Journal of
Mathematical
Inequalities
Volume 3, Number 3 (2009), 417–426

MORE ON THE TWO–POINT OSTROWSKI INEQUALITY

MARKO MATIĆ AND ŠIME UNGAR

Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. We improve the previous results of [7] on the L_p-version of an inequality similar to the two-point Ostrowski inequality of Matić and Pečarić [3].

1. Introduction

Given a function $f\colon [a,b] \to \mathbb{R}$ satisfying the Lipschitz condition with constant $M > 0$, and $a \leqslant c < d \leqslant b$, Matić and Pečarić [3] proved the following two-point Ostrowski inequality:

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c} \int_c^d f(x)\,\mathrm{d}x \right| \leqslant \frac{(c-a)^2+(b-d)^2}{2\,(c-a+b-d)} \cdot M\,.$$

This result was generalized by Pečarić, Perić and Vukelić in [5]. Further generalizations were done by Aglić Aljinović, Pečarić and Perić in [1], where they consider also the L_p-cases, $1 \leqslant p \leqslant \infty$, as well as the general case when $[c,d] \not\subseteq [a,b]$. Among other things, they proved that for $a \leqslant c < b \leqslant d$ and for a function f such that $|f'|^p$ is R-integrable on $[a,d]$, the following inequality holds:

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c} \int_c^d f(t)\,\mathrm{d}t \right|$$
$$\leqslant \left(\frac{1}{(q+1)(a-b+d-c)} \cdot \left(\frac{(d-b)^{q+1}}{(d-c)^{q-1}} - \frac{(c-a)^{q+1}}{(b-a)^{q-1}} \right) \right)^{\frac{1}{q}} \cdot \|f'\|_p\,.$$

Next, Dragomir [2] proved the following Ostrowski type inequality for a continuous function $f\colon [a,b] \to \mathbb{R}$ which is differentiable on (a,b):

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right| \leqslant \frac{b-a}{|x|} \left(\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right) \cdot \|f - \iota f'\|_\infty\,,$$

where $\iota(t) = t$, $t \in [a,b]$. These results have been generalized by Pečarić and Ungar in [6] and [7]. Here we will improve on these results, generally giving better estimates.

Mathematics subject classification (2000): 26D15, 26D10.
Keywords and phrases: Integral inequality, two-point Ostrowski, p-norm.

2. The main result

We will first consider the case of a function $f\colon [a,b] \to \mathbb{R}$ and a sub-segment $[c,d] \subseteq [a,b]$. The case of 'overlapping' intervals, i. e. when the intersection $[a,b] \cap [c,d]$ equals $[c,b]$ or $[a,d]$, will be dealt with in Section 4.

Now we state our main result. Throughout the paper, by $\iota\colon [a,b] \to \mathbb{R}$ we will denote the inclusion function $\iota(x) = x$.

THEOREM 1. *Let the function $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0 < a < b$. Then for $\frac{1}{p} + \frac{1}{q} = 1$, with $1 \leqslant p,q \leqslant \infty$, and numbers $a \leqslant c < d \leqslant b$, the following inequality holds:*

$$\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x - (d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \frac{1}{2}\,\|K\|_q \cdot \|f - \iota f'\|_p \tag{1}$$

where

$$K(u) = \begin{cases} (d^2-c^2)\left(\dfrac{a^2}{u^2} - 1\right) & a \leqslant u \leqslant c \\ b^2 - a^2 + c^2 - d^2 - \dfrac{b^2c^2 - a^2d^2}{u^2} & c \leqslant u \leqslant d \\ (d^2-c^2)\left(\dfrac{b^2}{u^2} - 1\right) & d \leqslant u \leqslant b \end{cases}. \tag{2}$$

First we state a simple lemma (for the proof see [6]):

LEMMA 2. *Let the function $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $a \cdot b > 0$. Then*

$$t\,f(x) - x\,f(t) = xt\int_x^t \big(f(u) - u\,f'(u)\big)\,\frac{1}{u^2}\,\mathrm{d}u \tag{3}$$

for all $x,t \in [a,b]$.

