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HERMITE–HADAMARD–TYPE INEQUALITIES FOR RADAU–TYPE QUADRATURE RULES

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Dedicated to Professor Josip Pečarić on the occasion of his 60th birthday

Abstract. Hermite-Hadamard-type inequalities are given for Radau-type quadrature rules and k-convex functions (where $k = 2,3,5$). Furthermore, the best possible error estimates for the Radau-type quadrature rules and functions with low degree of smoothness are obtained.

1. Introduction

The well-known Hermite-Hadamard inequality states that for any convex function $f : [-1,1] \to \mathbf{R}$, the following pair of inequalities holds:

$$f(0) \leqslant \frac{1}{2}\int_{-1}^{1} f(t)dt \leqslant \frac{f(-1)+f(1)}{2}. \tag{1.1}$$

If f is concave, the inequalities in (1.1) are reversed.

The aim of this paper is to give this type of inequalities for the Radau-type quadratures, i.e. quadrature formulas which involve one end of the interval as a node (cf. [4]):

$$\int_{-1}^{1} f(t)dt \approx (2-w(x))f(-1)+w(x)f(x)$$

and

$$\int_{-1}^{1} f(t)dt \approx w(x)f(x)+(2-w(x))f(1).$$

The main tool used is the extended Euler formula, obtained in [5]: if $f : [a,b] \to \mathbf{R}$ is such that $f^{(n-1)}$ is continuous and of bounded variation on $[a,b]$ for some $n \geqslant 1$, then for every $y \in [a,b]$ we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t &= f(y) - \sum_{k=1}^{n-1}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{y-a}{b-a}\right)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right] \\ &\quad + \frac{(b-a)^{n-1}}{n!}\int_a^b \left(B_n^*\left(\frac{y-t}{b-a}\right)-B_n\left(\frac{y-a}{b-a}\right)\right)df^{(n-1)}(t)\end{aligned} \tag{1.2}$$

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where $B_k(t)$ is the k-th Bernoulli polynomial and $B_k^*(t) = B_k(t - \lfloor t \rfloor)$, $t \in \mathbf{R}$.

For the reader's convenience, let us recall some basic properties of Bernoulli polynomials. Bernoulli polynomials $B_k(t)$ are uniquely determined by

$$B_k'(x) = kB_{k-1}(x), \qquad B_k(t+1) - B_k(t) = kt^{k-1},\ k \geqslant 0, \qquad B_0(t) = 1.$$

For the kth Bernoulli polynomial we have

$$B_k(1-x) = (-1)^k B_k(x), \quad x \in \mathbf{R}, \quad k \geqslant 1. \tag{1.3}$$

The kth Bernoulli number B_k is defined by $B_k = B_k(0)$. From (1.3) it follows that for $k \geqslant 2$, we have $B_k(1) = B_k(0) = B_k$. Note that $B_{2k-1} = 0$, $k \geqslant 2$ and $B_1(1) = -B_1(0) = 1/2$.

$B_k^*(x)$ are periodic functions of period 1 and are related to Bernoulli polynomials as $B_k^*(x) = B_k(x)$, $0 \leqslant x < 1$. $B_0^*(x)$ is a constant equal to 1, while $B_1^*(x)$ is a discontinuous function with a jump of -1 at each integer. For $k \geqslant 2$, $B_k^*(t)$ is a continuous function. For further details on Bernoulli polynomials see [1] and [6].

2. Main results

Let $x \in (-1,1]$ and $f : [-1,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is continuous and of bounded variation on $[-1,1]$ for some $n \geqslant 1$. Take $y = -1$ and $y = x$ in (1.2), multiply by $2 - w(x)$, $w(x)$ respectively and add. The following formula is produced:

$$\int_{-1}^{1} f(t)dt - Q(-1,x) + T_{n-1}(x) = \frac{2^{n-1}}{n!} \int_{-1}^{1} F_n(x,t)df^{(n-1)}(t), \tag{2.1}$$

where

$$Q(-1,x) = (2 - w(x))f(-1) + w(x)f(x) \tag{2.2}$$

$$T_{n-1}(x) = \sum_{k=1}^{n-1} \frac{2^{k-1}}{k!}\, G_k(x,1)\, [f^{(k-1)}(1) - f^{(k-1)}(-1)],\ T_0(x) = 0 \tag{2.3}$$

$$G_n(x,t) = (2 - w(x))B_n^*\left(\frac{1-t}{2}\right) + w(x)B_n^*\left(\frac{x-t}{2}\right), \tag{2.4}$$

$$F_n(x,t) = G_n(x,t) - G_n(x,1). \tag{2.5}$$

Note that

$$\frac{\partial^k G_n(x,t)}{\partial t^k} = \frac{n!}{(-2)^k (n-k)!}\, G_{n-k}(x,t) \quad \text{and} \quad G_n(x,-1) = G_n(x,1).$$

The following theorem gives the best possible estimate of error for this type of quadrature formulas.

