_i\| \neq \|x_k\|$ and a state φ of $\mathscr{A}$ such that*

$$\operatorname{sgn}(\|x_i\| - \|x_j\|)\operatorname{sgn}(\|x_i\| - \|x_k\|)\varphi(\langle x_j, x_k\rangle) = \|x_j\|\|x_k\|$$

for all $j \in \{1,\dots,n\} \setminus \{k\}$ such that $\|x_j\| \neq \|x_i\|$.

THEOREM 3.7. ([17]) *Let V be a pre-Hilbert $\mathscr{A}$-module and $x_1,\dots,x_n$ nonzero elements of V such that $\sum_{j=1}^n \frac{x_j}{\|x_j\|} \neq 0$. Then equality holds in* (12) *if and only if $\|x_1\| = \cdots = \|x_n\|$ or there exist $i \in \{1,\dots,n\}$ and a state φ of $\mathscr{A}$ satisfying*

$$\operatorname{sgn}(\|x_k\| - \|x_i\|) \sum_{j=1}^n \varphi\left(\left\langle \frac{x_j}{\|x_j\|}, x_k \right\rangle\right) = \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \|x_k\|$$

for all $k \in \{1,\dots,n\}$ such that $\|x_k\| \neq \|x_i\|$.

THEOREM 3.8. ([17]) *Let V be a pre-Hilbert $\mathscr{A}$-module and $x_1,\dots,x_n$ nonzero elements of V such that $\sum_{j=1}^n \frac{x_j}{\|x_j\|} = 0$. Then equality holds in* (12) *if and only if $\|x_1\| = \cdots = \|x_n\|$ or there exist $i,k \in \{1,\dots,n\}$ satisfying $\|x_i\| \neq \|x_k\|$ and a state φ of $\mathscr{A}$ such that*

$$\operatorname{sgn}(\|x_i\| - \|x_j\|)\operatorname{sgn}(\|x_i\| - \|x_k\|)\varphi(\langle x_j, x_k\rangle) = \|x_j\|\|x_k\|$$

for all $j \in \{1,\dots,n\} \setminus \{k\}$ such that $\|x_j\| \neq \|x_i\|$.

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