Proof of Theorem 1. Applying Lemma 2 to our function f and integrating on t over $[a,b]$, gives

$$\frac{b^2-a^2}{2}\,f(x) - x\int_a^b f(t)\,\mathrm{d}t$$
$$= x\int_a^b \left(t\int_x^t \big(f(u) - u\,f'(u)\big)\frac{1}{u^2}\,\mathrm{d}u\right)\mathrm{d}t \tag{4a}$$

and changing the order of integration we obtain

$$= -\int_a^x \left(\int_a^u \big(f(u) - uf'(u)\big)\frac{xt}{u^2}\,\mathrm{d}t\right)\mathrm{d}u + \int_x^b \left(\int_u^b \big(f(u) - uf'(u)\big)\frac{xt}{u^2}\,\mathrm{d}t\right)\mathrm{d}u \tag{4b}$$

$$= \frac{x}{2}\left(\int_a^x \left(\frac{a^2}{u^2} - 1\right)\big(f(u) - uf'(u)\big)\,\mathrm{d}u + \int_x^b \left(\frac{b^2}{u^2} - 1\right)\big(f(u) - uf'(u)\big)\,\mathrm{d}u\right). \tag{4c}$$

Integrating this identity on x over $[c,d]$, multiplying by 2, and again changing the order of integration, gives

$$(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t = \int_a^c I_1(u)\,\mathrm{d}u+\int_c^d I_2(u)\,\mathrm{d}u+\int_c^d I_3(u)\,\mathrm{d}u+\int_d^b I_4(u)\,\mathrm{d}u \tag{5}$$

where

$$I_1(u)=\int_c^d x\Big(\frac{a^2}{u^2}-1\Big)\big(f(u)-uf'(u)\big)\,\mathrm{d}x \tag{6a}$$

$$I_2(u)=\int_u^d x\Big(\frac{a^2}{u^2}-1\Big)\big(f(u)-uf'(u)\big)\,\mathrm{d}x \tag{6b}$$

$$I_3(u)=\int_c^u x\Big(\frac{b^2}{u^2}-1\Big)\big(f(u)-uf'(u)\big)\,\mathrm{d}x \tag{6c}$$

$$I_4(u)=\int_c^d x\Big(\frac{b^2}{u^2}-1\Big)\big(f(u)-uf'(u)\big)\,\mathrm{d}x. \tag{6d}$$

Evaluating these integrals, from (5) we obtain

$$(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t=\frac{1}{2}\int_a^b K(u)\big(f(u)-uf'(u)\big)\,\mathrm{d}u \tag{7}$$

where $K(u)$ is given by (2).

Applying the Hölder's inequality to (7) gives (1), proving the theorem. □

In order to consider the special cases $p=1$, $q=\infty$ and $p=\infty$, $q=1$, note that the function K is continuous and decreasing on $[a,c]$, increasing on $[c,d]$ and again decreasing on $[d,b]$, being zero at a, b, and $u_0=\sqrt{\frac{b^2c^2-a^2d^2}{b^2-a^2+c^2-d^2}}$ (see Fig. 1).

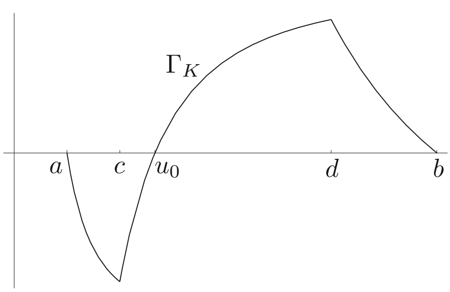

Figure 1. Function K

Therefore

$$\|K\|_\infty=\max_{u\in[a,b]}|K(u)|=\max\{-K(c),K(d)\}$$

$$=(d^2-c^2)\cdot\max\Big\{\frac{c^2-a^2}{c^2},\frac{b^2-d^2}{d^2}\Big\} \tag{8}$$

and

$$\begin{aligned}\|K\|_1 &= \int_a^b |K(u)|\,\mathrm{d}u \\ &= 2(a+b)(c+d)(b-a+c-d) - 4\sqrt{(b^2c^2-a^2d^2)(b^2-a^2+c^2-d^2)},\end{aligned} \tag{9}$$

and for $p = q = 2$ we obtain

$$\begin{aligned}\|K\|_2 &= \Big(\int_a^b |K(u)|^2\,\mathrm{d}u\Big)^{\frac{1}{2}} \\ &= \frac{2(d-c)}{\sqrt{3cd}}\sqrt{(b^2-a^2)(a^2d+b^2c)-3cd(b^2-a^2)+2cd(b-a)(c+d)}.\end{aligned} \tag{10}$$

