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# GENERAL FOUR–POINT QUADRATURE FORMULAE WITH APPLICATIONS FOR $\alpha$–$L$–HÖLDER TYPE FUNCTIONS

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*Dedicated to Professor Josip Pečarić*
*on the occasion of his 60th birthday*

*Abstract.* In this paper we establish a variant of general four-point weighted quadrature formula. This new formula is used to present several Ostrowski type inequalities for $\alpha$-$L$-Hölder functions.

## 1. Introduction

The most elementary quadrature rules in four nodes are Simpson's $3/8$ rule based on the following four point formula

$$\int_a^b f(t)\mathrm{d}t = \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{(b-a)^5}{6480}f^{(4)}(\xi), \tag{1.1}$$

where $\xi \in [a,b]$, and Lobatt's rule based on the formula

$$\int_{-1}^1 f(t)\mathrm{d}t = \frac{1}{6}\left[f(-1)+5f\left(-\frac{\sqrt{5}}{5}\right)+5f\left(\frac{\sqrt{5}}{5}\right)+f(1)\right]-\frac{2}{23625}f^{(6)}(\eta), \tag{1.2}$$

where $\eta \in [-1,1]$. Formula (1.1) is valid for any function $f$ with continuous fourth derivative $f^{(4)}$ on $[a,b]$ and formula (1.2) for any function $f$ with continuous sixth derivative $f^{(6)}$ on $[-1,1]$.

Let $f:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$ and $f':[a,b]\to\mathbb{R}$ integrable on $[a,b]$. Then the Montgomery identity holds [4]

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \int_a^b P(x,t)\,f'(t)\,\mathrm{d}t, \tag{1.3}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t) = \begin{cases} \dfrac{t-a}{b-a}, & a \leqslant t \leqslant x \\ \dfrac{t-b}{b-a}, & x < t \leqslant b \end{cases}.$$

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Now, let us suppose that $w : [a,b] \to [0,\infty\rangle$ is some probability density function satisfying $\int_a^b w(t)\,\mathrm{d}t = 1$, and $W(t) = \int_a^t w(x)\,\mathrm{d}x$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. In [5] J. E. Pečarić proved a weighted generalization of the well known Montgomery identity

$$f(x) = \int_a^b w(t) f(t)\,\mathrm{d}t + \int_a^b P_w(x,t) f'(t)\,\mathrm{d}t,$$

where the weighted Peano kernel is defined by

$$P_w(x,t) = \begin{cases} W(t), & a \leqslant t \leqslant x \\ W(t) - 1, & x < t \leqslant b \end{cases}.$$

In [2] G. A. Anastassiou used the following equality (which is an immediate consequence of the well known Taylor's formula):

$$g(y) - g(x) - \sum_{i=1}^{n} \frac{g^{(i)}(x)}{i!}(y-x)^i = \frac{1}{(n-1)!}\int_x^y \left(g^{(n)}(t) - g^{(n)}(x)\right)(y-t)^{n-1}\,\mathrm{d}t,$$

where $g : I \subseteq \mathbb{R} \to \mathbb{R}$ is such that for some $n \in \mathbb{N}$ the derivative $g^{(n)}$ exists for all $t \in [a,b] \subset I$ $(a < b)$ and $x,y$ belong to $[a,b]$.

These two identities were used in the recent paper [1], where A. Aglić Aljinović and J. Pečarić introduced two new extensions of the weighted Montgomery identity.

In this paper we continue our work which has been started in [3]. Namely, we use one of those new weighted Montgomery identities to establish for each $x \in (a,(a+b)/2]$ a general four-point quadrature formula of the type

$$\int_a^b w(t) f(t)\,\mathrm{d}t = \left(\frac{1}{2} - A(x)\right)[f(a) + f(b)] + A(x)[f(x) + f(a+b-x)] + R(f,w;x), \tag{1.4}$$

where $R(f,w;x)$ is the reminder and $A : (a,(a+b)/2] \to \mathbb{R}$ a real function. The obtained formula is used to prove several Ostrowski-type inequalities for $\alpha$-$L$-Hölder functions.

