Journal of
Mathematical
Inequalities
Volume 3, Number 3 (2009), 383–393

AN INEQUALITY FOR A LINEAR DISCRETE OPERATOR INVOLVING CONVEX FUNCTIONS

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Dedicated to Professor Josip Pečarić on the occasion of his 60th birthday

Abstract. For the functional $A[f]=\sum_{k=1}^{m} a_k f(z_k)$, we give necessary and sufficient conditions over the real numbers z_k, such that, the inequality $A[f]\geqslant 0$, holds for some classes of convex functions. Then, we deduce an inequality related to Alzer's inequality and a weighted majorization inequality.

1. Introduction

In [3], J.-Ch. Kuang proved the following inequality

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) > \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) > \int_0^1 f(x)dx,$$

where f is a strictly increasing convex (or concave) function in $(0,1]$.

In [6], it was proved that for $(a_n)_{n\in\mathbb{N}}$, a positive increasing sequence of real numbers such that $\left(n\left(1-\frac{a_n}{a_{n+1}}\right)\right)_{n\in\mathbb{N}}$ is increasing ($\left(n\left(\frac{a_{n+1}}{a_n}-1\right)\right)_{n\in\mathbb{N}}$ is increasing), we have

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{a_k}{a_n}\right) \geqslant \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{a_k}{a_{n+1}}\right)$$

for every increasing convex (concave) function $f:[0,1]\to\mathbb{R}$.

In [2], using positive linear operators of Bernstein-Stancu type, I. Gavrea obtained the inequality

$$\frac{1}{n}\sum_{k=1}^{n} f\left(x_{k-1,n-1}\right) - \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(x_{k-1,n}\right) \geqslant 0, \tag{1}$$

for an increasing convex function f and for the nodes $x_{i,n}$, $i=0,1,\ldots,n$ from $[0,1]$, which satisfy the properties

$$0\leqslant x_{0,n}\leqslant x_{1,n}<\ldots<x_{n,n}\leqslant 1$$

Mathematics subject classification (2000): Primary 05C38, 15A15; Secondary 05A15, 15A18.
Keywords and phrases: linear discrete operator, convex functions, inequality, majorization.

$$x_{k-1,n} \leqslant x_{k-1,n-1} \leqslant x_{k,n}$$

$$x_{0,n-1} \geqslant x_{0,n} \text{ and } x_{n-1,n-1} \geqslant x_{n,n}$$

$$(n-k)(x_{k,n-1} - x_{k,n}) \geqslant k(x_{k,n} - x_{k-1,n-1}),$$

for $n \geqslant 1$ and every $k = 1,2,\ldots,n$.

For related inequalities see [1], [2], [4], [5]. In [1], the authors presented, in a chronological order, these inequalities and some recent results related to them.

In this paper, we want to prove an inequality for a discrete linear operator and obtain necessary and sufficient conditions over the points $x_{i,n}$ in order to obtain inequality (1). We deduce, also, a weighted majorization inequality.

2. Main result

Let $m \geqslant 3$ be an integer and let $I_m = \{1,2,\ldots,m\}$. Consider $(z_k)_{k\in I_m}$ a strictly decreasing sequence of real numbers from $[0,1]$. Let A be the linear functional defined by

$$A[f] = \sum_{k=1}^{m} a_k f(z_k), \tag{2}$$

where a_k are real numbers and f is a real function defined on $[0,1]$. We want to find necessary and sufficient conditions over z_k, such that

$$A[f] \geqslant 0, \tag{3}$$

holds for some classes of convex functions. Let e_i be the functions defined by $e_i(t) = t^i$. Then we have the following result