This proves

COROLLARY 3. *Let the function $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0 < a < b$, and let $a \leqslant c < d \leqslant b$. Then*

$$\begin{aligned}&\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x - (d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right| \\ &\leqslant \Big((a+b)(c+d)(b-a+c-d)-2\sqrt{(b^2c^2-a^2d^2)(b^2-a^2+c^2-d^2)}\Big)\cdot\|f-\iota f'\|_\infty,\end{aligned} \tag{11}$$

$$\begin{aligned}&\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x - (d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right| \\ &\qquad \leqslant \frac{1}{2}(d^2-c^2)\cdot\max\left\{\frac{c^2-a^2}{c^2}, \frac{b^2-d^2}{d^2}\right\}\cdot\|f-\iota f'\|_1,\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x - (d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right| \\ &\leqslant \frac{d-c}{\sqrt{3cd}}\sqrt{(b^2-a^2)(a^2d+b^2c)-3cd(b^2-a^2)+2cd(b-a)(c+d)}\cdot\|f-\iota f'\|_2.\end{aligned} \tag{13}$$

3. Limit cases: $a = c$, $c = d$, and $d = b$

Let us first consider the case $a = c$. Then the left hand side in (7) becomes

$$\begin{aligned}&(b^2-a^2)\int_a^d f(x)\,\mathrm{d}x - (d^2-a^2)\int_a^b f(t)\,\mathrm{d}t \\ &\qquad = a^2\int_d^b f(x)\,\mathrm{d}x + b^2\int_a^d f(x)\,\mathrm{d}x - d^2\int_a^b f(x)\,\mathrm{d}x,\end{aligned} \tag{14}$$

and we get

$$a^2\int_d^b f(x)\,\mathrm{d}x+b^2\int_a^d f(x)\,\mathrm{d}x-d^2\int_a^b f(x)\,\mathrm{d}x=\frac{1}{2}\int_a^b G(u)\big(f(u)-uf'(u)\big)\,\mathrm{d}u \tag{15}$$

where

$$G(u)=\begin{cases}(b^2-d^2)\Big(1-\dfrac{a^2}{u^2}\Big) & a\leqslant u\leqslant d\\ (d^2-a^2)\Big(\dfrac{b^2}{u^2}-1\Big) & d\leqslant u\leqslant b\end{cases}. \tag{16}$$

Applying the Hölder's inequality, gives

COROLLARY 4. *Let the function $f\colon [a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$. Then for $\frac{1}{p}+\frac{1}{q}=1$, with $1\leqslant p,q\leqslant\infty$, and $a\leqslant d\leqslant b$, the following inequality holds:*

$$\left|a^2\int_d^b f(x)\,\mathrm{d}x+b^2\int_a^d f(x)\,\mathrm{d}x-d^2\int_a^b f(x)\,\mathrm{d}x\right|\leqslant\frac{1}{2}\,\|G\|_q\cdot\|f-\iota f'\|_p \tag{17}$$

where the function G is given by (16).

For the special cases $(p,q)=(1,\infty)$, $(p,q)=(\infty,1)$, and $(p,q)=(2,2)$ we have

COROLLARY 5. *Let the function $f\colon [a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$, and let $a\leqslant d\leqslant b$. Then the following three inequalities hold:*

$$\left|a^2\int_d^b f(x)\,\mathrm{d}x+b^2\int_a^d f(x)\,\mathrm{d}x-d^2\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\leqslant(b-d)(d-a)(b-a)\cdot\|f-\iota f'\|_\infty$$
$$\left|a^2\int_d^b f(x)\,\mathrm{d}x+b^2\int_a^d f(x)\,\mathrm{d}x-d^2\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\leqslant\frac{1}{2}(b^2-d^2)\Big(1-\frac{a^2}{b^2}\Big)\cdot\|f-\iota f'\|_1$$
$$\left|a^2\int_d^b f(x)\,\mathrm{d}x+b^2\int_a^d f(x)\,\mathrm{d}x-d^2\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\leqslant\frac{1}{\sqrt{3d}}(d-a)(b-d)\sqrt{(b-a)(a+b+2d)}\cdot\|f-\iota f'\|_2.$$

The case $d=b$ differs from the case $a=c$ only in that both sides in (14) and (15) change signs.