THEOREM 1. *Let* $p,q \in \mathbf{R}$ *be such that* $1 \leqslant p,\ q \leqslant \infty$ *and* $1/p + 1/q = 1$. *If* $f : [-1,1] \to \mathbf{R}$ *is such that* $f^{(n)} \in L_p[-1,1]$ *for some* $n \geqslant 1$, *then*

$$\left| \int_{-1}^{1} f(t)dt - Q(-1,x) + T_{n-1}(x) \right| \leqslant \frac{2^{n-1}}{n!} \left[\int_{-1}^{1} |F_n(x,t)|^q \, dt \right]^{\frac{1}{q}} \|f^{(n)}\|_p \, . \tag{2.6}$$

The inequality is sharp for $1 < p \leqslant \infty$ *and the best possible for* $p = 1$.

Proof. Inequality (2.6) follows immediately after applying Hölder's inequality to the remainder in (2.1). To prove the inequality is sharp, take

$$f^{(n)}(t) = \operatorname{sgn} F_n(x,t) \cdot |F_n(x,t)|^{1/(p-1)} \quad \text{for} \quad 1 < p < \infty$$
$$\text{and} \;\; f^{(n)}(t) = \operatorname{sgn} F_n(x,t) \quad \text{for} \quad p = \infty\,;$$

then $\left| \int_{-1}^{1} F_n(x,t) f^{(n)}(t) dt \right| = \int_{-1}^{1} |F_n(x,t) f^{(n)}(t)| dt = \left(\int_{-1}^{1} |F_n(x,t)|^q \, dt \right)^{\frac{1}{q}} \|f^{(n)}\|_p$. To prove the inequality is the best possible for $p = 1$ and $n \geqslant 2$, first assume $|F_n(x,t)|$ achieves its maximal value at $t_0 \in (-1,1)$ and $F_n(x,t_0) > 0$. For a small enough ε define function f_ε such that

$$f_\varepsilon^{(n-1)}(t) = \begin{cases} 0, & t \leqslant t_0 \\ \frac{1}{\varepsilon}(t - t_0), & t_0 \leqslant t \leqslant t_0 + \varepsilon \\ 1, & t \geqslant t_0 + \varepsilon \end{cases} \tag{2.7}$$

Then

$$\left| \int_{-1}^{1} F_n(x,t) f_\varepsilon^{(n)}(t) dt \right| = \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} F_n(x,t) dt \leqslant \frac{F_n(x,t_0)}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} dt = \max_{t \in (-1,1)} |F_n(x,t)|$$

and since $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} F_n(x,t) dt = F_n(x,t_0)$, the statement follows. If $F_n(x,t_0) < 0$, take g_ε such that $g_\varepsilon^{(n-1)}(t) = 1 - f_\varepsilon^{(n-1)}(t)$.

As for the case $n = 1$, note that $F_1(x,t)$ is a piecewise decreasing and piecewise linear function in t, with a jump at -1 and x on $[-1,1]$. So we have

$$\sup_{t \in [-1,1]} |F_1(x,t)| = \max\{\, |2 - w(x)|,\ |1 - x - w(x)|,\ 1 - x \}.$$

Assume $\sup_{t \in [-1,1]} |F_1(x,t)| = |1 - x - w(x)|$. If $1 - x - w(x) > 0$, define f_ε such that

$$f_\varepsilon^{(n-1)}(t) = \begin{cases} 0, & t \leqslant x - \varepsilon \\ \frac{1}{\varepsilon}(t - x + \varepsilon), & x - \varepsilon \leqslant t \leqslant x \\ 1, & t \geqslant x \end{cases}$$

The claim now follows analogously as above. If $1 - x - w(x) < 0$, take g_ε such that $g_\varepsilon^{(n-1)}(t) = 1 - f_\varepsilon^{(n-1)}(t)$. If $\sup_{t \in [-1,1]} |F_1(x,t)| = |2 - w(x)|$, take $f_\varepsilon^{(n-1)}$ as in (2.7) with $t_0 = -1$, and finally, if $\sup_{t \in [-1,1]} |F_1(x,t)| = 1 - x$, take $f_\varepsilon^{(n-1)}$ as in (2.7) with $t_0 = x$. $\square$

2.1. Hermite-Hadamard-Type Inequality For 2-Convex Functions

As the coefficient $w(x)$ is arbitrary, it can be chosen so that $G_1(x,1)=0$, and then

$$G_1(x,1)=0 \quad \Leftrightarrow \quad w(x)=\frac{2}{x+1}\,. \tag{2.8}$$

This coefficient removes the values of the function at the end points of the interval out of $T_{n-1}(x)$ and thus provides the highest possible degree of exactness (namely, such a quadrature rule is exact for all first degree polynomials), without the values of the derivatives being included in the quadrature. To emphasize the coefficient we are working with, we denote expressions (2.2)–(2.5) by $Q_{R1}(-1,x)$, $T_{n-1}^{R1}(x)$, $G_n^{R1}(x,t)$ and $F_n^{R1}(x,t)$.