THEOREM 1. *Consider the following conditions*

$$A[e_0] = 0 \tag{4}$$

$$A[e_1] \geqslant 0 \tag{5}$$

$$A[e_1] \leqslant 0 \tag{6}$$

$$\sum_{i=1}^{k} a_i(z_i - z_{k+1}) \geqslant 0, \textit{ for every } k \in I_{m-2}. \tag{7}$$

$$\sum_{i=k}^{m} a_i(z_i - z_{k-1}) \geqslant 0, \textit{ for every } k \in I_m \setminus \{1,2\}. \tag{8}$$

Then

a) (3) holds for every increasing convex function f iff (4), (5) and (7) hold;
b) (3) holds for every decreasing convex function f iff (4), (6) and (7) hold;
c) (3) holds for every convex function f iff (4), (5), (6) and (7) hold;
d) (3) holds for every increasing concave function f iff (4), (5) and (8) hold;
e) (3) holds for every decreasing concave function f iff (4), (6) and (8) hold;
f) (3) holds for every concave function f iff (4), (5), (6) and (8) hold.

Proof. $a)$ Suppose that (4), (5) and (7) hold.
We set $S_0 = 0$ and $S_k = a_1 + a_2 + \ldots + a_k$, for $k \in I_m$. We define also $T_0 = 0$ and for $k \in I_{m-1}$

$$T_k = \sum_{i=1}^{k} S_i(z_i - z_{i+1}).$$

Because $S_m = A[e_0]$ and $A[e_0] = 0$, we obtain $S_m = 0$. Using Abel summation formula and the notation $[x,y;f] = [f(x) - f(y)]/(x-y)$, for the divided difference, we have

$$\begin{aligned} A[f] &= \sum_{k=1}^{m} (S_k - S_{k-1}) f(z_k) \\ &= \sum_{k=1}^{m} [S_k f(z_k) - S_{k-1} f(z_{k-1})] + \sum_{k=1}^{m} S_{k-1}[f(z_{k-1}) - f(z_k)] \\ &= S_m f(z_m) + \sum_{k=1}^{m-1} S_k [f(z_k) - f(z_{k+1})] = \sum_{k=1}^{m-1} S_k (z_k - z_{k+1})[z_k, z_{k+1}; f]. \end{aligned}$$

In the same way, we deduce

$$\begin{aligned} A[f] &= \sum_{k=1}^{m-1} (T_k - T_{k-1})[z_k, z_{k+1}; f] = \sum_{k=1}^{m-1} (T_k[z_k, z_{k+1}; f] - T_{k-1}[z_{k-1}, z_k; f]) \\ &\quad + \sum_{k=1}^{m-1} T_{k-1}([z_{k-1}, z_k; f] - [z_k, z_{k+1}; f]) \\ &= T_{m-1}[z_{m-1}, z_m; f] + \sum_{k=1}^{m-2} T_k (z_k - z_{k+2})[z_k, z_{k+1}, z_{k+2}; f]. \end{aligned}$$

Computing T_k we obtain

$$\begin{aligned} T_k &= \sum_{i=1}^{k} S_i(z_i - z_{i+1}) = \sum_{i=1}^{k} (S_i z_i - S_{i+1} z_{i+1}) + \sum_{i=1}^{k} (S_{i+1} - S_i) z_{i+1} \\ &= S_1 z_1 - S_{k+1} z_{k+1} + \sum_{i=1}^{k} a_{i+1} z_{i+1} = a_1 z_1 - z_{k+1} \sum_{i=1}^{k+1} a_i + \sum_{i=2}^{k+1} a_i z_i \\ &= \sum_{i=1}^{k+1} a_i z_i - z_{k+1} \sum_{i=1}^{k+1} a_i = \sum_{i=1}^{k} a_i z_i - z_{k+1} \sum_{i=1}^{k} a_i = \sum_{i=1}^{k} a_i (z_i - z_{k+1}). \end{aligned}$$

So, the conditions (7) are equivalent with the inequalities $T_k \geqslant 0$, for all $k \in I_{m-2}$. Computing $T_{m-1} = A[e_1] - z_m A[e_0] = A[e_1] \geqslant 0$, we deduce that

$$A[f] = A[e_1][z_{m-1}, z_m; f] + \sum_{k=1}^{m-2} T_k (z_k - z_{k+2})[z_k, z_{k+1}, z_{k+2}; f]. \tag{9}$$

Because the divided differences which appear in the last equality are all non-negative for an increasing convex function f, we deduce $A[f] \geqslant 0$.

