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ON YOUNG'S INEQUALITY AND ITS REVERSE FOR BOUNDING THE LORENZ CURVE AND GINI MEAN

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Dedicated to Professor Josip Pečarić on the occasion of his 60th birthday

Abstract. The performance of the Young integral inequality is investigated for bounding the Lorenz curve and the Gini index. The study relies on a comparison of reverse Young type integral inequalities. The resulting approximation and bounds for the Lorenz curve and the Gini index are compared with previous results.

1. Introduction

Let $f:\mathbb{R}\to[0,\infty)$ be a *probability density function* (pdf), meaning that f is integrable on $\mathbb{R}$ and $\int_{-\infty}^{\infty} f(t)\,dt=1$, and define

$$F(x):=\int_{-\infty}^{x} f(t)\,dt,\ x\in\mathbb{R} \quad\text{and}\quad E(f):=\int_{-\infty}^{\infty} xf(x)\,dx, \tag{1.1}$$

to be its *cumulative function* or distribution and the *expectation* provided that the integrals exist and are finite.

The *mean difference*

$$R_G(f):=\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x-y|\,dF(x)\,dF(y) \tag{1.2}$$

was proposed by Gini in 1912 [11], after whom it is usually named, but it was discussed by Helmert and other German writers in the 1870's (cf. H.A. David [8], see also [15, p. 48]). The mean difference has a certain theoretical attraction, being dependent on the spread of the variate values among themselves rather than on the deviations from some central value ([15, p. 48]). Further, its defining integral (1.2) may converge when the *variance* $\sigma^2(f)$,

$$\sigma^2(f):=\int_{-\infty}^{\infty}(x-E(f))^2\,dF(x), \tag{1.3}$$

does not. It can, however, be more difficult to compute than (1.3).

Another useful concept is the *mean deviation* $M_D(f)$, defined by [15, p. 48]

$$M_D(f):=\int_{-\infty}^{\infty}|x-E(f)|\,dF(x)=2\int_{\mu}^{\infty}(x-\mu)\,dF(x). \tag{1.4}$$

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As G.M. Giorgi noted in [12], some of the many reasons for the success and the relevance of the Gini mean difference or *Gini index* $I_G(f)$,

$$I_G(f) = \frac{R_G(f)}{E(f)}, \tag{1.5}$$

are their simplicity, certain interesting properties and useful decomposition possibilities, and these attributes have been analysed in an earlier work by Giorgi [13]. For a bibliographic portrait of the Gini index, see [12] where numerous references are given.

The Gini index given by (1.5) is a measure of relative inequality since it is a ratio of the Gini mean difference, a measure of dispersion, to the average value $\mu = E(f)$. Other measures are the coefficient of variation $V = \frac{\sigma}{\mu}$ and half the relative mean deviation $\frac{M_D(f)}{2\mu}$ where $M_D(f)$ is as defined in (1.4).

From (1.1), $F(x)$ is assumed to increase on its support and its mean $\mu = E(f)$ exist. These assumptions imply that $F^{-1}(p)$ is well defined and is the population's p^{th} quantile. The theoretical Lorenz curve (Gastwirth [10]) corresponding to a given $F(x)$ is defined by

$$L(p) = \frac{1}{\mu}\int_0^p F^{-1}(x)\,dx, \qquad 0 \leqslant p \leqslant 1. \tag{1.6}$$

Now $F^{-1}(x)$ is nondecreasing and so from (1.6) $L(p)$ is convex and $L'(p) = 1$ at $p = F(\mu)$.

The area between the Lorenz curve and the line p, is known as the area of concentration.

The most common measure of inequality is the Gini index defined by (1.5) which may be shown to be equivalent to twice the area of concentration ([10])

$$C = \int_0^1 c(p)\,dp, \quad c(p) = p - L(p). \tag{1.7}$$

$c(p)$ vanishes at $p = 0$ or 1 and is concave since $L(p)$ is convex. Further, there is a *point of maximum discrepancy* p^* between the Lorenz curve and the line of equality which satisfies

$$c(p^*) \geqslant c(p) \qquad \text{for all } p \in [0,1]. \tag{1.8}$$

The point $p^* = F(\mu)$ and $c(p^*) = \frac{M_D(f)}{2\mu}$ where $M_D(f)$ is given by (1.4).