LEMMA 1 *For $x\in(-1,0]\cup\{1\}$, $F_2^{R1}(x,t)$ has no zeros in the variable t on $(-1,1)$. The sign of the function is determined by:*

$$F_2^{R1}(x,t)>0 \textit{ for } x\in(-1,0] \quad \textit{and} \quad F_2^{R1}(1,t)<0.$$

Proof. We have:

$$F_2^{R1}(x,t)=\frac{2x}{x+1}\left[B_2\left(\frac{1-t}{2}\right)-\frac{1}{6}\right]+\frac{2}{x+1}\left[B_2^*\left(\frac{x-t}{2}\right)-B_2\left(\frac{x+1}{2}\right)\right].$$

It is obvious that $F_2^{R1}(x,-1)=F_2^{R1}(x,1)=0$. Assume: $-1<t\leqslant x\leqslant 1$. Then:

$$F_2^{R1}(x,t)=\frac{1+t}{2(1+x)}(t(1+x)-3x+1)=0 \;\Leftrightarrow\; t^*=\frac{3x-1}{x+1}\,.$$

It is elementary to see that $t^*\leqslant x$, but $t^*>-1 \;\Leftrightarrow x>0$. Also, $x=1\Leftrightarrow t^*=1$. If $-1<x\leqslant t<1$, $F_2^{R1}(x,t)=\frac{1}{2}(1-t)^2>0$, so the assertion is proved. □

THEOREM 2. *Let $f:[-1,1]\to\mathbf{R}$ be such that f'' is continuous on $[-1,1]$ and let $x\in(-1,0]\cup\{1\}$. Then there exists $\xi\in[-1,1]$ such that*

$$\int_{-1}^{1}f(t)dt-\frac{2x}{x+1}f(-1)-\frac{2}{x+1}f(x)=\frac{1}{3}(1-3x)f''(\xi) \tag{2.9}$$

and

$$\int_{-1}^{1}f(t)dt-\frac{2}{x+1}f(-x)-\frac{2x}{x+1}f(1)=\frac{1}{3}(1-3x)f''(-\xi). \tag{2.10}$$

Proof. (2.9) follows after applying the Mean Value Theorem for integrals and Lemma 1 to the remainder in (2.1) for $n=2$, with the coefficient as in (2.8). Note that $\int_{-1}^{1}F_2^{R1}(x,t)dt=-\frac{2}{3}\int_{-1}^{1}\frac{\partial G_3^{R1}(x,t)}{\partial t}dt-2G_2^{R1}(x,1)=-2G_2^{R1}(x,1)$. (2.10) follows analogously for $f(-x)$. □

REMARK 1. When considering the limit process $x \to -1$, we obtain the following quadrature rules:

$$\int_{-1}^{1} f(t)dt - 2f(-1) - 2f'(-1) = \frac{4}{3}f''(\xi)$$

and

$$\int_{-1}^{1} f(t)dt - 2f(1) + 2f'(1) = \frac{4}{3}f''(-\xi).$$

THEOREM 3. *If* $f : [-1,1] \to \mathbf{R}$ *is such that* $f' \in L_\infty[-1,1]$, *then for* $x \in (-1,0]$

$$\left| \int_{-1}^{1} f(t)dt - \frac{2x}{x+1}f(-1) - \frac{2}{x+1}f(x) \right| \leqslant (1-x)^2 \|f'\|_\infty \tag{2.11}$$

while for $x \in [0,1]$

$$\left| \int_{-1}^{1} f(t)dt - \frac{2x}{x+1}f(-1) - \frac{2}{x+1}f(x) \right| \leqslant \left(\frac{1+x^2}{1+x} \right)^2 \|f'\|_\infty \tag{2.12}$$

The node which provides the smallest error here is $x = \sqrt{2} - 1 \approx 0.4142$ *and we have*

$$\left| \int_{-1}^{1} f(t)dt - (2-\sqrt{2})\, f(-1) - \sqrt{2}\, f(\sqrt{2}-1) \right| \leqslant (12 - 8\sqrt{2})\, \|f'\|_\infty$$

$(12 - 8\sqrt{2} \approx 0.6863)$.