For the necessity part, we apply the inequality $A[f] \geqslant 0$ for the increasing convex functions e_0 and $-e_0$ and we deduce $A[e_0] = 0$. The inequality $A[e_1] \geqslant 0$ is true since e_1 is increasing and convex. For the functions $f_k(x) = \max(0, x - z_{k+1})$, which are increasing and convex for every $k \in I_{m-2}$, we obtain

$$T_k = \sum_{i=1}^{k} a_i(z_i - z_{k+1}) = A[f_k] \geqslant 0.$$

$b)$ Let f be a decreasing and convex function. Then

$$[z_{m-1}, z_m; f] \leqslant 0 \text{ and } [z_k, z_{k+1}, z_{k+2}; f] \geqslant 0, \text{ for } k \in I_{m-2}.$$

Because $A[e_1] \leqslant 0$ and $T_k \geqslant 0$, we deduce by the equality (9) that (3) holds.
For the part of necessity take e_0 and $-e_0$ and obtain $A[e_0] = 0$. If we take $-e_1$ in (3) we obtain $A[-e_1] \geqslant 0$ which is equivalent with $A[e_1] \leqslant 0$. The inequalities (7) can be obtained from the relations $T_k = A[g_k] \geqslant 0$, for the decreasing convex functions $g_k(x) = \max(0, z_{k+1} - x)$.

$c)$ The sufficiency can be proved using the equality (9) with $A[e_1] = 0$. The necessity of inequalities (7) and the equality (4) can be obtained using the same argument as in $a)$ or in $b)$. The equality $A[e_1] = 0$ can be obtained from the inequality $A[f] \geqslant 0$, for the convex functions e_1 and $-e_1$.

$d)$, $e)$, $f)$ can be obtained from $b)$, $a)$, and $c)$ replacing f by $-f$ and a_k by $-a_k$ and using the following relation for every $k \in I_m \setminus \{1, 2\}$

$$\begin{aligned}
\sum_{i=k}^{m} a_i(z_i - z_{k-1}) &= \sum_{i=k-1}^{m} a_i z_i - z_{k-1} \sum_{i=k-1}^{m} a_i \\
&= A[e_1] - \sum_{i=1}^{k-2} a_i z_i - z_{k-1}\left(A[e_0] - \sum_{i=1}^{k-2} a_i\right) \\
&= A[e_1] - z_{k-1}A[e_0] - \sum_{i=1}^{k-2} a_i(z_i - z_{k-1}). \quad \square
\end{aligned}$$

REMARK 2. If we look at the equality (9) in the proof of the theorem, we must have $z_k - z_{k+2} \geqslant 0$, in order to prove the sufficiency of the conditions in the theorem. This condition is guaranteed by $z_k > z_{k+1}$, which was given as a property of $(z_k)_{k \in I_m}$. This weaker condition ($z_k - z_{k+2} \geqslant 0$) is equivalent with the property that $(z_{2k})_k$ and $(z_{2k+1})_k$ are decreasing.

REMARK 3. In [4], it is presented the case c) from the Theorem 1. The author, also, generalizes the result to the class of convex functions of order n.

