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A NOTE ON GENERALIZATION OF WEIGHTED ČEBYŠEV AND OSTROWSKI INEQUALITIES

A. AGLIĆ ALJINOVIĆ

Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. A recently obtained generalization of weighted Montgomery identity is considered and used to obtain the new weighted generalizations of the Čebyšev and Ostrowski inequality.

1. Introduction

N. Boukerrioua and A. Guezane-Lakoud in the recent paper [2] established the following generalization of weighted Montgomery identity:

THEOREM 1. *Let* $f:[a,b]\to\mathbb{R}$ *be differentiable on* $[a,b]$, *and* $f':[a,b]\to\mathbb{R}$ *integrable on* $[a,b]$, $w:[a,b]\to[0,\infty\rangle$ *is some normalized weighted function, i.e. integrable function satisfying* $\int_a^b w(t)\,dt=1$, $\varphi:[0,1]\to\mathbb{R}$ *differentiable on* $[0,1]$ *with* $\varphi(0)=0$, $\varphi(1)\neq 0$ *and* $\varphi':[0,1]\to\mathbb{R}$ *integrable on* $[0,1]$. *Then the following identity holds*

$$f(x)=\frac{1}{\varphi(1)}\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right)f(t)\,dt+\frac{1}{\varphi(1)}\int_a^b P_{w,\varphi}(x,t)\,f'(t)\,dt \tag{1.1}$$

where $P_{w,\varphi}(x,t)$ *is the generalization of weighted Peano kernel, defined by*

$$P_{w,\varphi}(x,t)=\begin{cases}\varphi(W(t)), & a\leqslant t\leqslant x,\\ \varphi(W(t))-\varphi(1), & x<t\leqslant b,\end{cases} \tag{1.2}$$

and $W(t)=\int_a^t w(x)\,dx$ *for* $t\in[a,b]$.

In the special case, for $\varphi(t)=t$, $t\in[0,1]$, (1.1) reduces to *weighted Montgomery identity* (obtained by J. Pečarić in [4])

$$f(x)=\int_a^b w(t)\,f(t)\,dt+\int_a^b P_w(x,t)\,f'(t)\,dt \tag{1.3}$$

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where the weighted Peano kernel is

$$P_w(x,t) = \begin{cases} W(t), & a \leqslant t \leqslant x, \\ W(t) - 1, & x < t \leqslant b. \end{cases}$$

Finally, for the uniform normalized weighted function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ (1.3) reduces to *Montgomery identity* (see for instance [3])

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \int_a^b P(x,t) f'(t)\,dt \tag{1.4}$$

where $P(x,t)$ is the Peano kernel, defined by

$$P(x,t) = \begin{cases} \dfrac{t-a}{b-a}, & a \leqslant t \leqslant x, \\ \dfrac{t-b}{b-a}, & x < t \leqslant b. \end{cases}$$

Identity (1.1) was used in the same paper [2] to obtain the following weighted generalization of the Čebyšev inequality:

THEOREM 2. *Let $f,g : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, $f', g' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$, and w, φ as in the Theorem 1. Then*

$$\left|T\left(w,f,g,\varphi'\right)\right| \leqslant \frac{1}{\varphi(1)}\left(\int_a^b w(t) H^2(t)\,dt\right) \left\|f'\right\|_\infty \left\|g'\right\|_\infty \left\|\varphi'\right\|_\infty \tag{1.5}$$

where

$$T\left(w,f,g,\varphi'\right) = \int_a^b w(x)\,\varphi'\left(\int_a^x w(s)\,ds\right) f(x)\,g(x)\,dx$$
$$-\frac{1}{\varphi(1)}\left[\int_a^b w(x)\,\varphi'\left(\int_a^x w(s)\,ds\right) f(x)\,dx\right]\left[\int_a^b w(x)\,\varphi'\left(\int_a^x w(s)\,ds\right) g(x)\,dx\right]$$

and

$$H(x) = \int_a^b \left|P_{w,\varphi}(x,t)\right| dt.$$

N. S. Barnett and S. S. Dragomir in [1] also used the same identity (1.1) to obtain the next weighted generalization of the Ostrowski inequality, with the following notation for Lebesgue norm

$$\|f\|_{[a,b],\infty} = ess \sup_{t\in[a,b]} |f(t)|$$

and for $1 \leqslant p < \infty$

$$\|f\|_{[a,b],p} = \left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}}.$$