In a sequence of four papers, Cerone and Dragomir ([3] – [6]) developed approximation and bounds from identities involving the Gini mean difference $R_G(f)$. Some of these results involved using the well known Sonin and Korkine identities. Cerone [2] procured some approximations and bounds utilising the Steffensen and Karamata inequalities.

It is the intention of the current article to utilise characteristics of the Lorenz curve, $L(p)$ and its connection to the Gini index via (1.7) to obtain upper and lower bounds for both $L(p)$ and $I_G(f)$. This will be accomplished by utilising the well known Young's integral inequality and some less well known reverse inequalities. These will be discussed in Section 2 and applied in Section 3.

2. Young's Integral Inequality and Reverses

The famous Young's integral inequality states that:

THEOREM 1. *If* $h:[0,A]\to\mathbb{R}$ *is continuous and a strictly increasing function satisfying* $h(0)=0$, *then for every positive* $0<a\leqslant A$ *and* $0<b\leqslant h(A)$

$$Y(h;a,b):=\int_0^a h(t)\,dt+\int_0^b h^{-1}(t)\,dt\geqslant ab \tag{2.1}$$

holds with equality if and only if $b=h(a)$.

In the 1912 paper in fact Young [21] proved (2.1) assuming differentiability of the functions. The inequality (2.1) has a geometric interpretation involving the areas of the two functions and the rectangular area. There has been much work on different proofs and generalisations of (2.1) (see for example, Diaz and Metcalf [9], Bullen [1], Páles [19] and Mitrinović et al. [18]).

We notice that in (2.1), ab is a lower bound for the Young functional $Y(h;a,b)$. In 1974, Merkle [17] showed that there cannot be an upper bound to $Y(h;a,b)$ which is independent of h. He proves the following theorem which provides a reverse inequality.

THEOREM 2. *Suppose the conditions of Theorem* 1 *hold. Then*

$$Y(h;a,b)\leqslant \max\left\{ah(a),bh^{-1}(b)\right\}. \tag{2.2}$$

REMARK 1. The proof of Merkle uses the fact that for $h(a)\geqslant b$, $Y(h;a,b)\leqslant Y(h;a,h(a))=ah(a)$ and for $h(a)\leqslant b$ and interchanging a and b and, h and h^{-1} gives $Y(h;a,b)\leqslant Y\left(h;h^{-1}(b),b\right)=bh^{-1}(b)$.

In 2007, Witkowski [20] gave two simple proofs for Theorem 1. The first utitlises the fact that since h is strictly increasing, then its anti-derivative is strictly convex. The second uses the Mean Value Theorem. The second proof will be replicated here to highlight the fact that this approach does not just provide a proof for Young's inequality (2.1) but it also gives its reverse.

THEOREM 3. *Let the conditions of Theorem* 1 *hold. Then*

$$ab\leqslant Y(h;a,b)\leqslant ah(a)+h^{-1}(b)(b-h(a)) \tag{2.3}$$

with equality if and only if $b=h(a)$.

Proof. Since h is strictly increasing, we have by the Mean Value Theorem that for $a<h^{-1}(b)$ $(h(a)<b)$

$$h(a)<\frac{\int_0^{h^{-1}(b)}h(t)\,dt-\int_0^a h(t)\,dt}{h^{-1}(b)-a}<h\left(h^{-1}(b)\right)=b. \tag{2.4}$$

That is, on noting that

$$\int_0^{h^{-1}(b)} h(t)\,dt = bh^{-1}(b) - \int_0^b h^{-1}(t)\,dt \tag{2.5}$$

then since $h^{-1}(b) - a > 0,$ from (2.4) we obtain

$$h(a)\left(h^{-1}(b) - a\right) < bh^{-1}(b) - Y(h;a,b) < b\left(h^{-1}(b) - a\right),$$

which upon simplification gives (2.3) for $a < h^{-1}(b)$. A similar argument follows for $a > h^{-1}(b)$ $(h(a) > b)$. □