Let us now consider the limit case $d=c=:x$. By the Mean value Theorem it is reasonable to assume that $\frac{1}{d-c}\int_c^d f(s)\,\mathrm{d}s$ has the value $f(x)$. It will be more convenient, both for taking the required limits and for comparing the results with those in [7], to divide (1) by $(b-a)(d^2-c^2)$ and rewrite it in the following form:

$$\begin{aligned}\frac{a+b}{c+d}\cdot\frac{1}{d-c}\int_c^d f(x)\,\mathrm{d}x-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\ =\frac{1}{2(b-a)(d^2-c^2)}\int_a^b K(u)\,(f(u)-uf'(u))\,\mathrm{d}u\end{aligned}\tag{18}$$

where the function K is as in (2). Taking the appropriate limits in (18), we obtain

$$\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t=\frac{1}{2}\int_a^b k_x(u)\,(f(u)-uf'(u))\,\mathrm{d}u\tag{19}$$

where

$$k_x(u)=\begin{cases}\dfrac{1}{b-a}\left(\dfrac{a^2}{u^2}-1\right) & a\leqslant u\leqslant x\\ \dfrac{1}{b-a}\left(\dfrac{b^2}{u^2}-1\right) & x<u\leqslant b\end{cases}.\tag{20}$$

Note that the function k_x is discontinuous at x.

Aplying the Hölder's inequality we obtain

COROLLARY 6. *Let the function* $f\colon[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $0<a<b$. *Then for* $\frac{1}{p}+\frac{1}{q}=1$, *with* $1\leqslant p,q\leqslant\infty$, *and* $a\leqslant x\leqslant b$, *the following inequality holds:*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\leqslant\frac{1}{2}\,\|k_x\|_q\cdot\|f-\iota f'\|_p\,.$$

In general case, the integrals involved in calculating $\|k_x\|_q$ can be expressed in closed form only using Gamma and the Gauss's hypergeometric function ${}_2F_1$, giving

$$\begin{aligned}\|k_x\|_q=\Bigg(&\frac{b\,\Gamma(\frac{1}{2}-q)\,\Gamma(1+q)}{\sqrt{\pi}}-\frac{a\sqrt{\pi}\,\Gamma(1+q)}{\Gamma(\frac{1}{2}+q)}+x\,{}_2F_1\left(-\frac{1}{2},-q;\tfrac{1}{2};\frac{a^2}{x^2}\right)\\ &+x\frac{1}{2q-1}\left(\frac{b}{x}\right)^{2q}{}_2F_1\left(\frac{1}{2}-q,-q;\frac{3}{2}-q;\frac{x^2}{b^2}\right)\Bigg)^{\frac{1}{q}}.\end{aligned}\tag{21}$$

But in some special cases the norm $\|k_x\|_q$ can be calculated, and in particular we get:

COROLLARY 7. *Let the function* $f\colon[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $0<a<b$, *and let* $a\leqslant x\leqslant b$. *Then the following three inequalities hold:*

$$\begin{aligned}&\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\\ &\qquad\leqslant\frac{1}{b-a}\left(\frac{a^2+b^2}{2x}-(a+b)+x\right)\cdot\|f-\iota f'\|_\infty\end{aligned}\tag{22}$$

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \frac{1}{2(b-a)}\max\left\{1-\frac{a^2}{x^2},\frac{b^2}{x^2}-1\right\}\cdot\|f-\iota f'\|_1 \tag{23}$$

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \frac{1}{2(b-a)\sqrt{3}}\left(\left(1-\frac{a}{x}\right)^3(a+3x)+\left(\frac{b}{x}-1\right)^3(b+3x)\right)^{\frac{1}{2}}\cdot\|f-\iota f'\|_2. \tag{24}$$

The estimate (21) coincides with those in [2], [6] and [7], whereas (22) and (23) are better than those in [6] and [7].

4. Case of overlapping intervals

We turn now to the case when the line segments $[a,b]$ and $[c,d]$ overlap, i.e. $[a,b]\cap[b,d]$ equals $[c,b]$ or $[a,d]$. It suffices to consider the first case, $a\leqslant c<b\leqslant d$. The other one is obtained by interchanging $a\leftrightarrow c$ and $b\leftrightarrow d$.