Furthermore, if $f : [-1,1] \to \mathbf{R}$ *is such that* $f'' \in L_\infty[-1,1]$, *then for* $x \in (-1,0] \cup \{1\}$ *we have*

$$\left| \int_{-1}^{1} f(t)dt - \frac{2x}{x+1}f(-1) - \frac{2}{x+1}f(x) \right| \leqslant \frac{1}{3}|1-3x| \cdot \|f''\|_\infty \tag{2.13}$$

while for $x \in (0,1)$

$$\left| \int_{-1}^{1} f(t)dt - \frac{2x}{x+1}f(-1) - \frac{2}{x+1}f(x) \right| \leqslant \frac{1-6x^2+24x^3-3x^4}{3(1+x)^3} \|f''\|_\infty \tag{2.14}$$

The node which provides the smallest error in this case is $x^* := 2\sqrt{2} - 1 - 2\sqrt{2-\sqrt{2}} \approx 0.2977$ *and we have:*

$$\left| \int_{-1}^{1} f(t)dt - 0.4588 \cdot f(-1) - 1.5412 \cdot f(x^*) \right| \leqslant 0.1644 \cdot \|f''\|_\infty$$

Proof. (2.11) and (2.12) follow after taking $p = \infty$ and $n = 1$ in (2.6) with the coefficient from (2.8). (2.13) and (2.14) follow similarly, for $n = 2$.

In order to find the nodes which provide the smallest error, the functions on the right-hand sides of all four inequalities have to be minimized. Routine calculation confirms the claims. When trying to minimize the function on the right-hand side of (2.14), note that $x^4 + 4x^3 - 26x^2 + 4x + 1 = (x+1)^4 - 32x^2$, so the zeros can be found analytically. $\square$

THEOREM 4. *Let* $f : [-1,1] \to \mathbf{R}$ *be* 2*-convex and such that* f'' *is continuous on* $[-1,1]$. *Let* $x \in (-1,0]$. *Then*

$$\frac{x}{x+1} f(-1) + \frac{1}{x+1} f(x) \leqslant \frac{1}{2} \int_{-1}^{1} f(t)dt \leqslant \frac{f(-1)+f(1)}{2}. \tag{2.15}$$

If f *is* 2*-concave, the inequalities in (*2.15*) are reversed.*

Proof. For a 2-convex function f we have $f'' \geqslant 0$, while for a 2-concave function f we have $f'' \leqslant 0$, so the statement follows easily from (2.9). □

As a special case, we now obtain the classical Hermite-Hadamard inequality.

COROLLARY 1 *If* $f : [-1,1] \to \mathbf{R}$ *is* 2*-convex and such that* f'' *is continuous on* $[-1,1]$*, then*

$$f(0) \leqslant \frac{1}{2} \int_{-1}^{1} f(t)dt \leqslant \frac{f(-1)+f(1)}{2}.$$

If f *is* 2*-concave, the inequalities are reversed.*

Proof. Take $x = 0$ in (2.15). □

REMARK 2. All the results obtained here easily follow for the quadrature rule with the right-end of the interval as the preassigned node, therefore we do not state them explicitly.

2.2. Hermite-Hadamard-Type Inequality For 3-Convex Functions

Suppose we want to obtain a quadrature rule exact for all polynomials of order $\leqslant 2$, instead of $\leqslant 1$, as were (2.9) and (2.10). Observe (2.1) again. We considered the case when $G_1(x,1) = 0$. Now, impose another condition and choose the coefficient so that $G_2(x,1) = 0$:

$$G_2(x,1) = 0 \;\Leftrightarrow\; w(x) = \frac{4}{3(1-x^2)} \tag{2.16}$$

This will produce a quadrature rule with the desired degree of exactness. However, as a downside, the value of the function at the right end of the interval will now also be included in the quadrature. To emphasize the coefficient we are working with, we denote expressions (2.2)–(2.5) by $Q_{R2}(-1,x)$, $T_{n-1}^{R2}(x)$, $G_n^{R2}(x,t)$ and $F_n^{R2}(x,t)$ for this specific coefficient.

