GENERAL FOUR–POINT QUADRATURE FORMULAE WITH APPLICATIONS FOR α–L–HÖLDER TYPE FUNCTIONS

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. In this paper we establish a variant of general four-point weighted quadrature formula. This new formula is used to present several Ostrowski type inequalities for α-L-Hölder functions.

1. Introduction

The most elementary quadrature rules in four nodes are Simpson's $3/8$ rule based on the following four point formula

$$\int_a^b f(t)\mathrm{d}t = \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480}f^{(4)}(\xi),$$
(1.1)

where $\xi \in [a,b]$, and Lobatt's rule based on the formula

$$\int_{-1}^1 f(t)\mathrm{d}t = \frac{1}{6}\left[f(-1) + 5f\left(-\frac{\sqrt{5}}{5}\right) + 5f\left(\frac{\sqrt{5}}{5}\right) + f(1)\right] - \frac{2}{23625}f^{(6)}(\eta), \quad (1.2)$$

where $\eta \in [-1,1]$. Formula (1.1) is valid for any function f with continuous fourth derivative $f^{(4)}$ on $[a,b]$ and formula (1.2) for any function f with continuous sixth derivative $f^{(6)}$ on $[-1,1]$.

Let $f:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$ and $f':[a,b]\to\mathbb{R}$ integrable on $[a,b]$. Then the Montgomery identity holds [4]

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \int_a^b P(x,t)f'(t)\,\mathrm{d}t,$$
(1.3)

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t) = \begin{cases} \dfrac{t-a}{b-a}, & a \leqslant t \leqslant x \\ \dfrac{t-b}{b-a}, & x < t \leqslant b \end{cases}.$$

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Now, let us suppose that $w : [a,b] \to [0,\infty\rangle$ is some probability density function satisfying $\int_a^b w(t)\,dt = 1$, and $W(t) = \int_a^t w(x)\,dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. In [5] J. E. Pečarić proved a weighted generalization of the well known Montgomery identity

$$f(x) = \int_a^b w(t)f(t)\,dt + \int_a^b P_w(x,t)f'(t)\,dt,$$

where the weighted Peano kernel is defined by

$$P_w(x,t) = \begin{cases} W(t), & a \leqslant t \leqslant x \\ W(t) - 1, & x < t \leqslant b \end{cases}.$$

In [2] G. A. Anastassiou used the following equality (which is an immediate consequence of the well known Taylor's formula):

$$g(y) - g(x) - \sum_{i=1}^{n} \frac{g^{(i)}(x)}{i!}(y-x)^i = \frac{1}{(n-1)!}\int_x^y \left(g^{(n)}(t) - g^{(n)}(x)\right)(y-t)^{n-1}\,dt,$$

where $g : I \subseteq \mathbb{R} \to \mathbb{R}$ is such that for some $n \in \mathbb{N}$ the derivative $g^{(n)}$ exists for all $t \in [a,b] \subset I$ $(a < b)$ and x,y belong to $[a,b]$.

These two identities were used in the recent paper [1], where A. Aglić Aljinović and J. Pečarić introduced two new extensions of the weighted Montgomery identity.

In this paper we continue our work which has been started in [3]. Namely, we use one of those new weighted Montgomery identities to establish for each $x \in (a,(a+b)/2]$ a general four-point quadrature formula of the type

$$\int_a^b w(t)f(t)\,dt = \left(\frac{1}{2} - A(x)\right)[f(a) + f(b)] + A(x)[f(x) + f(a+b-x)] + R(f,w;x),$$

(1.4)

where $R(f,w;x)$ is the reminder and $A : (a,(a+b)/2] \to \mathbb{R}$ a real function. The obtained formula is used to prove several Ostrowski-type inequalities for α-L-Hölder functions.