REMARK 2. We note that the upper bound in (2.3) provides a reverse of Young's integral inequality (2.1). Equation (2.3) can be written in the appealing form

$$ab \leqslant Y(h;a,b) \leqslant ab + (b - h(a))\left(h^{-1}(b) - a\right) \tag{2.6}$$

or

$$0 \leqslant Y(h;a,b) - ab \leqslant (b - h(a))\left(h^{-1}(b) - a\right) \tag{2.7}$$

We notice that $(b - h(a))\left(h^{-1}(b) - a\right) \geqslant 0$ with equality holding only for $b = h(a)$ (equivalently, $a = h^{-1}(b)$).

THEOREM 4. *Let the conditions of Theorem 1 persist. Then the inequality*

$$\alpha(a,b)\int_0^a h(t)\,dt + \beta(a,b)\int_0^b h^{-1}(t)\,dt \leqslant ab \tag{2.8}$$

holds, where

$$\alpha(a,b) = \min\left\{1, \frac{b}{h(a)}\right\} \quad and \quad \beta(a,b) = \min\left\{1, \frac{a}{h^{-1}(b)}\right\} \tag{2.9}$$

with equality holding if and only if $b = h(a)$.

REMARK 3. Witkowski uses strict convexity arguments of $H(x) = \int_0^x h(t)\,dt$ for $a < h^{-1}(b)$ to produce

$$\int_0^a h(t)\,dt + \frac{a}{h^{-1}(b)}\int_0^b h^{-1}(t)\,dt < ab$$

and of $G(x) = \int_0^x h^{-1}(t)\,dt$ for $a > h^{-1}(b)$ to get

$$\frac{b}{h(a)}\int_0^a h(t)\,dt + \int_0^b h^{-1}(t)\,dt < ab.$$

Witkowski sees (2.8) as a reverse of (2.1). In a sense the upper bound for (2.8) is $Y(h;a,b)$. He did not highlight the fact that the upper bound in (2.3) is a reverse of (2.1).

LEMMA 1. *The upper bound obtained by Witkowski*

$$U_W = ah(a) + h^{-1}(b)(b - h(a)) \tag{2.10}$$

given in (2.3) *is always better than that of Merkle*

$$U_M = \begin{cases} ah(a), & b < h(a); \\ bh^{-1}(b), & b > h(a) \end{cases} \tag{2.11}$$

which is equivalent to the upper bound in (2.2)*. That is, the reverse Young's inequality of Theorem* 4 *is always tighter than that of Theorem* 2*.*

Proof. Obvious and is thus omitted. □

It is instructive to compare the upper bounds for $\int_0^b h^{-1}(t)\,dt$ provided from the results of Witkowski from (2.3) and (2.8)–(2.9). These will be utilised to examine the bounds for $L(p)$ via (1.6) in the next section.

LEMMA 2. *From Theorems* 2 *and* 4*, the following upper bounds are tighter. Namely,*

$$\int_0^b h^{-1}(t)\,dt < \begin{cases} \frac{b}{h(a)}\left[ah(a) - \int_0^a h(t)\,dt\right] & \text{for } \Delta > 0,\ b < h(a); \\ ab + \left(h^{-1}(b) - a\right)(b - h(a)) - \int_0^a h(t)\,dt & \text{for } \Delta < 0 \text{ or } b > h(a), \end{cases} \tag{2.12}$$

where $\Delta := ah(a) - \int_0^a h(t)\,dt - h(a)h^{-1}(b)$.