## 2. General four-point quadrature formula

Let $I$ be an open interval in $\mathbb{R}$, $[a,b] \subset I$ and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous function for some $n \geqslant 2$. In the recent paper [1] the following extension of the Montgomery identity was proved for each $x \in [a,b]$:

$$\int_a^b w(t) f(t)\,\mathrm{d}t = f(x) - \sum_{i=0}^{n-1} \frac{f^{(i+1)}(a)}{i!}\int_a^x W(t)(t-a)^i\,\mathrm{d}t$$
$$+ \sum_{i=0}^{n-1} \frac{f^{(i+1)}(b)}{i!}\int_x^b (1 - W(t))(t-b)^i\,\mathrm{d}t$$

$$+\frac{1}{(n-2)!}\left\{\int_a^x W(t)\left[\int_a^t \left(f^{(n)}(a)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\right.$$
$$\left.+\int_x^b (1-W(t))\left[\int_t^b \left(f^{(n)}(b)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\right\}, \tag{2.1}$$

where $w:[a,b]\to[0,\infty\rangle$ is some probability density function.

In this section we use (2.1) to study for each number $x\in\left(a,\frac{a+b}{2}\right]$ the general four-point quadrature formula of the type (1.4).

Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ exists on $[a,b]$ for some $n\geqslant 2$. We introduce the following notation for each $x\in\left(a,\frac{a+b}{2}\right]$

$$D(x)=\left(\frac{1}{2}-A(x)\right)[f(a)+f(b)]+A(x)[f(x)+f(a+b-x)].$$

Further, we define

$$t_n(x)=\left(\frac{1}{2}-A(x)\right)\left\{\sum_{i=0}^{n-1}\frac{f^{(i+1)}(b)}{i!}\left[\int_a^b (1-W(t))(t-b)^i\,\mathrm{d}t\right]\right.$$
$$\left.-\sum_{i=0}^{n-1}\frac{f^{(i+1)}(a)}{i!}\left[\int_a^b W(t)(t-a)^i\,\mathrm{d}t\right]\right\}$$
$$+A(x)\left\{\sum_{i=0}^{n-1}\frac{f^{(i+1)}(b)}{i!}\left[\int_x^b (1-W(t))(t-b)^i\,\mathrm{d}t+\int_{a+b-x}^b (1-W(t))(t-b)^i\,\mathrm{d}t\right]\right.$$
$$\left.-\sum_{i=0}^{n-1}\frac{f^{(i+1)}(a)}{i!}\left[\int_a^x W(t)(t-a)^i\,\mathrm{d}t+\int_a^{a+b-x} W(t)(t-a)^i\,\mathrm{d}t\right]\right\}$$

and

$$T_n(x)=\left(\frac{1}{2}-A(x)\right)\left[T_n^a(b)+T_n^b(a)\right]$$
$$+A(x)\left[T_n^a(x)+T_n^b(x)+T_n^a(a+b-x)+T_n^b(a+b-x)\right],$$

where

$$T_n^a(x)=\frac{1}{(n-2)!}\int_a^x W(t)\left[\int_a^t \left(f^{(n)}(a)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t,$$
$$T_n^b(x)=\frac{1}{(n-2)!}\int_x^b (1-W(t))\left[\int_t^b \left(f^{(n)}(b)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t.$$

In the next theorem we establish our variant of generalized four-point quadrature formula based on the extended Montgomery identity which will play the key role in this paper.

THEOREM 1. *Let $I$ be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $w : [a,b] \to [0,\infty\rangle$ be some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous function for some $n \geqslant 2$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following identity holds*

$$\int_a^b w(t) f(t)\,\mathrm{d}t = D(x) + t_n(x) + T_n(x). \tag{2.2}$$

*Proof.* We put $x \equiv a, x \equiv x, x \equiv a+b-x$ and $x \equiv b$ in (2.1) to obtain four new formulae. After multiplying these four formulae by $1/2 - A(x), A(x), A(x)$ and $1/2 - A(x)$ respectively and adding we get (2.2). □