THEOREM 3. *Let* $\varphi : [0,1] \to \mathbb{R}$ *be continuous on* $[0,1]$, *differentiable on* $\langle a,b\rangle$, *with* $\varphi(0)=0$, $\varphi(1)\neq 0$ *and* $w:[a,b]\to[0,\infty\rangle$ *is some normalized weighted function. Then for any* $f:[a,b]\to\mathbb{R}$ *an absolutely continuous function and* $x\in[a,b]$ *we have*

$$\begin{aligned}\left|f(x)-\frac{1}{\varphi(1)}\int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right)f(t)\,dt\right| &\leqslant \frac{1}{\varphi(1)}\int_a^x \left|\varphi\left(\int_a^t w(s)\,ds\right)\right|\left|f'(t)\right|dt \\ &+\frac{1}{\varphi(1)}\int_x^b \left|\varphi\left(\int_a^t w(s)\,ds\right)-\varphi(1)\right|\left|f'(t)\right|dt. \qquad (1.6)\end{aligned}$$

If

$$H_1(x)=\int_a^x \left|\varphi\left(\int_a^t w(s)\,ds\right)\right|\left|f'(t)\right|dt$$

and

$$H_2(x)=\int_x^b \left|\varphi\left(\int_a^t w(s)\,ds\right)-\varphi(1)\right|\left|f'(t)\right|dt,$$

then for (p,q) *a pair of conjugate exponents* $1\leqslant p,q\leqslant\infty$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $f'\in L_q[a,x]$

$$H_1(x)\leqslant\left\|\varphi\left(\int_a^{\cdot} w(s)\,ds\right)\right\|_{[a,x],p}\left\|f'\right\|_{[a,x],q}$$

and for (r,s) *a pair of conjugate exponents* $1\leqslant r,s\leqslant\infty$, $\frac{1}{r}+\frac{1}{s}=1$ *and* $f'\in L_s[x,b]$

$$H_2(x)\leqslant\left\|\varphi\left(\int_a^{\cdot} w(s)\,ds\right)-\varphi(1)\right\|_{[x,b],r}\left\|f'\right\|_{[x,b],s}$$

The aim of this paper is to give another proof of the generalization of weighted Montgomery identity (1.1) from the paper [2], as well as another weighted generalization of the Čebyšev inequality (1.5) and Ostrowski inequality (1.6).

2. Some further results for Čebyšev-type inequalities

REMARK 1. N. Boukerrioua and A. Guezane-Lakoud obtained the new generalization of weighted Montgomery identity (1.1) by using the integration by parts of the term $\int_a^b P_{w,\varphi}(x,t)\,f'(t)\,dt$. It can also be obtained in a different way, directly from the weighted Montgomery identity (1.3). Indeed, if we take

$$\widetilde{w}(x)=\frac{1}{\varphi(1)}w(x)\,\varphi'\left(\int_a^x w(t)\,dt\right),\quad x\in[a,b]$$

in the weighted Montgomery identity (1.3)

$$f(x)=\int_a^b \widetilde{w}(t)\,f(t)\,dt+\int_a^b P_{\widetilde{w}}(x,t)\,f'(t)\,dt,$$

we obtain

$$\int_a^b P_{\widetilde{w}}(x,t) f'(t)\,dt = \int_a^x \widetilde{W}(t) f'(t)\,dt + \int_x^b \left(\widetilde{W}(t) - 1\right) f'(t)\,dt.$$

Since $\widetilde{w}(x)$ is a normalized weighted function:

$$\int_a^b \widetilde{w}(t)\,dt = \frac{1}{\varphi(1)} \int_a^b w(x)\,\varphi'\left(\int_a^x w(s)\,ds\right) dt = \frac{1}{\varphi(1)} \varphi(W(b) - W(a)) = 1,$$

and for $t \in [a,x]$ we have

$$\begin{aligned}\widetilde{W}(t) &= \int_a^t \widetilde{w}(x)\,dx = \frac{1}{\varphi(1)} \int_a^t w(x)\,\varphi'\left(\int_a^x w(t)\,dt\right) dx \\ &= \frac{1}{\varphi(1)} \int_a^t W'(x)\,\varphi'(W(x))\,dx = \frac{1}{\varphi(1)} \varphi(W(t)),\end{aligned}$$