Proof. From (2.3) and (2.6), we have

$$\begin{aligned} \int_0^b h^{-1}(t)\,dt &< ah(a) + h^{-1}(b)(b - h(a)) - \int_0^a h(t)\,dt \qquad (2.13) \\ &= ab + \left(h^{-1}(b) - a\right)(b - h(a)) - \int_0^a h(t)\,dt =: U_W. \end{aligned}$$

From (2.9) we have

$$\alpha(a,b) = \begin{cases} \frac{b}{h(a)}, & b < h(a); \\ 1, & b > h(a) \end{cases} \quad \text{and} \quad \beta(a,b) = \begin{cases} 1, & b < h(a); \\ \frac{a}{h^{-1}(b)}, & b > h(a) \end{cases}$$

so that from (2.8)

$$\int_0^b h^{-1}(t)\,dt < \begin{cases} ab - \frac{b}{h(a)} \int_0^a h(t)\,dt =: U_{W_1} & b < h(a); \\ bh^{-1}(b) - \frac{h^{-1}(b)}{a} \int_0^a h(t)\,dt =: U_{W_2} & b > h(a). \end{cases} \tag{2.14}$$

For $b > h(a)$ consider from (2.13) and (2.14)

$$\begin{aligned} U_W - U_{W_2} &= ah(a) + h^{-1}(b)(b - h(a)) - \int_0^a h(t)\,dt - bh^{-1}(b) + \frac{h^{-1}(b)}{a} \int_0^a h(t)\,dt \\ &= \frac{1}{a}\left(h^{-1}(b) - a\right)\left[\int_0^a h(t)\,dt - ah(a)\right] < 0. \end{aligned}$$

This indicates that U_W is a tighter upper bound than U_{W_2} for $b > h(a)$.

Now, for $b < h(a)$, we consider from (2.13) and (2.14)

$$\begin{aligned} U_W - U_{W_1} &= \left(h^{-1}(b) - a\right)(b - h(a)) - \left(1 - \frac{b}{h(a)}\right)\int_0^a h(t)\,dt \\ &= (h(a) - b)\left[a - h^{-1}(b) - \frac{1}{h(a)}\int_0^a h(t)\,dt\right] \end{aligned}$$

so that for $b < h(a)$

$$U_W - U_{W_1} \quad \text{is:} \quad \begin{cases} > 0 \text{ for } \Delta > 0; \\ < 0 \text{ for } \Delta < 0, \end{cases}$$

where Δ is as given by (2.12). That is, for $b < h(a)$ and $\Delta > 0$, then U_{W_1} is tighter than U_W and vice versa for $\Delta < 0$. □

3. Bounds for the Lorenz Curve and Gini Index

Some identities for the Gini Mean Difference, $R_G(f)$ through which results for the Gini index $I_G(f)$ may be procured via the relationship (1.5) will be stated here. These have been used in [3] – [7] to obtain approximations and bounds. The reader is referred to the book [15], Exercise 2.9, p. 94 or [3].

The following result holds (see for instance [15, p. 54] or [3]).

THEOREM 5. *With the above notation, the identities*

$$R_G(f) = \int_{-\infty}^{\infty}(1 - F(y))F(y)\,dy = 2\int_{-\infty}^{\infty} xf(x)F(x)\,dx - E(f) \tag{3.1}$$

hold.

The following result was obtained in [4] using the well known Sonin identity (see [18, p. 246]) for the case of univariate real functions.

THEOREM 6. *With the above assumptions for f and F, we have the identity:*

$$\begin{aligned} R_G(f) &= 2\int_{-\infty}^{\infty}(x - E(f))(F(x) - \gamma)f(x)\,dx \\ &= 2\int_{-\infty}^{\infty}(x - \delta)\left(F(x) - \frac{1}{2}\right)f(x)\,dx \end{aligned} \tag{3.2}$$

for any $\gamma, \delta \in \mathbb{R}$.

The following result was developed in [5] using the Korkine identity (see [18, p. 242]) for the case of univariate real functions.

THEOREM 7. *With the above assumptions for f and F, we have the following representation for the Gini mean difference:*

$$R_G(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-y)(F(x)-F(y))f(x)f(y)\,dxdy. \tag{3.3}$$

The following lemma will be proven here since it will be crucial for the current work in bounding the Gini index via the Lorenz curve and the area of concentration C. The identity is also proven in [15, p. 49] in a different way.