3. Applications

COROLLARY 4. *Let $n \geqslant 1$ be an integer and let x_i, $i \in I_n$ and y_j, $j \in I_{n+1}$ be two increasing sequences of points from $[0,1]$. Let A be the linear functional defined by*

$$A[f] = \frac{1}{n}\sum_{k=1}^{n} f(x_k) - \frac{1}{n+1}\sum_{k=1}^{n+1} f(y_k). \tag{10}$$

If

$$\begin{aligned} x_1 &\geqslant y_1, \\ x_n &\geqslant y_{n+1}, \\ (n-i)(x_{i+1}-y_{i+1}) &\geqslant i(y_{i+1}-x_i), \text{ for } i \in I_{n-1}, \\ (n+1-i)(x_i-y_i) &\geqslant i(y_{i+1}-x_i), \text{ for } i \in I_n, \end{aligned}$$

then $A[f] \geqslant 0$, for every increasing convex or concave function $f:[0,1] \to \mathbb{R}$.

Proof. We apply Theorem 1 and Remark 2 for $m = 2n+1$, $z_{2k} = x_{n+1-k}$ and $a_{2k} = 1/n$, with $k \in I_n$, $z_{2k-1} = y_{n+2-k}$ and $a_{2k-1} = -1/(n+1)$, with $k \in I_{n+1}$. Because (x_i) and (y_j) are increasing sequences, we obtain $z_k - z_{k+2} \geqslant 0$. Computing T_k we have $T_1 = (x_n - y_{n+1})/(n+1) \geqslant 0$, and for $k \neq 1$, $k \in I_n$

$$\begin{aligned} T_{2k-1} &= \sum_{i=1}^{k} S_{2i-1}(z_{2i-1}-z_{2i}) + \sum_{i=1}^{k-1} S_{2i}(z_{2i}-z_{2i+1}) \\ &= C\sum_{i=1}^{k} -(n+1-i)(y_{n+2-i}-x_{n+1-i}) + C\sum_{i=1}^{k-1} i(x_{n+1-i}-y_{n+1-i}) \\ &= C\sum_{i=n+1-k}^{n-1} [(n-i)(x_{i+1}-y_{i+1}) - i(y_{i+1}-x_i)] + Cn(x_n-y_{n+1}) \geqslant 0, \end{aligned}$$

where $C = [n(n+1)]^{-1}$. For $k \in I_n$

$$\begin{aligned} T_{2k} &= \sum_{i=1}^{k} S_{2i-1}(z_{2i-1}-z_{2i}) + \sum_{i=1}^{k} S_{2i}(z_{2i}-z_{2i+1}) \\ &= C\sum_{i=1}^{k} -(n+1-i)(y_{n+2-i}-x_{n+1-i}) + C\sum_{i=1}^{k} i(x_{n+1-i}-y_{n+1-i}) \\ &= C\sum_{i=n+1-k}^{n} [(n+1-i)(x_i-y_i) - i(y_{i+1}-x_i)] \geqslant 0. \end{aligned}$$

Because $T_k \geqslant 0$, we deduce $A[f] \geqslant 0$, for every increasing convex function f. Denoting by Q_k the sum $\sum_{i=k}^{m} P_i(z_i - z_{i-1})$, for every $k \in I_m$, $k \geqslant 2$, where P_k is the sum

$\sum_{i=k}^{m} a_i$, we have

$$\begin{aligned}Q_k &= \sum_{i=k}^{m}(P_i z_i - P_{i-1} z_{i-1}) + \sum_{i=k}^{m}(P_{i-1} - P_i) z_{i-1} \\ &= P_m z_m - P_{k-1} z_{k-1} + \sum_{i=k}^{m} a_{i-1} z_{i-1} = A[e_0] z_m + \sum_{i=k-1} a_i z_i - z_k \sum_{i=k-1} a_i \\ &= \sum_{i=k}^{m} a_i (z_i - z_{k-1})\end{aligned}$$