LEMMA 2 *For* $x \in (-1,-1/3] \cup [1/3,1)$*,* $F_3^{R2}(x,t)$ *has no zeros in* t *on* $(-1,1)$*. The sign of this function is determined by:*

$$F_3^{R2}(x,t) > 0 \;\; \textit{for} \;\; x \in [1/3,1)$$
$$F_3^{R2}(x,t) < 0 \;\; \textit{for} \;\; x \in (-1,-1/3].$$

Proof. For $-1 < t \leqslant x < 1$, we have

$$F_3^{R2}(x,t) = (1+t)^2\left(\frac{2x}{1+x} - t\right) = 0 \Leftrightarrow t^* = \frac{2x}{1+x},$$

and $-1 < t^* \leqslant x \Leftrightarrow -1/3 < x \leqslant 0$. If $-1 < x \leqslant t < 1$,

$$F_3^{R2}(x,t) = \frac{(1-t)^2}{4}\left(\frac{2x}{1-x} - t\right) = 0 \Leftrightarrow t^{**} = \frac{2x}{1-x}.$$

Now, $x \leqslant t^{**} < 1 \Leftrightarrow 0 \leqslant x < 1/3$. Therefore, the claim follows. □

THEOREM 5. *Let $f : [-1,1] \to \mathbf{R}$ be such that f''' is continuous on $[-1,1]$ and let $x \in (-1,-1/3] \cup [1/3,1)$. Then there exists $\xi \in [-1,1]$ such that*

$$\int_{-1}^{1} f(t)dt - \frac{1+3x}{3(1+x)} f(-1) - \frac{4}{3(1-x^2)} f(x) - \frac{1-3x}{3(1-x)} f(1) = \frac{2x}{9} f'''(\xi). \quad (2.17)$$

Proof. Analogous to the proof of Theorem 2. □

REMARK 3. For $x = 1/3$ and $x = -1/3$, from (2.17) we get the Radau 2-point formulas:

$$\int_{-1}^{1} f(t)dt - \frac{1}{2} f(-1) - \frac{3}{2} f\left(\frac{1}{3}\right) = \frac{2}{27} f'''(\xi)$$

and

$$\int_{-1}^{1} f(t)dt - \frac{3}{2} f\left(-\frac{1}{3}\right) - \frac{1}{2} f(1) = -\frac{2}{27} f'''(-\xi)$$

REMARK 4. When considering the limit processes $x \to 1$ and $x \to -1$, the following quadrature rules are produced:

$$\int_{-1}^{1} f(t)dt - \frac{2}{3} f(-1) - \frac{4}{3} f(1) + \frac{2}{3} f'(1) = \frac{2}{9} f'''(\xi)$$

and

$$\int_{-1}^{1} f(t)dt - \frac{4}{3} f(-1) - \frac{2}{3} f(1) - \frac{2}{3} f'(-1) = -\frac{2}{9} f'''(-\xi).$$

Next, we consider the error estimates for this type of quadrature rules.

THEOREM 6. *If $f : [-1,1] \to \mathbf{R}$ is such that $f'' \in L_\infty[-1,1]$, then for $x \in (-1,-1/3] \cup [1/3,1)$*

$$\left|\int_{-1}^{1} f(t)dt - \frac{1+3x}{3(1+x)} f(-1) - \frac{4}{3(1-x^2)} f(x) - \frac{1-3x}{3(1-x)} f(1)\right| \leqslant \frac{4}{81}\left(\frac{1+3|x|}{1+|x|}\right)^3 \|f''\|_\infty \quad (2.18)$$

while for $x \in (-1/3, 1/3)$

$$\left| \int_{-1}^{1} f(t)dt - \frac{1+3x}{3(1+x)} f(-1) - \frac{4}{3(1-x^2)} f(x) - \frac{1-3x}{3(1-x)} f(1) \right| \leqslant \frac{8(1-3x^2)(1+3x^2)^2}{81(1-x^2)^3} \, \|f''\|_\infty \tag{2.19}$$

Further, if $f : [-1,1] \to \mathbf{R}$ *is such that* $f''' \in L_\infty[-1,1]$*, then for* $x \in (-1,-1/3] \cup [1/3,1)$*,*

$$\left| \int_{-1}^{1} f(t)dt - \frac{1+3x}{3(1+x)} f(-1) - \frac{4}{3(1-x^2)} f(x) - \frac{1-3x}{3(1-x)} f(1) \right| \leqslant \frac{2|x|}{9} \, \|f'''\|_\infty \tag{2.20}$$

while for $x \in (-1/3, 1/3)$

$$\left| \int_{-1}^{1} f(t)dt - \frac{1+3x}{3(1+x)} f(-1) - \frac{4}{3(1-x^2)} f(x) - \frac{1-3x}{3(1-x)} f(1) \right| \leqslant \frac{8|x|^5 + 49x^4 - 60|x|^3 + 22x^2 - 4|x| + 1}{36(1-|x|)^4} \, \|f'''\|_\infty \tag{2.21}$$

In both cases, the node which provides the smallest error is $x = 0$*. The quadrature rule thus obtained is the classical Simpson's rule. More precisely, we have:*

$$\left| \int_{-1}^{1} f(t)dt - \frac{1}{3} f(-1) - \frac{4}{3} f(0) - \frac{1}{3} f(1) \right| \leqslant \frac{8}{81} \|f''\|_\infty$$

and

$$\left| \int_{-1}^{1} f(t)dt - \frac{1}{3} f(-1) - \frac{4}{3} f(0) - \frac{1}{3} f(1) \right| \leqslant \frac{1}{36} \|f'''\|_\infty$$

Proof. (2.18) and (2.19) follow after taking $p = \infty$ and $n = 2$ in (2.6) with the coefficient from (2.16). (2.20) and (2.21) follow similarly, for $n = 3$.