LEMMA 3. *The following identity holds*

$$R_G(f) = \mu I_G(f) = 2\mu C, \tag{3.4}$$

where the quantities are defined by (1.2), (1.5), (1.6)–(1.7).

Proof. From (1.6) and (1.7) we have

$$2\mu C = 2\int_0^1\left[p\mu - \int_0^p F^{-1}(x)\,dx\right]dp = 2\int_0^1\int_0^p\left[E(f) - F^{-1}(x)\right]dxdp.$$

An interchange of the order of integration and a substitution $x = F(t)$ produces

$$2\mu C = 2\int_{-\infty}^{\infty}(t - E(f))F(t)\,dF. \tag{3.5}$$

Now (3.5) is equivalent to identity (3.2) with $\gamma = 0$ and so $2\mu C = R_G(f)$ and hence the identity (3.4) is proved. □

We are now in a position to investigate bounds for both the Lorenz curve and through the relationship (3.4) for the Gini index using the results of Section 2 based on Young type inequalities. Firstly, however, we state a result of Gastwirth [10] for bounding the Lorenz curve.

THEOREM 8. *Let $F(x)$ be a distribution function with mean μ and support (a,b). Then its Lorenz curve, $L(p)$ satisfies*

$$B(p) \leqslant L(p) \leqslant p \tag{3.6}$$

where

$$B(p) = \begin{cases} \frac{ap}{\mu}, & p < r; \\ \frac{ar}{\mu} + \frac{b}{\mu}(p-r), & p > r \end{cases} \tag{3.7}$$

and r is determined by the relation $ra + (1-r)b = \mu$. Here the random variable X generating the Lorenz curve $B(p)$ takes on the value a with probability r and b with probability $(1-r)$.

The following technical lemma will prove useful subsequently.

LEMMA 4. *Let $F(\cdot)$ be a distribution function defined on $(0,A]$ and its inverse $F^{-1}(\cdot)$ exists, then for $a \in (0,A]$*

$$\int_0^1 (p-F(a))\left(F^{-1}(p)-a\right)dp = \frac{1}{2}\left[A-a-\int_0^A F^2(t)\,dt\right] + (a-\mu)F(a). \tag{3.8}$$

Proof. Firstly, we note that for $h(0)=0$,

$$\int_0^A h(t)\,dt = Ah(A) - \int_0^A th'(t)\,dt = Ah(A) - \int_0^{h(A)} h^{-1}(t)\,dt. \tag{3.9}$$

If we associate $h(\cdot)$ with $F(\cdot)$, noting that $F(A)=1$, then from (3.9):

$$\mu L(1) = \int_0^1 F^{-1}(p)\,dp = A - \int_0^A F(t)\,dt = \mu \tag{3.10}$$

since $L(1)=1$.

Further, a substitution of $p=F(t)$ and integration by parts gives

$$\int_0^1 pF^{-1}(p)\,dp = \int_0^A tF(t)F'(t)\,dt = \frac{A}{2} - \frac{1}{2}\int_0^A F^2(t)\,dt. \tag{3.11}$$

Now,

$$\begin{aligned}\int_0^1 (p-F(a))\left(F^{-1}(p)-a\right)dp \\ = \int_0^1 pF^{-1}(p)\,dp + aF(a) - F(a)\int_0^1 F^{-1}(p)\,dp - \frac{a}{2}.\end{aligned} \tag{3.12}$$

Substitution of (3.10) and (3.11) into (3.12) gives the stated result (3.8). □

The following theorem uses the results of Witkowski [20] as given by (2.3) to procure bounds for the Lorenz curve.