2. General four-point quadrature formula

Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$ and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous function for some $n \geqslant 2$. In the recent paper [1] the following extension of the Montgomery identity was proved for each $x \in [a,b]$:

$$\int_a^b w(t)f(t)\,dt = f(x) - \sum_{i=0}^{n-1} \frac{f^{(i+1)}(a)}{i!}\int_a^x W(t)(t-a)^i\,dt$$

$$+ \sum_{i=0}^{n-1} \frac{f^{(i+1)}(b)}{i!}\int_x^b (1 - W(t))(t-b)^i\,dt$$

$$+ \frac{1}{(n-2)!} \left\{ \int_a^x W(t) \left[\int_a^t \left(f^{(n)}(a) - f^{(n)}(s) \right) (t-s)^{n-2} \, ds \right] dt \right.$$

$$\left. + \int_x^b (1 - W(t)) \left[\int_t^b \left(f^{(n)}(b) - f^{(n)}(s) \right) (t-s)^{n-2} \, ds \right] dt \right\},$$

$$(2.1)$$

where $w : [a,b] \to [0,\infty\rangle$ is some probability density function.

In this section we use (2.1) to study for each number $x \in \left(a, \frac{a+b}{2} \right]$ the general four-point quadrature formula of the type (1.4).

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ exists on $[a,b]$ for some $n \geqslant 2$. We introduce the following notation for each $x \in \left(a, \frac{a+b}{2} \right]$

$$D(x) = \left(\frac{1}{2} - A(x) \right) [f(a) + f(b)] + A(x) [f(x) + f(a+b-x)].$$

Further, we define

$$t_n(x) = \left(\frac{1}{2} - A(x) \right) \left\{ \sum_{i=0}^{n-1} \frac{f^{(i+1)}(b)}{i!} \left[\int_a^b (1 - W(t)) (t-b)^i \, dt \right] \right.$$

$$- \sum_{i=0}^{n-1} \frac{f^{(i+1)}(a)}{i!} \left[\int_a^b W(t) (t-a)^i \, dt \right] \right\}$$

$$+ A(x) \left\{ \sum_{i=0}^{n-1} \frac{f^{(i+1)}(b)}{i!} \left[\int_x^b (1 - W(t)) (t-b)^i \, dt + \int_{a+b-x}^b (1 - W(t)) (t-b)^i \, dt \right] \right.$$

$$- \sum_{i=0}^{n-1} \frac{f^{(i+1)}(a)}{i!} \left[\int_a^x W(t) (t-a)^i \, dt + \int_a^{a+b-x} W(t) (t-a)^i \, dt \right] \right\}$$

and

$$T_n(x) = \left(\frac{1}{2} - A(x) \right) \left[T_n^a(b) + T_n^b(a) \right]$$

$$+ A(x) \left[T_n^a(x) + T_n^b(x) + T_n^a(a+b-x) + T_n^b(a+b-x) \right],$$

where

$$T_n^a(x) = \frac{1}{(n-2)!} \int_a^x W(t) \left[\int_a^t \left(f^{(n)}(a) - f^{(n)}(s) \right) (t-s)^{n-2} \, ds \right] dt,$$

$$T_n^b(x) = \frac{1}{(n-2)!} \int_x^b (1 - W(t)) \left[\int_t^b \left(f^{(n)}(b) - f^{(n)}(s) \right) (t-s)^{n-2} \, ds \right] dt.$$

In the next theorem we establish our variant of generalized four-point quadrature formula based on the extended Montgomery identity which will play the key role in this paper.

THEOREM 1. *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $w : [a,b] \to [0,\infty\rangle$ be some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous function for some $n \geqslant 2$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following identity holds*

$$\int_a^b w(t) f(t) \, dt = D(x) + t_n(x) + T_n(x). \tag{2.2}$$

Proof. We put $x \equiv a, x \equiv x, x \equiv a+b-x$ and $x \equiv b$ in (2.1) to obtain four new formulae. After multiplying these four formulae by $1/2 - A(x), A(x), A(x)$ and $1/2 - A(x)$ respectively and adding we get (2.2). $\quad\square$

Before we give an estimation of the term

$$\left| \int_a^b w(t) f(t) \, dt - D(x) - t_n(x) \right|,$$

let us recall that a function $\varphi : [a,b] \to \mathbb{R}$ is said to be of α-L-Hölder type if $|\varphi(x) - \varphi(y)| \leqslant L|x-y|^\alpha$ for every $x, y \in [a,b]$, where $L > 0$ and $\alpha \in (0,1]$. We will also make use of the Beta function of Euler type which is for $x, y > 0$ defined by

$$B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt.$$

In what follows for $x \in \left(a, \frac{a+b}{2}\right]$ we denote

$$W(x,t) = \begin{cases} W(t), & a \leqslant t \leqslant x \\ 1 - W(t), & x < t \leqslant b \end{cases},$$

$$U_n(x,t) = \begin{cases} (t-a)^{\alpha+n-1}, & a \leqslant t \leqslant x \\ (b-t)^{\alpha+n-1}, & x < t \leqslant b \end{cases}.$$