As for finding the node which provides the smallest error, the functions on the right-hand sides of all four inequalities have to be minimized. The claim follows after somewhat lengthy but routine calculation. □

COROLLARY 2 *Let* $f : [-1,1] \to \mathbf{R}$ *be such that* $f^{(n)} \in L_\infty[-1,1]$ *for* $n = 1, 2$ *or* 3*. Then we have:*

$$\left| \int_{-1}^{1} f(t)dt - \frac{1}{2} f(-1) - \frac{3}{2} f\left(\frac{1}{3}\right) \right| \leqslant C_n^\infty \, \|f^{(n)}\|_\infty, \quad n = 1,2,3 \tag{2.22}$$

where

$$C_1^\infty = \frac{25}{36}, \quad C_2^\infty = \frac{1}{6}, \quad C_3^\infty = \frac{2}{27}.$$

Proof. For $n=2$ and $n=3$ the assertions follow directly after taking $x=1/3$ in (2.18) and (2.20). As for $n=1$, take $n=1$ and $p=\infty$ in (2.6). $\square$

COROLLARY 3 *Let $f:[-1,1]\to\mathbf{R}$ be such that $f^{(n)}\in L_1[-1,1]$ for $n=1,2$ or 3. Then we have:*

$$\left|\int_{-1}^{1} f(t)dt-\frac{1}{2}f(-1)-\frac{3}{2}f\left(\frac{1}{3}\right)\right|\leqslant C_n^1\,\|f^{(n)}\|_1,\quad n=1,2,3 \tag{2.23}$$

where

$$C_1^1=\frac{5}{6},\quad C_2^1=\frac{2}{9},\quad C_3^1=\frac{1}{12}.$$

Proof. Take $p=1$ and $n=1,2,3$, respectively, in (2.6) and then find $\sup\limits_{t\in[-1,1]}|F_n(1/3,t)|$. $\square$

THEOREM 7. *Let $f:[-1,1]\to\mathbf{R}$ be 3-convex and such that f''' is continuous on $[-1,1]$. Let $x\in(-1,1/3]$ and $y\in[1/3,1)$. Then*

$$\begin{aligned}&\frac{1+3y}{3(1+y)}f(-1)+\frac{4}{3(1-y^2)}f(y)+\frac{1-3y}{3(1-y)}f(1)\\&\quad\leqslant\int_{-1}^{1} f(t)dt\ \leqslant\ \frac{1+3x}{3(1+x)}f(-1)+\frac{4}{3(1-x^2)}f(x)+\frac{1-3x}{3(1-x)}f(1)\end{aligned} \tag{2.24}$$

If f is 3-concave, the inequalities in (2.24) are reversed.

Proof. Analogous to the proof of Theorem 4. $\square$

COROLLARY 4 *If $f:[-1,1]\to\mathbf{R}$ is 3-convex and f''' is continuous on $[-1,1]$, then*

$$\frac{1}{2}f(-1)+\frac{3}{2}f\left(\frac{1}{3}\right)\leqslant\int_{-1}^{1} f(t)dt\leqslant\frac{3}{2}f\left(-\frac{1}{3}\right)+\frac{1}{2}f(1) \tag{2.25}$$

If f is 3-concave, the inequalities in (2.25) are reversed.

Proof. Take $x=-1/3$ and $y=1/3$ in (2.24). $\square$

REMARK 5. Using another, more general approach, the inequality (2.25) was also obtained in [2], i.e. [3].

2.3. Hermite-Hadamard-Type Inequality and Radau 3-point Formulas

Since we have obtained the Radau 2-point formula as a special case in the previous section, it is natural to consider if similar results can be derived for Radau 3-point formula.