First let us introduce a notation. For real numbers $\alpha\leqslant\gamma<\delta\leqslant\beta$ and a real function $\varphi\in L_p[\alpha,\beta]$, $1\leqslant p\leqslant\infty$, denote by

$$\|\varphi\|_{p,[\gamma,\delta]}:=\left(\int_\gamma^\delta|\varphi(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}$$

the L_p-norm of the restriction of φ to the sub-interval $[\gamma,\delta]\subseteq[\alpha,\beta]$. Obviously, for $[\gamma',\delta']\subseteq[\gamma,\delta]$, the following holds:

$$\|\varphi\|_{p,[\gamma',\delta']}\leqslant\|\varphi\|_{p,[\gamma,\delta]}\,. \tag{25}$$

We can now state our main result for overlapping intervals:

THEOREM 8. *Let $0<a\leqslant c<b\leqslant d$ and let the function $f\colon[a,d]\to\mathbb{R}$ be continuous on $[a,d]$ and differentiable on (a,d). Then for $\frac{1}{p}+\frac{1}{q}=1$, with $1\leqslant p,q\leqslant\infty$, the following inequality holds:*

$$\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^d f(t)\,\mathrm{d}t\right|\leqslant\frac{1}{2}\|L\|_{p,[a,d]}\cdot\|f-\iota f'\|_{q,[a,d]} \tag{26}$$

where

$$L(u)=\begin{cases}(d^2-c^2)\left(\dfrac{a^2}{u^2}-1\right) & a\leqslant u\leqslant c\\ b^2-a^2+c^2-d^2-\dfrac{b^2c^2-a^2d^2}{u^2} & c\leqslant u\leqslant b\\ (b^2-a^2)\left(1-\dfrac{d^2}{u^2}\right) & b\leqslant u\leqslant d\end{cases}. \tag{27}$$

Proof. We proceed as in the proof of Theorem 1: apply Lemma 2 to f, integrate on t over $[a,b]$, and change the order of integration to obtain (4c):

$$\begin{aligned}&\frac{b^2-a^2}{2}f(x)-x\int_a^b f(t)\,\mathrm{d}t\\&\quad=\frac{x}{2}\left(\int_a^x\left(\frac{a^2}{u^2}-1\right)\big(f(u)-uf'(u)\big)\mathrm{d}u+\int_x^b\left(\frac{b^2}{u^2}-1\right)\big(f(u)-uf'(u)\big)\mathrm{d}u\right).\end{aligned} \tag{4c}$$

Multiplying (4c) by 2, integrating on x over $[c,d]$, and changing the order of integration, gives

$$\begin{aligned}&(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\\&\quad=\int_a^c I_1(u)\,\mathrm{d}u+\int_c^d I_2(u)\,\mathrm{d}u+\int_c^d I_3(u)\,\mathrm{d}u-\int_b^d J_4(u)\,\mathrm{d}u\end{aligned} \tag{28}$$

where I_1, I_2, and I_3 are as in (6) (except that in (6b) and (6c) I_2 and I_3 are integrated over different segments), and

$$J_4(u)=\int_c^d x\left(\frac{d^2}{u^2}-1\right)\big(f(u)-u\,f'(u)\big)\,\mathrm{d}x. \tag{29}$$

As in the proof of Theorem 1, evaluating these integrals gives

$$(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t=\frac{1}{2}\int_a^d L(u)\big(f(u)-u\,f'(u)\big)\,\mathrm{d}u \tag{30}$$

where $L(u)$ is given by (26). Now apply the Hölder's inequality to obtain (25), which proves the theorem. □

In order to consider the special cases $p=1$, $q=\infty$ and $p=\infty$, $q=1$, note that the function L is continuous and negative, decreasing on $[a,c]$ from $L(a)=0$ to $L(c)$, on $[c,b]$ increasing/decreasing from $L(c)$ to $L(b)$, depending on a, b, c, and d, and increasing on $[b,d]$ from $L(b)$ to $L(d)=0$ (see Fig. 2).