THEOREM 2. *Suppose that all the assumptions of Theorem 1 hold and additionally assume that for some $L > 0$ and $\alpha \in (0,1]$ $f^{(n)} : [a,b] \to \mathbb{R}$ is an α-L-Hölder type function. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following inequalities hold*

$$\left| \int_a^b w(t) f(t) \, dt - D(x) - t_n(x) \right|$$

$$\leqslant \frac{B(\alpha+1, n-1)}{(n-2)!} L \left\{ \left| \frac{1}{2} - A(x) \right| \left[\int_a^b W(t)(t-a)^{\alpha+n-1} \, dt \right. \right.$$

$$+ \int_a^b (1 - W(t))(b-t)^{\alpha+n-1} \, dt \Big]$$

$$+ |A(x)| \left[\int_a^b W(x,t) U_n(x,t) \, dt + \int_a^b W(a+b-x,t) U_n(a+b-x,t) \, dt \right] \right\}$$

$$\leqslant \frac{2B(\alpha+1, n-1)}{(\alpha+n)(n-2)!} L \left\{ \left| \frac{1}{2} - A(x) \right| (b-a)^{\alpha+n} + |A(x)| \left[(x-a)^{\alpha+n} + (b-x)^{\alpha+n} \right] \right\}.$$

Proof. From (2.2) we have

$$\left| \int_a^b w(t) f(t) \, dt - D(x) - t_n(x) \right|$$

$$= \left| \left(\frac{1}{2} - A(x) \right) \left[T_n^a(b) + T_n^b(a) \right] \right.$$

$$+ A(x) \left[T_n^a(x) + T_n^b(x) + T_n^a(a+b-x) + T_n^b(a+b-x) \right] \bigg|$$

$$\leqslant \left| \frac{1}{2} - A(x) \right| \left[\left| T_n^a(b) \right| + \left| T_n^b(a) \right| \right]$$

$$+ \left| A(x) \right| \left[\left| T_n^a(x) \right| + \left| T_n^b(x) \right| + \left| T_n^a(a+b-x) \right| + \left| T_n^b(a+b-x) \right| \right] \qquad (2.3)$$

Since $f^{(n)}$ is an α-L-Hölder type function, from (2.3) we obtain

$$\left| \int_a^b w(t) f(t) \, dt - D(x) - t_n(x) \right|$$

$$\leqslant \frac{\left| \frac{1}{2} - A(x) \right|}{(n-2)!} L \left\{ \int_a^b W(t) \left[\int_a^t (s-a)^\alpha (t-s)^{n-2} \, ds \right] dt \right.$$

$$+ \int_a^b (1 - W(t)) \left[\int_t^b (b-s)^\alpha (s-t)^{n-2} \, ds \right] dt$$

$$+ \frac{|A(x)|}{(n-2)!} L \left\{ \int_a^x W(t) \left[\int_a^t (s-a)^\alpha (t-s)^{n-2} \, ds \right] dt \right.$$

$$+ \int_a^{a+b-x} W(t) \left[\int_a^t (s-a)^\alpha (t-s)^{n-2} \, ds \right] dt$$

$$+ \int_x^b (1 - W(t)) \left[\int_t^b (b-s)^\alpha (s-t)^{n-2} \, ds \right] dt$$

$$+ \int_{a+b-x}^b (1 - W(t)) \left[\int_t^b (b-s)^\alpha (s-t)^{n-2} \, ds \right] dt \right\} \qquad (2.4)$$