while for $t \in \langle x,b]$

$$\begin{aligned}\widetilde{W}(t) - 1 &= -\int_t^b \widetilde{w}(x)\,dx = \frac{-1}{\varphi(1)} \int_t^b w(x)\,\varphi'\left(\int_a^x w(t)\,dt\right) dx \\ &= \frac{-1}{\varphi(1)} \int_t^b W'(x)\,\varphi'(W(x))\,dx = \frac{1}{\varphi(1)} \left(\varphi(W(t)) - \varphi(W(b))\right),\end{aligned}$$

this result coincides with the generalization of weighted Peano kernel $P_{w,\varphi}(x,t)$ defined by (1.2)

$$\int_a^b P_{\widetilde{w}}(x,t) f'(t)\,dt = \frac{1}{\varphi(1)} \int_a^b P_{w,\varphi}(x,t) f'(t)\,dt.$$

Thus, the identity obtained in a such way is equivalent to (1.1).

The following identity has been proved by J. Pečarić in [5]:

$$T(f,g,p) = \int_a^b \overline{P}(x) \int_a^x P(t)\,dg(t)\,df(x) + \int_a^b P(x) \int_x^b \overline{P}(t)\,dg(t)\,df(x), \tag{2.1}$$

where $f,g : [a,b] \to \mathbb{R}$ are two differentiable functions on $[a,b]$ and $p : [a,b] \to \mathbb{R}$ integrable, $P(x) = \int_a^x P(t)\,dt$, $\overline{P}(x) = \int_x^b P(t)\,dt$, for which the following holds $0 \leqslant P(x) \leqslant P(b)$, $\forall x \in [a,b]$ and

$$T(f,g,p) = \int_a^b p(x)\,dx \int_a^b p(x) f(x) g(x)\,dx - \int_a^b p(x) f(x)\,dx \int_a^b p(x) g(x)\,dx.$$

Using this result we can obtain the following result of the same type as inequality (1.5) in Theorem 2.

THEOREM 4. *Let $f,g : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f',g' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$, and w,φ as in the Theorem 1. Then*

$$\left|T\left(w,f,g,\varphi'\right)\right| \leqslant \frac{1}{\varphi(1)} \left(\int_a^b H(t)\,dt\right) \left\|f'\right\|_\infty \left\|g'\right\|_\infty \left\|\varphi'\right\|_\infty \tag{2.2}$$

where

$$H(x) = \int_a^b \left|P_{w,\varphi}(x,t)\right| dt.$$

Proof. If we take $P(x) = \frac{1}{\varphi(1)}\varphi(W(x))$, $p(x) = \frac{1}{\varphi(1)} w(x)\varphi'\left(\int_a^x w(t)\,dt\right)$ and $T(f,g,p) = \frac{1}{\varphi(1)} T(w,f,g,\varphi')$ in (2.1), we have

$$\begin{aligned} T(f,g,p) = &\frac{1}{\varphi(1)^2}\int_a^b (\varphi(1)-\varphi(W(x)))\left[\int_a^x \varphi(W(t))\,g'(t)\,dt\right] f'(x)\,dx \\ &+ \frac{1}{\varphi(1)^2}\int_a^b \varphi(W(x))\left[\int_x^b (\varphi(1)-\varphi(W(t)))\,g'(t)\,dt\right] f'(x)\,dx \end{aligned}$$

By mean value theorem we have

$$\begin{gathered} \varphi(1)-\varphi(W(x)) = \varphi'(\eta_x)(1-W(x)) \quad \text{for some } \eta_x \in [0,1], \\ \varphi(W(x)) = \varphi'(\xi_x)\,W(x) \quad \text{for some } \xi_x \in [0,1], \end{gathered}$$