THEOREM 9. *Let $L(p)$ be the Lorenz curve defined by (1.6) corresponding to a given distribution (cumulative) function $F(a)$ with $F(0)=0,\ 0<a\leqslant A$ and $0<p\leqslant F(A)=1$. Then*

$$\frac{1}{\mu}\left[ap-\int_0^a F(t)\,dt\right] \leqslant L(p) \leqslant \frac{1}{\mu}\left[ap-\int_0^a F(t)\,dt\right] + \frac{1}{\mu}(p-F(a))\left(F^{-1}(p)-a\right) \tag{3.13}$$

with equality if and only if $p=F(a)$.

Proof. From Theorem 3, if we associate $h(t)$ with $F(t)$, then the conditions of the theorem are satisfied and so from (2.4),

$$ap - \int_0^a F(t)\,dt \leqslant \int_0^p F^{-1}(p)\,dp \leqslant ap - \int_0^a F(t)\,dt + (p-F(a))\left(F^{-1}(p)-a\right)$$

and the result follows from (1.6). Equality is obvious from (2.5). □

REMARK 4. The lower bound is only useful for $p > \frac{1}{a}\int_0^a F(t)\,dt$ since zero is a lower bound for $L(p)$. The upper bound is useful if it is less than p.

COROLLARY 1. *Let the condition of Theorem 9 hold. Then*

$$l(p) \leqslant L(p) \leqslant u(p), \tag{3.14}$$

where

$$l(p) = \begin{cases} 0, & p < 1 - \frac{\mu}{A}; \\ \frac{A}{\mu}\left[p - \left(1 - \frac{\mu}{A}\right)\right], & p > 1 - \frac{\mu}{A} \end{cases} \tag{3.15}$$

and

$$u(p) = \begin{cases} p, & 0 < p < p^*; \\ 1 + \frac{F^{-1}(p)}{\mu}(p-1), & p^* < p < 1, \end{cases} \tag{3.16}$$

where $p^* = F(\mu)$ *is the point of maximum discrepancy satisfying* (1.8).

Proof. Taking $a = A$ in (3.13) and noting that $F(A) = 1$ and $\int_0^A F(t)\,dt = A - \mu$ produces upon simplification

$$1 + \frac{A}{\mu}(p-1) \leqslant L(p) \leqslant 1 + \frac{F^{-1}(p)}{\mu}(p-1). \tag{3.17}$$

Now, it is well known that $0 \leqslant L(p) \leqslant p$ for $0 \leqslant p \leqslant 1$.

Consider

$$p - \left[1 + \frac{F^{-1}(p)}{\mu}(p-1)\right] = (1-p)\left[\frac{F^{-1}(p)}{\mu} - 1\right] > 0$$

for $p^* = F(\mu) < p < 1$ and negative otherwise. This gives the upper bound in (3.16).

Now the lower bound in (3.17) is useful for $1 + \frac{A}{\mu}(p-1) > 0$, that is, for $p > \left(1 - \frac{\mu}{A}\right)$. The lower bound given by (3.15) is thus procured. □

REMARK 5. It may be noticed that by taking $a = 0$ and $b = A$ in Theorem 8 we have $r = 1 - \frac{\mu}{A}$ and so Corollary 1 recaptures the lower bound obtained by Gastwirth [10]. The upper bound obtained in (3.16) gives a refinement of that given in (3.6).

THEOREM 10. *Let the conditions of Theorem 9 hold. Then the Gini index defined by* (1.5) *or equivalently,* (1.7) *satisfies*

$$\left(1 - \frac{a}{\mu}\right) + \frac{2}{\mu}\int_0^a F(t)\,dt + 2\left(1 - \frac{a}{\mu}\right)F(a) - \frac{1}{\mu}\left[A - a - \int_0^A F^2(t)\,dt\right] \\ \leqslant I_G(f) \leqslant \left(1 - \frac{a}{\mu}\right) + \frac{2}{\mu}\int_0^a F(t)\,dt. \tag{3.18}$$