Let $x_1, x_2 \in (-1,1]$ and $x_1 < x_2$, and suppose $f:[-1,1] \to \mathbf{R}$ is such that $f^{(n-1)}$ is continuous and of bounded variation on $[-1,1]$ for some $n \geqslant 1$. Take $y=-1$, $y=x_1$ and $y=x_2$ in (1.2), multiply by $2-w_1(x_1,x_2)-w_2(x_1,x_2)$, $w_1(x_1,x_2)$, $w_2(x_1,x_2)$ respectively and add. The following formula is produced:

$$\int_{-1}^{1} f(t)dt - Q(-1,x_1,x_2) + T_{n-1}(x_1,x_2) = \frac{2^{n-1}}{n!}\int_{-1}^{1} F_n(x_1,x_2,t)df^{(n-1)}(t), \tag{2.26}$$

where

$$\begin{aligned} Q(-1,x_1,x_2) &= (2-w_1(x_1,x_2)-w_2(x_1,x_2))\,f(-1) \\ &\quad +w_1(x_1,x_2)f(x_1)+\ w_2(x_1,x_2)f(x_2) \end{aligned} \tag{2.27}$$

$$T_{n-1}(x_1,x_2) = \sum_{k=1}^{n-1}\frac{2^{k-1}}{k!}\,G_k(x_1,x_2,1)\,[f^{(k-1)}(1)-f^{(k-1)}(-1)], \tag{2.28}$$

$$\begin{aligned} G_n(x_1,x_2,t) &= (2-w_1(x_1,x_2)-w_2(x_1,x_2))\,B_n^*\left(\frac{1-t}{2}\right) \\ &\quad +w_1(x_1,x_2)B_n^*\left(\frac{x_1-t}{2}\right)+\ w_2(x_1,x_2)B_n^*\left(\frac{x_2-t}{2}\right) \end{aligned} \tag{2.29}$$

$$F_n(x_1,x_2,t) = G_n(x_1,x_2,t)-G_n(x_1,x_2,1). \tag{2.30}$$

Now, impose conditions:

$$G_1(x_1,x_2,1)=G_2(x_1,x_2,1)=G_3(x_1,x_2,1)=G_4(x_1,x_2,1)=0.$$

The unique solution of this system

$$x_1=\frac{1-\sqrt{6}}{5},\ \ x_2=\frac{1+\sqrt{6}}{5},\ \ w_1(x_1,x_2)=\frac{16+\sqrt{6}}{18},\ \ w_2(x_1,x_2)=\frac{16-\sqrt{6}}{18} \tag{2.31}$$

are the nodes and the coefficients of the Radau 3-point formula.

To emphasize the nodes and the coefficients we are going to be working with in this subsection, denote expressions (2.28)–(2.30) by T_{n-1}^{R3}, $G_n^{R3}(t)$, $F_n^{R3}(t)$ and

$$Q_{R3}=\frac{2}{9}f(-1)+\frac{16+\sqrt{6}}{18}f\left(\frac{1-\sqrt{6}}{5}\right)+\frac{16-\sqrt{6}}{18}f\left(\frac{1+\sqrt{6}}{5}\right).$$

LEMMA 3 *$F_5^{R3}(t)$ has no zeros in $(-1,1)$ and its sign is determined by $F_5^{R3}(t)>0$.*

Proof. For $-1 \leqslant t \leqslant (1-\sqrt{6})/5$, we have $F_5^{R3}(t)=\frac{1}{144}(1+t)^4(1-9t)$ so the claim is obvious. As is for $(1+\sqrt{6})/5 \leqslant t < 1$, since then $F_5^{R3}(t)=\frac{1}{16}(1-t)^5$. For

$(1-\sqrt{6})/5 \leqslant t \leqslant (1+\sqrt{6})/5$, the function is a bit more complicated:

$$F_5^{R3}(t) = \frac{1}{288}\, k(t)$$

where

$$k(t) = -18t^5 + 5(\sqrt{6}+2)t^4 + 20(3\sqrt{6}-7)t^3 - 30(\sqrt{6}-2)t^2 + 10(2\sqrt{6}-5)t + 10 - 3\sqrt{6}.$$

We have to prove that $k(t) > 0$. From $k'''(t) = -1080t^2 + 120(\sqrt{6}+2)t + 120(3\sqrt{6}-7)$ we conclude that k'' increases on (t_1,t_2) and decreases on $[\frac{1-\sqrt{6}}{5}, t_1) \cup (t_2, \frac{1+\sqrt{6}}{5}]$, where $t_1 \approx -0.068755$ and $t_2 \approx 0.563143$. This, together with the fact that $k''(\frac{1-\sqrt{6}}{5}) < 0$, $k''(t_1) < 0$, $k''(t_2) > 0$, $k''(\frac{1+\sqrt{6}}{5}) > 0$, shows that k'' has only one zero $t^{**} \in (t_1,t_2)$. This means k' is decreasing on $[\frac{1-\sqrt{6}}{5}, t^{**})$ and increasing on $(t^{**}, \frac{1+\sqrt{6}}{5}]$. Since $k'(\frac{1-\sqrt{6}}{5}) > 0$ and $k'(\frac{1+\sqrt{6}}{5}) < 0$, it follows that k' has only one zero $t^* \in (\frac{1-\sqrt{6}}{5}, t^{**})$. From there we conclude that k increases on $[\frac{1-\sqrt{6}}{5}, t^*)$ and decreases on $(t^*, \frac{1+\sqrt{6}}{5}]$. Since $k(\frac{1-\sqrt{6}}{5}) > 0$ and $k(\frac{1+\sqrt{6}}{5}) > 0$, the claim follows. □