In order to use Theorem 1 we have to prove that $Q_k \geqslant 0$. For every $k \in I_n$

$$\begin{aligned}Q_{2k} &= \sum_{i=2k}^{2n+1} P_i (z_i - z_{i-1}) = \sum_{i=k}^{n} P_{2i}(z_{2i} - z_{2i-1}) + \sum_{i=k}^{n} P_{2i+1}(z_{2i+1} - z_{2i}) \\ &= C \sum_{i=k}^{n} [(n+1-i)(x_{n+1-i} - y_{n+2-i}) - i(y_{n+1-i} - x_{n+1-i})] \\ &= C \sum_{i=1}^{n+1-k} [(n+1-i)(x_i - y_i) - i(y_{i+1} - x_i)] \geqslant 0.\end{aligned}$$

We have $Q_{2n+1} = (x_1 - y_1)/(n+1) \geqslant 0$ and for every $k \in I_{n-1}$

$$\begin{aligned}Q_{2k+1} &= \sum_{i=2k+1}^{2n+1} P_i (z_i - z_{i-1}) = \sum_{i=k+1}^{n} P_{2i}(z_{2i} - z_{2i-1}) + \sum_{i=k}^{n} P_{2i+1}(z_{2i+1} - z_{2i}) \\ &= C \sum_{i=k}^{n-1} [(n-i)(x_{n-i} - y_{n+1-i}) - i(y_{n+1-i} - x_{n+1-i})] + Cn(x_1 - y_1) \\ &= C \sum_{i=1}^{n-k} [(n-i)(x_{i+1} - y_{i+1}) - i(y_{i+1} - x_i)] + Cn(x_1 - y_1) \geqslant 0.\end{aligned}$$

Because $Q_k \geqslant 0$ we deduce $A[f] \geqslant 0$ for every increasing concave function f. □

COROLLARY 5. *Let $n \geqslant 1$ be an integer and let x_i, $i \in I_n$ and y_j, $j \in I_{n+1}$ be two increasing sequences of points from $[0,1]$ such that $x_k \geqslant y_k$ for $k \in I_n$. Let A be the linear functional defined by*

$$A[f] = \frac{1}{n} \sum_{k=1}^{n} f(x_k) - \frac{1}{n+1} \sum_{k=1}^{n+1} f(y_k), \tag{11}$$

a) If $x_n \geqslant y_{n+1}$ and

$$(n-i)(x_{i+1} - y_{i+1}) \geqslant i(y_{i+1} - x_i), \text{ for } i \in I_{n-1},$$

then $A[f] \geqslant 0$, for every increasing convex function $f : [0,1] \to \mathbb{R}$.

b) If

$$(n+1-i)(x_i - y_i) \geqslant i(y_{i+1} - x_i), \text{ for } i \in I_n,$$

then $A[f] \geqslant 0$, for every increasing concave function $f : [0,1] \to \mathbb{R}$.

Proof. $a)$ Using the arguments from the previous corollary we have

$$T_{2k-1} = C\sum_{i=n+1-k}^{n-1}[(n-i)(x_{i+1}-y_{i+1})-i(y_{i+1}-x_i)]+Cn(x_n-y_{n+1})\geqslant 0.$$

$T_{2k} = T_{2k-1}+Ci(x_{n+1-k}-y_{n+1-k})\geqslant 0$. We deduce that $A[f]\geqslant 0$ for every increasing convex function f.

$b)$ We have

$$Q_{2k} = C\sum_{i=1}^{n+1-k}[(n+1-i)(x_i-y_i)-i(y_{i+1}-x_i)]\geqslant 0.$$

$Q_{2k-1} = Q_{2k}+C(k-1)(x_{n+2-k}-y_{n+2-k})\geqslant 0$. We obtain $A[f]\geqslant 0$ for every increasing concave function f. □

COROLLARY 6. *Let $(a_n)_{n\in\mathbb{N}}$ be a positive increasing sequence of real numbers such that $\left(n\left(1-\frac{a_n}{a_{n+1}}\right)\right)_{n\in\mathbb{N}}$ is increasing ($\left(n\left(\frac{a_{n+1}}{a_n}-1\right)\right)_{n\in\mathbb{N}}$ is increasing). Then*

$$\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{a_k}{a_n}\right)\geqslant\frac{1}{n+1}\sum_{k=1}^{n+1}f\left(\frac{a_k}{a_{n+1}}\right)$$

for every $f:[0,1]\to\mathbb{R}$ increasing convex (concave) function.