Proof. From (3.13) we have, since, as shown in Lemma 3, the Gini index, $I_G(f)$ of (1.5) is equivalent to twice the area of concentration, namely, $2C$. Now, (3.13) gives

$$\left(1-\frac{a}{\mu}\right)p+\frac{1}{\mu}\int_0^a F(t)\,dt-\frac{1}{\mu}(p-F(a))\left(F^{-1}(p)-a\right) \leqslant p-L(p)\leqslant\left(1-\frac{a}{\mu}\right)p+\frac{1}{\mu}\int_0^a F(t)\,dt$$

so that from (3.4) and (1.7)

$$\left(1-\frac{a}{\mu}\right)+\frac{2}{\mu}\int_0^a F(t)\,dt-\frac{2}{\mu}\int_0^1 (p-F(a))\left(F^{-1}(p)-a\right)dp \leqslant I_G(f)\leqslant\left(1-\frac{a}{\mu}\right)+\frac{2}{\mu}\int_0^a F(t)\,dt.$$

Using (3.8) from Lemma 4 produces the inequality as stated in (3.18). □

COROLLARY 2. *Let the conditions of Theorems* 9 *and* 10 *hold. Then the Gini index bounds from* (3.18) *are the tightest bounds on* $(0,A]$ *at* $a=\mu$ *and* $a=m$ *for the lower and upper bounds, respectively. These are given by:*

$$\frac{1}{\mu}\int_0^\mu F(t)\,dt<I_G(f)<\left(1-\frac{m}{\mu}\right)+\frac{2}{\mu}\int_0^m F(t)\,dt, \tag{3.19}$$

where $m=F^{-1}\left(\frac{1}{2}\right)$ *is the median and* μ *is the mean.*

Proof. Since $F(t)$ is defined for $t\in[0,A]$ and $F(0)=0$, we have from (3.1) that

$$I_G(f)=\frac{1}{\mu}\int_0^A F(t)(1-F(t))\,dt. \tag{3.20}$$

We notice that the lower bound in (3.18) approaches $I_G(f)$ as $a\to 0^+$ and the upper bound tends to 1. Further, if we denote the lower bound in (3.18) by $\kappa(a)$, then

$$\kappa'(a)=2\left(1-\frac{a}{\mu}\right)f(a)\begin{cases}>0,\ 0<a<\mu\\ <0,\ \mu<a<A.\end{cases}$$

The maximum occurs at $a=\mu$ so that

$$\begin{aligned}\sup_{a\in(0,A)}\kappa(a)=\kappa(\mu)&=\frac{2}{\mu}\int_0^\mu F(t)\,dt-\frac{1}{\mu}\left[A-\mu-\int_0^a F^2(t)\,dt\right] \\ &=\frac{2}{\mu}\int_0^\mu F(t)\,dt-\frac{1}{\mu}\int_0^A F(t)(1-F(t))\,dt\end{aligned} \tag{3.21}$$

since from (3.10) $A-\mu=\int_0^A F(t)\,dt$. We now have from (3.21) and using (3.20) that

$$\sup_{a\in(0,A)} \kappa(a)=\kappa(\mu)=\frac{2}{\mu}\int_0^{\mu} F(t)\,dt-I_G(f), \tag{3.22}$$

as the best choice for the lower bounds in (3.18) from which the lower bound in (3.19) results.

Further, the minimum upper bound in (3.18) occurs when $2F(a)-1=0$, namely, at $a=m=F^{-1}\left(\frac{1}{2}\right)$ producing the upper bound in (3.19) from (3.18). □

REMARK 6. In Cerone [2, Theorem 13] the Steffensen inequality was utilised together with the property that $F(x)$ is nondecreasing to obtain

$$\frac{1}{\mu}\int_a^{a+\lambda} F(x)\,dx \leqslant I_G(f) \leqslant \frac{1}{\mu}\int_{b-\lambda}^{b} F(x)\,dx,$$

where $\lambda=\mu-a$ and f is supported on $[a,b]$. That is, taking $a=0$ and $b=A$, we have

$$\frac{1}{\mu}\int_0^{\mu} F(x)\,dx \leqslant I_G(f) \leqslant \frac{1}{\mu}\int_{A-\mu}^{A} F(x)\,dx. \tag{3.23}$$

We notice that the lower bound here is recaptured by (3.19), however, the upper bounds differ.