THEOREM 8. *If $f : [-1,1] \to \mathbf{R}$ is such that $f^{(5)}$ is continuous on $[-1,1]$, then there exists $\xi \in [-1,1]$ such that*

$$\int_{-1}^{1} f(t)dt - Q_{R3} = \frac{1}{1125}\, f^{(5)}(\xi)$$

and

$$\int_{-1}^{1} f(t)dt - \frac{16-\sqrt{6}}{18} f\left(-\frac{1+\sqrt{6}}{5}\right) - \frac{16+\sqrt{6}}{18} f\left(-\frac{1-\sqrt{6}}{5}\right) - \frac{2}{9} f(1)$$
$$= -\frac{1}{1125}\, f^{(5)}(-\xi).$$

Proof. Analogous to the proof of Theorem 2. □

THEOREM 9. *If $f : [-1,1] \to \mathbf{R}$ is 5-convex on $[-1,1]$ and such that $f^{(5)}$ is continuous on $[-1,1]$, then*

$$\frac{2}{9} f(-1) + \frac{16+\sqrt{6}}{18} f\left(\frac{1-\sqrt{6}}{5}\right) + \frac{16-\sqrt{6}}{18} f\left(\frac{1+\sqrt{6}}{5}\right)$$
$$\leqslant \int_{-1}^{1} f(t)dt \leqslant \frac{16-\sqrt{6}}{18} f\left(-\frac{1+\sqrt{6}}{5}\right) + \frac{16+\sqrt{6}}{18} f\left(-\frac{1-\sqrt{6}}{5}\right) + \frac{2}{9} f(1)$$

Proof. Follows immediately from Theorem 8. □

THEOREM 10. *Let* $p,q \in \mathbf{R}$ *be such that* $1 \leqslant p,\ q \leqslant \infty$ *and* $1/p+1/q=1$. *If* $f:[-1,1] \to \mathbf{R}$ *is such that* $f^{(n)} \in L_p[-1,1]$ *for some* $n \geqslant 1$, *then*

$$\left| \int_{-1}^{1} f(t)dt - Q_{R3} + T_{n-1}^{R3} \right| \leqslant \frac{2^{n-1}}{n!} \left[\int_{-1}^{1} \left| F_n^{R3}(t) \right|^q dt \right]^{\frac{1}{q}} \| f^{(n)} \|_p . \tag{2.32}$$

The inequality is sharp for $1 < p \leqslant \infty$ *and the best possible for* $p = 1$.

Proof. Analogous to the proof of Theorem 1. $\square$

COROLLARY 5 *Let* $f:[-1,1] \to \mathbf{R}$ *be such that* $f^{(k)} \in L_\infty[-1,1]$ *for* $k = 1,2,3,4$ *or* 5. *Then we have*

$$\left| \int_{-1}^{1} f(t)dt - Q_{R3} \right| \leqslant C_k^\infty \, \| f^{(k)} \|_\infty$$

where

$$C_1^\infty \approx 0.434014, \quad C_2^\infty \approx 0.0566841, \quad C_3^\infty \approx 0.0106218,$$
$$C_4^\infty \approx 0.00247235, \quad C_5^\infty = \frac{1}{1125} \approx 0.000888889.$$

Proof. Take $p = \infty$ and $n = 1,2,3,4,5$ in (2.32). $\square$

COROLLARY 6 *Let* $f:[-1,1] \to \mathbf{R}$ *be such that* $f^{(k)} \in L_1[-1,1]$ *for* $k = 1,2,3,4$ *or* 5. *Then we have*

$$\left| \int_{-1}^{1} f(t)dt - Q_{R3} \right| \leqslant C_k^1 \, \| f^{(k)} \|_1$$

where

$$C_1^1 = \left| F_1^{R3}\left((1-\sqrt{6})/5 \right) \right| \approx 0.537092, \quad C_2^1 = \left| F_2^{R3}\left((1-\sqrt{6})/5 \right) \right| \approx 0.094322,$$
$$C_3^1 \approx 0.0131784, \quad C_4^1 = \left| F_4^{R3}(-1/3) \right| \approx 0.00274348, \quad C_5^1 \approx 0.00123618.$$

Proof. Take $p = 1$ and $n = 1,2,3,4,5$ in (2.32). $\square$

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