so

$$\begin{aligned} T(f,g,p) \leqslant &\frac{1}{\varphi(1)^2}\left\|g'\right\|_\infty \int_a^b (\varphi(1)-\varphi(W(x)))\,f'(x)\left[\int_a^x |\varphi(W(t))|\,dt\right] dx \\ &+ \frac{1}{\varphi(1)^2}\left\|g'\right\|_\infty \int_a^b \varphi(W(x))\,f'(x)\left[\int_x^b |(\varphi(1)-\varphi(W(t)))|\,dt\right] dx \\ = &\frac{1}{\varphi(1)^2}\left\|g'\right\|_\infty \int_a^b \left(\varphi'(\eta_x)(1-W(x))\right) f'(x)\left[\int_a^x |\varphi(W(t))|\,dt\right] dx \\ &+ \frac{1}{\varphi(1)^2}\left\|g'\right\|_\infty \int_a^b \varphi'(\xi_x)\,W(x)\,f'(x)\left[\int_x^b |(\varphi(1)-\varphi(W(t)))|\,dt\right] dx \\ \leqslant &\frac{1}{\varphi(1)^2}\left\|\varphi'\right\|_\infty\left\|f'\right\|_\infty\left\|g'\right\|_\infty \int_a^b (1-W(x))\left[\int_a^x |\varphi(W(t))|\,dt\right] dx \\ &+ \frac{1}{\varphi(1)^2}\left\|\varphi'\right\|_\infty\left\|f'\right\|_\infty\left\|g'\right\|_\infty \int_a^b W(x)\left[\int_x^b |(\varphi(1)-\varphi(W(t)))|\,dt\right] dx \\ \leqslant &\frac{1}{\varphi(1)^2}\left\|\varphi'\right\|_\infty\left\|f'\right\|_\infty\left\|g'\right\|_\infty \int_a^b \left(\left[\int_a^x |\varphi(W(t))|\,dt\right] dx\right. \\ &\left.+ \int_x^b |(\varphi(1)-\varphi(W(t)))|\,dt\right) dx \\ = &\frac{1}{\varphi(1)^2}\left\|\varphi'\right\|_\infty\left\|f'\right\|_\infty\left\|g'\right\|_\infty \int_a^b \left(\int_a^b \left|P_{w,\varphi}(x,t)\right| dt\right) dx \\ = &\frac{1}{\varphi(1)^2}\left\|\varphi'\right\|_\infty\left\|f'\right\|_\infty\left\|g'\right\|_\infty \int_a^b H(x)\,dx. \end{aligned}$$

Consequently

$$T\left(w,f,g,\varphi'\right) \leqslant \frac{1}{\varphi(1)} \left\|\varphi'\right\|_{\infty} \left\|f'\right\|_{\infty} \left\|g'\right\|_{\infty} \int_a^b H(x)\,dx. \quad \square$$

3. Some further results for Ostrowski-type inequalities

THEOREM 5. *Let $f:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$, $w:[a,b]\to[0,\infty\rangle$ normalized weighted function, $\varphi:[0,1]\to\mathbb{R}$ differentiable on $[0,1]$ with $\varphi(0)=0$, $\varphi(1)\neq 0$ and $\varphi':[0,1]\to\mathbb{R}$ integrable on $[0,1]$. Additionally let (p,q) be a pair of conjugate exponents, that is $1\leqslant p,q\leqslant\infty$, $\frac{1}{p}+\frac{1}{q}=1$ and $f'\in L_p$. Then for any $x\in[a,b]$ the following inequality holds*

$$\begin{aligned}
&\left| f(x) - \frac{1}{\varphi(1)} \int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt \right| \\
&\qquad \leqslant \frac{1}{|\varphi(1)|} \left\|P_{w,\varphi}(x,\cdot)\right\|_q \left\|f'\right\|_p .
\end{aligned} \tag{3.1}$$

Proof. Using the formula (1.1) we have

$$\left| f(x) - \frac{1}{\varphi(1)} \int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt \right| = \left| \frac{1}{\varphi(1)} \int_a^b P_{w,\varphi}(x,t)\, f'(t)\,dt \right|$$

By applying the Hölder inequality we obtain the proof. $\square$

COROLLARY 1. *Let w and φ be as in the Theorem 5, $f'\in L_1$ and φ is a monotonic non-decreasing on $[0,1]$. Then for any $x\in[a,b]$ the following inequality holds*

$$\begin{aligned}
&\left| f(x) - \frac{1}{\varphi(1)} \int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt \right| \\
&\qquad \leqslant \left[\frac{1}{2} + \left| \frac{\varphi\left(\int_a^x w(t)\,dt\right)}{\varphi(1)} - \frac{1}{2} \right| \right] \left\|f'\right\|_1 .
\end{aligned}$$