The first integral over ds in (2.4) can be written as

$$\int_a^t (s-a)^\alpha (t-s)^{n-2} \, ds = (t-a)^{\alpha+n-2} \int_a^t \left(\frac{s-a}{t-a} \right)^\alpha \left(\frac{t-s}{t-a} \right)^{n-2} \, ds$$

$$= \left[u = \frac{s-a}{t-a} \right] = (t-a)^{\alpha+n-1} \int_0^1 u^\alpha (1-u)^{n-2} \, du$$

$$= (t-a)^{\alpha+n-1} B(\alpha+1, n-1).$$

Similarly can be done with other integrals in (2.4), so we obtain

$$\left| \int_a^b w(t) f(t) \, dt - D(x) - t_n(x) \right|$$

$$\leqslant \frac{B(\alpha+1, n-1)}{(n-2)!} L \left\{ \left| \frac{1}{2} - A(x) \right| \left[\int_a^b W(t) (t-a)^{\alpha+n-1} \, dt \right. \right.$$

$$+ \int_a^b \left(1 - W\left(t\right)\right) \left(b-t\right)^{\alpha+n-1} dt \right] \tag{2.5}$$

$$+ |A\left(x\right)| \left[\int_a^b W\left(x,t\right) U_n\left(x,t\right) dt + \int_a^b W\left(a+b-x,t\right) U_n\left(a+b-x,t\right) dt \right] \right\}.$$

Since we have

$$0 \leqslant W\left(t\right) \leqslant 1, \quad t \in [a,b],$$

from (2.5) we obtain

$$\left| \frac{1}{2} - A\left(x\right) \right| \left[\int_a^b W\left(t\right) \left(t-a\right)^{\alpha+n-1} dt + \int_a^b \left(1 - W\left(t\right)\right) \left(b-t\right)^{\alpha+n-1} dt \right]$$

$$+ |A\left(x\right)| \left[\int_a^b W\left(x,t\right) U_n\left(x,t\right) dt + \int_a^b W\left(a+b-x,t\right) U_n\left(a+b-x,t\right) dt \right]$$

$$\leqslant \frac{2}{\alpha+n} \left\{ \left| \frac{1}{2} - A\left(x\right) \right| \left(b-a\right)^{\alpha+n} + |A\left(x\right)| \left[\left(x-a\right)^{\alpha+n} + \left(b-x\right)^{\alpha+n} \right] \right\},$$

which completes the proof. $\square$

3. Nonweighted four-point quadrature formula and applications

Here we define

$$\widehat{t}_n\left(x\right) = \left(\frac{1}{2} - A\left(x\right) \right) \sum_{i=0}^{n-1} \left[\left(-1\right)^i f^{(i+1)}\left(b\right) - f^{(i+1)}\left(a\right) \right] \frac{\left(b-a\right)^{i+1}}{i!\left(i+2\right)}$$

$$+ A\left(x\right) \sum_{i=0}^{n-1} \left[\left(-1\right)^i f^{(i+1)}\left(b\right) - f^{(i+1)}\left(a\right) \right] \frac{\left(x-a\right)^{i+2} + \left(b-x\right)^{i+2}}{i!\left(i+2\right)\left(b-a\right)},$$

and

$$\widehat{T}_n\left(x\right) = \left(\frac{1}{2} - A\left(x\right) \right) \left[\widehat{T}_n^a\left(b\right) + \widehat{T}_n^b\left(a\right) \right]$$

$$+ A\left(x\right) \left[\widehat{T}_n^a\left(x\right) + \widehat{T}_n^b\left(x\right) + \widehat{T}_n^a\left(a+b-x\right) + \widehat{T}_n^b\left(a+b-x\right) \right],$$

where

$$\widehat{T}_n^a\left(x\right) = \frac{1}{\left(n-2\right)!\left(b-a\right)} \int_a^x \left(t-a\right) \left[\int_a^t \left(f^{(n)}\left(a\right) - f^{(n)}\left(s\right) \right) \left(t-s\right)^{n-2} ds \right] dt,$$

$$\widehat{T}_n^b\left(x\right) = \frac{1}{\left(n-2\right)!\left(b-a\right)} \int_x^b \left(b-t\right) \left[\int_t^b \left(f^{(n)}\left(b\right) - f^{(n)}\left(s\right) \right) \left(t-s\right)^{n-2} ds \right] dt. \tag{3.1}$$

COROLLARY 1 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geqslant 2$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following identity holds*

$$\frac{1}{b-a} \int_a^b f\left(t\right) dt = D\left(x\right) + \widehat{t}_n\left(x\right) + \widehat{T}_n\left(x\right). \tag{3.2}$$