COROLLARY 3. *Let the conditions of Theorem* 9 *hold. The Gini index,* $I_G(f)$ *satisfies*

$$\frac{1}{2\mu}\int_0^{\mu} F(x)(2-F(x))\,dx \leqslant I_G(f) \leqslant 1-\frac{\mu}{A}. \tag{3.24}$$

Proof. From (3.14) we have $p-u(p)\leqslant p-L(p)\leqslant p-\ell(p)$ so that from (3.15) and (3.16)

$$\hat{\ell}(p)\leqslant p-L(p)\leqslant \hat{u}(p), \tag{3.25}$$

where with $p^*=F(\mu)$

$$\hat{\ell}(p)=\begin{cases}0, & 0<p<p^*\\ (p-1)\left[1-\frac{F^{-1}(p)}{\mu}\right], & p^*<p<1\end{cases} \tag{3.26}$$

and

$$\hat{u}(p)=\begin{cases}p, & 0<p<1-\frac{\mu}{A};\\ \left(\frac{A}{\mu}-1\right)(1-p), & 1-\frac{\mu}{A}<p<1.\end{cases} \tag{3.27}$$

Now, from (3.25) we have

$$\int_0^1 \hat{\ell}(p)\,dp \leqslant \int_0^1 (p-L(p))\,dp \leqslant \int_0^1 \hat{u}(p)\,dp$$

and so from Lemma 3 and equation (1.7) we have

$$2\int_0^1 \hat{\ell}(p)\,dp \leqslant I_G(f) \leqslant 2\int_0^1 \hat{u}(p)\,dp. \tag{3.28}$$

That is, from (3.26)

$$\int_0^1 \hat{\ell}(p)\,dp = \int_{p^*}^1 (p-1)\left[1-\frac{F^{-1}(p)}{\mu}\right]dp = \int_\mu^A \left(1-\frac{x}{\mu}\right)(F(x)-1)\,f(x)\,dx,$$

where the substitution $p = F(x)$ has been made and noting that $p^* = F(\mu)$. Also, integration by parts gives $\int_0^1 \hat{\ell}(p)\,dp = \frac{1}{2\mu}\int_\mu^A (F(x)-1)^2\,dx$. Now,

$$\begin{aligned}\int_\mu^A (F(x)-1)^2\,dx &= \int_\mu^A F(x)(F(x)-1)\,dx - \int_\mu^A F(x)\,dx + A - \mu\\ &= \int_0^A F(x)(F(x)-1)\,dx - \int_0^\mu F(x)(F(x)-1)\,dx + \int_0^\mu F(x)\,dx\\ &= -\mu I_G(f) - \int_0^\mu F^2(x)\,dx + 2\int_0^\mu F(x)\,dx\end{aligned}$$

and so

$$\int_0^1 \hat{\ell}(p)\,dp = -\frac{1}{2}I_G(f) - \frac{1}{2\mu}\int_0^\mu F^2(x)\,dx + \frac{1}{\mu}\int_0^\mu F(x)\,dx. \tag{3.29}$$

Further, from (3.27)

$$\begin{aligned}\int_0^1 \hat{u}(p)\,dp &= \int_0^{1-\frac{\mu}{A}} p\,dp + \left(\frac{A}{\mu}-1\right)\int_{1-\frac{\mu}{A}}^1 (1-p)\,dp \qquad (3.30)\\ &= \frac{1}{2}\left(1-\frac{\mu}{A}\right)^2 + \left(\frac{A}{\mu}-1\right)\cdot\frac{1}{2}\left(\frac{\mu}{A}\right)^2 = \frac{1}{2}\left(1-\frac{\mu}{A}\right).\end{aligned}$$

Substitution of (3.29) and (3.30) into (3.28) produces (3.24). □

REMARK 7. The upper bound given in (3.24) was also obtained in Gastwirth [10] using a result from Hardy et al. [14]. The lower bound obtained in [10] was zero which is smaller than that given in (3.24).

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