Proof. Letting $x_k=\frac{a_k}{a_n}$ and $y_k=\frac{a_k}{a_{n+1}}$ we apply the result of the previous corollary. □

REMARK 7. The result of the Corrolary 6 was obtained in [6].

COROLLARY 8. *Let $n\geqslant 1$ be an integer and let x_i, $i\in I_n$ and y_j, $j\in I_{n+1}$ be two strictly increasing sequences of points from $[0,1]$, with the property*

$$0\leqslant y_1<x_1<y_2<\ldots<y_n<x_n\leqslant y_{n+1}\leqslant 1. \tag{12}$$

Let A be the linear functional defined by

$$A[f]=\alpha\sum_{k=1}^{n}f(x_k)-\beta\sum_{k=1}^{n+1}f(y_k), \tag{13}$$

where α and β are positive real numbers. Then $A[f]\geqslant 0$ for every increasing convex or concave function $f:[0,1]\to\mathbb{R}$, if and only if

$$\alpha=\frac{c}{n}\text{ and }\beta=\frac{c}{n+1},\text{ where } c>0,$$
$$(n+1)(x_1+x_2+\ldots+x_k)-n(y_1+\ldots+y_k)\geqslant ky_{k+1},\text{ for every } k\in I_n,$$
$$(n+1)(x_k+\ldots+x_n)-n(y_{k+1}+\ldots+y_{n+1})\geqslant(n+1-k)x_k,\text{ for } k\in I_n.$$

Proof. The condition $A[e_0]=0$ is equivalent with $\alpha=c/n$ and $\beta=c/(n+1)$, for some $c>0$. We apply Theorem 1 for $m=2n+1$, $z_{2k}=x_{n+1-k}$ and $a_{2k}=1/n$, with $k\in I_n$, $z_{2k-1}=y_{n+2-k}$ and $a_{2k-1}=-1/(n+1)$, with $k\in I_{n+1}$, in the case $x_n<y_{n+1}$. Using $T_k=\sum_{i=1}^k a_iz_i-z_{k+1}\sum_{i=1}^k a_i$, we obtain for every $k\in I_n$

$$T_{2k-1}=\frac{(n+1)(x_{n+1-k}+\ldots+x_n)-n(y_{n+2-k}+\ldots+y_{n+1})-kx_{n+1-k}}{n(n+1)}.$$

Because $T_{2k-1}=T_{2k}-S_{2k}(z_{2k}-z_{2k+1})\leqslant T_{2k}$, we deduce that $T_k\geqslant 0$, for $k\in I_{2n}$, if and only if $T_{2k-1}\geqslant 0$, for $k\in I_n$, which is equivalent with

$$(n+1)(x_k+\ldots+x_n)-n(y_{k+1}+\ldots+y_{n+1})\geqslant(n+1-k)x_k,\ \text{for every } k\in I_n.$$

Using $Q_{k+1}=\sum_{i=k+1}^{2n+1}a_iz_i-z_k\sum_{i=k+1}^{2n+1}a_i$, we obtain for every $k\in I_n$

$$Q_{2k}=\frac{(n+1)(x_1+\ldots+x_{n+1-k})-n(y_1+\ldots+y_{n+1-k})-(n+1-k)y_{n+2-k}}{n(n+1)}.$$

Because $Q_{2k+1}=Q_{2k}-P_{2k}(z_{2k}-z_{2k-1})\geqslant Q_{2k}$, we deduce that $Q_{k+1}\geqslant 0$, for $k\in I_{2n}$, if and only if $Q_{2k}\geqslant 0$, for $k\in I_n$, which is equivalent with

$$(n+1)(x_1+\ldots+x_k)-n(y_1+\ldots+y_k)\geqslant ky_{k+1},\ \text{for every } k\in I_n.$$