Proof. This is a special case of Theorem 1 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. $\quad\square$

COROLLARY 2 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that that for some $n \geqslant 2$, $L > 0$ and $\alpha \in (0,1]$ the derivative $f^{(n-1)}$ is absolutely continuous and $f^{(n)} : [a,b] \to \mathbb{R}$ is an α-L-Hölder type function. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D(x) - \widehat{t}_n(x) \right|$$

$$\leqslant \frac{2B(\alpha+1, n-1)}{(b-a)(\alpha+n+1)(n-2)!}$$

$$\times L \left\{ \left| \frac{1}{2} - A(x) \right| (b-a)^{\alpha+n+1} + |A(x)| \left[(x-a)^{\alpha+n+1} + (b-x)^{\alpha+n+1} \right] \right\}.$$

Proof. This is a special case of Theorem 2 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. $\quad\square$

The next step is setting

$$A(x) = \frac{(b-a)^2}{12(x-a)(b-x)}.$$

This special choice of the function A enables us to establish our generalizations of the well known Simpson's 3/8 formula (1.1) and Lobatt's formula (1.2). We will also show how to apply the results of Section 2 to obtain some error estimates for these quadrature rules if they involve α-L-Hölder type functions.

3.1. $x = \frac{2a+b}{3}$

Suppose that all the assumptions of Corollary 1 hold. Then our generalization of Simpson's 3/8 formula states

$$\frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t = D\left(\frac{2a+b}{3}\right) + \widehat{t}_n\left(\frac{2a+b}{3}\right) + \widehat{T}_n\left(\frac{2a+b}{3}\right),$$

where

$$D\left(\frac{2a+b}{3}\right) = \frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right],$$

$$\widehat{t}_n\left(\frac{2a+b}{3}\right) = \frac{1}{8}\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b) - f^{(i+1)}(a)\right] \frac{\left(3^{i+1} + 2^{i+2} + 1\right)(b-a)^{i+1}}{3^{i+1} i!(i+2)}$$

and

$$\widehat{T}_n\left(\frac{2a+b}{3}\right) = \frac{1}{8}\left[\widehat{T}_n^a(b) + 3\widehat{T}_n^a\left(\frac{2a+b}{3}\right) + 3\widehat{T}_n^b\left(\frac{2a+b}{3}\right) \right.$$

$$\left. + 3\widehat{T}_n^a\left(\frac{a+2b}{3}\right) + 3\widehat{T}_n^b\left(\frac{a+2b}{3}\right) + \widehat{T}_n^b(a) \right].$$

Here $\widehat{T}_n^a(x)$ and $\widehat{T}_n^b(x)$ are as in (3.1).

COROLLARY 3 *Suppose that all the assumptions of Corollary 2 hold. Then we have*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - D\left(\frac{2a+b}{3}\right) - \widehat{t_n}\left(\frac{2a+b}{3}\right) \right|$$

$$\leqslant \frac{B(\alpha+1, n-1)\left(3^{\alpha+n} + 2^{\alpha+n+1} + 1\right)(b-a)^{\alpha+n}}{4 \cdot 3^{\alpha+n}(\alpha+n+1)(n-2)!} L.$$

Proof. This is a special case of Corollary 2 for $x = \frac{2a+b}{3}$. □

EXAMPLE 1 *Let us consider the special case $n = 2$ in Corollary 3 (that is if f' is absolutely continuous and f'' is of α-L-Hölder type). We have*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - D\left(\frac{2a+b}{3}\right) - \widehat{t_2}\left(\frac{2a+b}{3}\right) \right|$$

$$\leqslant \frac{\left(3^{\alpha+2} + 2^{\alpha+3} + 1\right)(b-a)^{\alpha+2}}{4 \cdot 3^{\alpha+2}(\alpha+1)(\alpha+3)} L,$$

where

$$D\left(\frac{2a+b}{3}\right) = \frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right],$$

$$\widehat{t_2}\left(\frac{2a+b}{3}\right) = \frac{b-a}{12}\left\{ 2\left[f'(b) - f'(a) \right] - \left[f''(b) + f''(a) \right](b-a) \right\}.$$