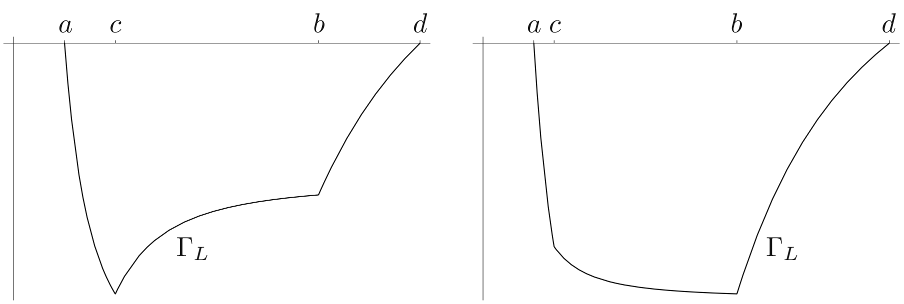

Figure 2. Function L

Therefore

$$\|L\|_{1,[a,d]} = \int_a^b |L(u)|\,\mathrm{d}u = 2(b-a)(d-c)(d-b+c-a) \tag{31}$$

and

$$\begin{aligned}\|L\|_{\infty,[a,d]} &= \max_{u\in[a,d]} |L(u)| = \max\{-L(c),-L(b)\}\\ &= \max\left\{\frac{(d^2-c^2)(c^2-a^2)}{c^2}, \frac{(b^2-a^2)(d^2-b^2)}{b^2}\right\}.\end{aligned} \tag{32}$$

In addition, for $p=q=2$ we have

$$\begin{aligned}\|L\|_{2,[a,d]} &= \left(\int_a^b |L(u)|^2\,\mathrm{d}u\right)^{\frac{1}{2}}\\ &= \frac{2}{\sqrt{3abc}}\sqrt{(b-a)(d-c)}\\ &\quad\times\Big(a^2b(d-c)(a+b)+bc\big(2b^3+(c+d)(d^2-2c^2-3b^2+3bc)\big)\\ &\quad+ac\big(2b^3-(c+d)(3b^2-3bc+d^2)\big)\Big)^{\frac{1}{2}}.\end{aligned} \tag{33}$$

This proves

COROLLARY 9. *Let the function $f\colon [a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$, and let $a\leqslant c<d\leqslant b$. Then*

$$\begin{aligned}&\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right|\\ &\qquad\leqslant (b-a)(d-c)(d-b+c-a)\cdot\|f-\iota f'\|_{\infty,[a,d]},\end{aligned} \tag{34}$$

and

$$\begin{aligned}&\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right|\\ &\quad\leqslant \frac{1}{2}\max\left\{\frac{(d^2-c^2)(c^2-a^2)}{c^2}, \frac{(b^2-a^2)(d^2-b^2)}{b^2}\right\}\cdot\|f-\iota f'\|_{1,[a,d]},\end{aligned} \tag{35}$$

where $\iota(t)=t$, $t\in[a,b]$.

REFERENCES

[1] A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ, I. PERIĆ, *Estimations of the difference of two weighted integral means via weighted Montgomery identity*, Mathematical Inequalities & Applications, **7**, 3 (2004), 315–336.

[2] S. S. DRAGOMIR, *An inequality of Ostrowski type via Pompeiu's mean value theorem*, Journal of Inequalities in Pure and Applied Mathematics, **6**, 3 (2005), Article 83, 1–9.

[3] M. MATIĆ, J. PEČARIĆ, *Two-point Ostrowski inequality*, Mathematical Inequalities & Applications, **4**, 2 (2001), 215–221.

[4] A. OSTROWSKI, *Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert*, Comment. Math. Hel., **10** (1938), 226–227.

[5] J. PEČARIĆ, I. PERIĆ, A. VUKELIĆ, *Estimations of the difference of two integral means via Euler-type identities*, Mathematical Inequalities & Applications, **7**, 3 (2004), 365–378.

[6] J. PEČARIĆ, Š. UNGAR, *On an inequality of Ostrowski type*, Journal of Inequalities in Pure and Applied Mathematics, **7**, 4 (2006), Article 151, 1–5.

[7] J. PEČARIĆ, Š. UNGAR, *On two-point Ostrowski inequality*, Mathematical Inequalities & Applications, to appear.

[8] D. POMPEIU, *Sur une proposition analogue au théorème des accroissements finis*, Mathematica (Cluj, Romania), **22** (1946), 143–146.

(Received November 24, 2008)

Marko Matić
FESB, Mathematical Department
University of Split
e-mail: mmatic@pmfst.hr

Šime Ungar
Department of Mathematics
University of Zagreb
e-mail: ungar@math.hr