Proof. We apply (3.1) with $p=1$

$$\left| f(x) - \frac{1}{\varphi(1)} \int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt \right| = \frac{1}{|\varphi(1)|} \left\|P_{w,\varphi}(x,\cdot)\right\|_\infty \left\|f'\right\|_1 .$$

Since

$$\begin{aligned}
\left\|P_{w,\varphi}(x,\cdot)\right\|_\infty &= \sup_{t\in[a,b]} \left|P_{w,\varphi}(x,t)\right| \\
&= \max\left\{ \sup_{t\in[a,x]} |\varphi(W(t))|,\ \sup_{t\in[x,b]} |\varphi(W(t)) - \varphi(1)| \right\}
\end{aligned}$$

and since φ is a monotonic non-decreasing on $[0,1]$, we have

$$\sup_{t\in[a,x]} |\varphi(W(t))| = \varphi(W(x))$$

$$\sup_{t\in[x,b]} |\varphi(W(t)) - \varphi(1)| = \varphi(1) - \varphi(W(x))$$

Using the formula $\max\{A,B\} = \frac{1}{2}(A+B+|A-B|)$ the proof follows. □

COROLLARY 2. *Let w and φ be as in the Theorem 5, $1 < p \leqslant \infty$, $f' \in L_p$. Then for any $x \in [a,b]$ the following inequality holds*

$$\left| f(x) - \frac{1}{\varphi(1)} \int_a^b w(t)\,\varphi'\left(\int_a^t w(s)\,ds\right) f(t)\,dt \right| \leqslant \frac{1}{|\varphi(1)|}\left(\int_a^b |t-x|\,w(t)\,dt\right)^{\frac{p-1}{p}} \|\varphi'\|_\infty \|f'\|_p .$$

Proof. We apply (3.1) with $p \neq 1$ $(q \neq \infty)$,

$$\left\|P_{w,\varphi}(x,\cdot)\right\|_q = \left(\int_a^b \left|P_{w,\varphi}(x,t)\right|^q dt\right)^{\frac{1}{q}} = \left(\int_a^x |\varphi(W(t))|^q\,dt + \int_x^b |\varphi(W(t)) - \varphi(1)|^q\,dt\right)^{\frac{1}{q}}.$$

By mean value theorem we have

$$\varphi(1) - \varphi(W(t)) = \varphi'(\eta_t)(1 - W(t)) \quad \text{for some } \eta_t \in [0,1],$$
$$\varphi(W(t)) = \varphi'(\xi_t)\,W(t) \quad \text{for some } \xi_t \in [0,1],$$

so

$$\int_a^x |\varphi(W(t))|^q\,dt = \int_a^x \left|\varphi'(\xi_t)\,W(t)\right|^q dt \leqslant \left(\|\varphi'\|_\infty\right)^q \int_a^x W(t)^q\,dt,$$

$$\int_x^b |\varphi(W(t)) - \varphi(1)|^q\,dt = \int_x^b \left|\varphi'(\eta_t)(1-W(t))\right|^q dt \leqslant \left(\|\varphi'\|_\infty\right)^q \int_x^b (1-W(t))^q\,dt.$$

Since $1 \leqslant q < \infty$ and $0 \leqslant W(t) \leqslant 1$ we have $W(t)^q \leqslant W(t)$, $t \in [a,b]$ and $(1-W(t))^q \leqslant (1-W(t))$, $t \in [a,b]$. Thus

$$\int_a^x W(t)^q\,dt \leqslant \int_a^x W(t)\,dt,$$

$$\int_x^b (1-W(t))^q\,dt \leqslant \int_x^b (1-W(t))\,dt.$$

By integration by parts we obtain

$$\int_a^x W(t)\,dt = \int_a^x (x-t)\,w(t)\,dt,$$

$$\int_x^b (1 - W(t))\, dt = \int_x^b \left(\int_t^b w(s)\, ds \right) dt = \int_x^b (t - x)\, w(t)\, dt.$$

Thus

$$\left\| P_{w,\varphi}(x,\cdot) \right\|_q \leqslant \left\| \varphi' \right\|_\infty \left(\int_a^b |t - x|\, w(t)\, dt \right)^{\frac{1}{q}}$$

and the proof follows. □

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A. Aglić Aljinović
Department of applied mathematics
Faculty of Electrical Engineering and Computing
University of Zagreb
Unska 3
10 000 Zagreb
Croatia
e-mail: andrea.aglic@fer.hr