3.2. $[a,b] = [-1, 1]$, $x = -\frac{\sqrt{5}}{5}$

Suppose that all the assumptions of Corollary 1 hold. Then our generalization of Lobatt's formula states

$$\frac{1}{2} \int_{-1}^1 f(t)\, dt = D\left(-\frac{\sqrt{5}}{5}\right) + \widehat{t_n}\left(-\frac{\sqrt{5}}{5}\right) + \widehat{T_n}\left(-\frac{\sqrt{5}}{5}\right),$$

where

$$D\left(-\frac{\sqrt{5}}{5}\right) = \frac{1}{12}\left[f(-1) + 5f\left(-\frac{\sqrt{5}}{5}\right) + 5f\left(\frac{\sqrt{5}}{5}\right) + f(1) \right],$$

$$\widehat{t_n}\left(-\frac{\sqrt{5}}{5}\right) = \frac{1}{12} \sum_{i=0}^{n-1} \left[(-1)^i f^{(i+1)}(1) - f^{(i+1)}(-1) \right]$$

$$\times \frac{\left[2^{i+2} \cdot 5^{i+1} + \left(5 - \sqrt{5}\right)^{i+2} + \left(5 + \sqrt{5}\right)^{i+2} \right]}{2 \cdot 5^{i+1} i! (i+2)}$$

and

$$\widehat{T}_n\left(-\frac{\sqrt{5}}{5}\right) = \frac{1}{12}\left[\widehat{T}_n^{-1}(1) + 5\widehat{T}_n^{-1}\left(-\frac{\sqrt{5}}{5}\right) + 5\widehat{T}_n^{1}\left(-\frac{\sqrt{5}}{5}\right)\right.$$
$$\left. + 5\widehat{T}_n^{-1}\left(\frac{\sqrt{5}}{5}\right) + 5\widehat{T}_n^{1}\left(\frac{\sqrt{5}}{5}\right) + \widehat{T}_n^{1}(-1)\right].$$

Here $\widehat{T}_n^a(x)$ and $\widehat{T}_n^b(x)$ are again as in (3.1).

COROLLARY 4 *Suppose that all the assumptions of Corollary 2 hold. Then we have*

$$\left|\frac{1}{2}\int_{-1}^{1} f(t)\,\mathrm{d}t - D\left(-\frac{\sqrt{5}}{5}\right) - \widehat{t}_n\left(-\frac{\sqrt{5}}{5}\right)\right|$$
$$\leqslant \frac{B(\alpha+1,n-1)\left(2^{\alpha+n+1}\cdot 5^{\alpha+n} + \left(5-\sqrt{5}\right)^{\alpha+n+1} + \left(5+\sqrt{5}\right)^{\alpha+n+1}\right)}{12\cdot 5^{\alpha+n}\,(\alpha+n+1)(n-2)!}L.$$

Proof. This is a special case of Corollary 2 for $[a,b] = [-1,1]$ and $x = -\frac{\sqrt{5}}{5}$. $\square$

EXAMPLE 2 *Let us consider again the special case $n = 2$ in Corollary 4 (that is if f' is absolutely continuous and f'' is of α-L-Hölder type). We have*

$$\left|\frac{1}{2}\int_{-1}^{1} f(t)\,\mathrm{d}t - D\left(-\frac{\sqrt{5}}{5}\right) - \widehat{t}_2\left(-\frac{\sqrt{5}}{5}\right)\right|$$
$$\leqslant \frac{2^{\alpha+3}\cdot 5^{\alpha+2} + \left(5-\sqrt{5}\right)^{\alpha+3} + \left(5+\sqrt{5}\right)^{\alpha+3}}{12\cdot 5^{\alpha+2}\,(\alpha+1)(\alpha+3)}L.$$

where

$$D\left(-\frac{\sqrt{5}}{5}\right) = \frac{1}{12}\left[f(-1) + 5f\left(-\frac{\sqrt{5}}{5}\right) + 5f\left(\frac{\sqrt{5}}{5}\right) + f(1)\right],$$

$$\widehat{t}_2\left(-\frac{\sqrt{5}}{5}\right) = \frac{1}{3}\left[f'(1) - f'(-1) - f''(1) - f''(-1)\right].$$

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