If $x_n=y_{n+1}$ we apply Theorem 1 for the same points z_k like above and with $a_1=0$, $a_2=1/n-1/(n+1)$ and the others a_k like above. We obtain $T_1=0$. □

REMARK 9. Let $n\geqslant 1$ be an integer and let x_i, $i\in I_n$ and y_j, $j\in I_{n+1}$ be two strictly increasing sequences of points from $[0,1]$, with the property

$$0\leqslant y_1\leqslant x_1<y_2<\ldots<y_n<x_n\leqslant y_{n+1}\leqslant 1.$$

The condition

$$(n+1)(x_k+\ldots+x_n)-n(y_{k+1}+\ldots+y_{n+1})\geqslant(n+1-k)x_k,\ \text{for } k\in I_n,$$

is equivalent with

$$\sum_{i=k}^{n-1}[(n-i)(x_{i+1}-y_{i+1})-i(y_{i+1}-x_i)]\geqslant 0,\ \text{for every } k\in I_{n-1}\quad\text{and}\quad x_n=y_{n+1},$$

and the condition

$$(n+1)(x_1+x_2+\ldots+x_k)-n(y_1+\ldots+y_k)\geqslant ky_{k+1},\ \text{for every } k\in I_n$$

is equivalent with

$$\sum_{i=1}^{k}[(n+1-i)(x_i-y_i)-i(y_{i+1}-x_i)]\geqslant 0,\ \text{for every } k\in I_n.$$

Proof. Indeed, from $T_1 \geqslant 0$ we deduce $x_n = y_{n+1}$. For $k \in I_n$, $k \neq 1$

$$T_{2k-1} = C \sum_{i=n+1-k}^{n-1} [(n-i)(x_{i+1} - y_{i+1}) - i(y_{i+1} - x_i)],$$

where $C = 1/[n(n+1)]$ and for $k \in I_n$ we have the formula

$$Q_{2k} = C \sum_{i=1}^{n+1-k} [(n+1-i)(x_i - y_i) - i(y_{i+1} - x_i)]. \quad \square$$

REMARK 10. Using Corollary 8 and Remark 9 we deduce the result obtained in [2], the one presented in the introduction.

REMARK 11. If $y_1 \leqslant y_2 \leqslant \ldots \leqslant y_{n+1}$ are the roots of a polynomial P of degree $n+1$ and $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$ the roots of the derivative of the polynomial P, then we have the inequalities

$$(n+1)(x_1 + x_2 + \ldots + x_k) - n(y_1 + \ldots + y_k) \geqslant k y_{k+1}, \text{ for every } k \in I_{n-1}.$$

See [4], for details.

COROLLARY 12. *Let $n \geqslant 1$ be an integer and let x_i, y_i, $i \in I_n$ be two decreasing sequences of points from $[0,1]$ and p_i, q_i, $i \in I_n$ be real numbers such that (p_i) majorizes (q_i) (i.e. $p_1 + \ldots + p_k \geqslant q_1 + \ldots + q_k$ for every $k \in I_{n-1}$ and $p_1 + \ldots + p_n = q_1 + \ldots + q_n$). If the following conditions are satisfied:*

$$\sum_{i=1}^{k} q_i x_i \geqslant \sum_{i=1}^{k} q_i y_i, \text{for every } k \in I_{n-1},$$

$$\sum_{i=1}^{k} p_i x_i \geqslant \sum_{i=1}^{k} p_i y_i, \text{ for every } k \in I_n,$$

and

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} q_i y_i,$$

then, for every convex function $f : [0,1] \to \mathbb{R}$, we have

$$\sum_{k=1}^{n} p_k f(x_k) \geqslant \sum_{k=1}^{n} q_k f(y_k).$$

Proof. We apply Theorem 1 $c)$ and Remark 2 for the functional

$$A[f] = \sum_{k=1}^{n} p_k f(x_k) - \sum_{k=1}^{n} q_k f(y_k)$$

and for the points $z_{2k-1}=x_k$ and $z_{2k}=y_k$, for every $k\in I_n$. We can notice that

$$A[e_0]=\sum_{k=1}^{n}p_k-\sum_{k=1}^{n}q_k=0,$$
$$A[e_1]=\sum_{k=1}^{n}p_kx_k-\sum_{i=1}^{n}q_ky_k=0,$$
$$S_{2k}=p_1-q_1+\ldots+p_k-q_k\geqslant 0.$$

So

$$\begin{aligned}
T_{2k}&=\sum_{i=1}^{k}[S_{2i-1}(z_{2i-1}-z_{2i})+S_{2i}(z_{2i}-z_{2i+1})]\\
&=\sum_{i=1}^{k}[(S_{2i}+q_i)(x_i-y_i)+S_{2i}(y_i-x_{i+1})]\\
&=\sum_{i=1}^{k}S_{2i}(x_i-x_{i+1})+\sum_{i=1}^{k}q_i(x_i-y_i)\geqslant 0,\ \text{for every } k\in I_{n-1}.\\
T_{2k-1}&=S_1(z_1-z_2)+\sum_{i=1}^{k-1}[S_{2i}(z_{2i}-z_{2i+1})+S_{2i+1}(z_{2i+1}-z_{2i+2})]\\
&=p_1(x_1-y_1)+\sum_{i=1}^{k-1}[S_{2i}(y_i-x_{i+1})+(S_{2i}+p_{i+1})(x_{i+1}-y_{i+1})]\\
&=\sum_{i=1}^{k-1}S_{2i}(y_i-y_{i+1})+\sum_{i=1}^{k}p_i(x_i-y_i)\geqslant 0,\ \text{for every } k\in I_n. \quad \square
\end{aligned}$$

COROLLARY 13. *If (x_i) and (y_i) are two decreasing sequences of points from $[0,1]$ and p_i, $i\in I_n$ are real numbers such that*

$$\sum_{i=1}^{k}p_ix_i\geqslant\sum_{i=1}^{k}p_iy_i,\ \textit{for every } k\in I_n, \tag{14}$$

and

$$\sum_{i=1}^{n}p_ix_i=\sum_{i=1}^{n}p_iy_i, \tag{15}$$

then, for every convex function $f:[0,1]\to\mathbb{R}$, we have

$$\sum_{k=1}^{n}p_kf(x_k)\geqslant\sum_{k=1}^{n}p_kf(y_k). \tag{16}$$

REMARK 14. If we suppose that p_k are positive real numbers, then also the converse of Corollary 13 is true. Indeed, applying inequality (16) to the convex functions $f_k(x)=\max(0,x-x_k)$ we obtain

$$\sum_{i=1}^{k}p_ix_i-x_k\sum_{i=1}^{k}p_k\geqslant\sum_{i=1}^{n}p_if_k(y_i).$$

Because $f_k(x) \geqslant 0$ and $f_k(x) \geqslant x - x_k$ we obtain

$$\sum_{i=1}^{n} p_i f_k(y_i) \geqslant \sum_{i=1}^{k} p_i y_i - x_k \sum_{i=1}^{k} p_i.$$

So, we have obtained the relations (14). The equality (15) can be obtained from inequality (16) for the convex functions e_1 and $-e_1$.

REMARK 15. For $p_i = 1$ this is the inequality of Hardy-Littlewood-Pólya, which is named also Karamata inequality or the majorization inequality.

Acknowledgement. The author would like to thank Professor Ioan Gavrea for suggesting the problem and for his very useful remarks.

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(Received October 25, 2008)

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