Before we give an estimation of the term

$$\left| \int_a^b w(t) f(t)\,\mathrm{d}t - D(x) - t_n(x) \right|,$$

let us recall that a function $\varphi : [a,b] \to \mathbb{R}$ is said to be of $\alpha$-$L$-Hölder type if $|\varphi(x) - \varphi(y)| \leqslant L|x-y|^{\alpha}$ for every $x, y \in [a,b]$, where $L > 0$ and $\alpha \in (0,1]$. We will also make use of the Beta function of Euler type which is for $x, y > 0$ defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

In what follows for $x \in \left(a, \frac{a+b}{2}\right]$ we denote

$$W(x,t) = \begin{cases} W(t), & a \leqslant t \leqslant x \\ 1 - W(t), & x < t \leqslant b \end{cases},$$

$$U_n(x,t) = \begin{cases} (t-a)^{\alpha+n-1}, & a \leqslant t \leqslant x \\ (b-t)^{\alpha+n-1}, & x < t \leqslant b \end{cases}.$$

THEOREM 2. *Suppose that all the assumptions of Theorem 1 hold and additionally assume that for some $L > 0$ and $\alpha \in (0,1]$ $f^{(n)} : [a,b] \to \mathbb{R}$ is an $\alpha$-$L$-Hölder type function. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following inequalities hold*

$$\begin{aligned} &\left| \int_a^b w(t) f(t)\,\mathrm{d}t - D(x) - t_n(x) \right| \\ &\quad \leqslant \frac{B(\alpha+1, n-1)}{(n-2)!} L \left\{ \left| \frac{1}{2} - A(x) \right| \left[ \int_a^b W(t)(t-a)^{\alpha+n-1}\,\mathrm{d}t \right.\right. \\ &\quad\quad \left. + \int_a^b (1 - W(t))(b-t)^{\alpha+n-1}\,\mathrm{d}t \right] \\ &\quad\quad \left. + |A(x)| \left[ \int_a^b W(x,t) U_n(x,t)\,\mathrm{d}t + \int_a^b W(a+b-x,t) U_n(a+b-x,t)\,\mathrm{d}t \right] \right\} \\ &\quad \leqslant \frac{2B(\alpha+1, n-1)}{(\alpha+n)(n-2)!} L \left\{ \left| \frac{1}{2} - A(x) \right| (b-a)^{\alpha+n} + |A(x)| \left[ (x-a)^{\alpha+n} + (b-x)^{\alpha+n} \right] \right\}. \end{aligned}$$

*Proof.* From (2.2) we have

$$\begin{aligned}
&\left|\int_a^b w(t)f(t)\,\mathrm{d}t - D(x) - t_n(x)\right| \\
&\quad = \left|\left(\frac{1}{2} - A(x)\right)\left[T_n^a(b) + T_n^b(a)\right]\right. \\
&\qquad \left. + A(x)\left[T_n^a(x) + T_n^b(x) + T_n^a(a+b-x) + T_n^b(a+b-x)\right]\right| \\
&\quad \leqslant \left|\frac{1}{2} - A(x)\right|\left[\left|T_n^a(b)\right| + \left|T_n^b(a)\right|\right] \\
&\qquad + |A(x)|\left[\left|T_n^a(x)\right| + \left|T_n^b(x)\right| + \left|T_n^a(a+b-x)\right| + \left|T_n^b(a+b-x)\right|\right] \qquad (2.3)
\end{aligned}$$

Since $f^{(n)}$ is an $\alpha$-$L$-Hölder type function, from (2.3) we obtain

$$\begin{aligned}
&\left|\int_a^b w(t)f(t)\,\mathrm{d}t - D(x) - t_n(x)\right| \\
&\quad \leqslant \frac{\left|\frac{1}{2} - A(x)\right|}{(n-2)!}L\left\{\int_a^b W(t)\left[\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\right. \\
&\qquad + \int_a^b (1-W(t))\left[\int_t^b (b-s)^\alpha (s-t)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t \\
&\qquad + \frac{|A(x)|}{(n-2)!}L\left\{\int_a^x W(t)\left[\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\right. \\
&\qquad + \int_a^{a+b-x} W(t)\left[\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t \\
&\qquad + \int_x^b (1-W(t))\left[\int_t^b (b-s)^\alpha (s-t)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t \\
&\qquad \left. + \int_{a+b-x}^b (1-W(t))\left[\int_t^b (b-s)^\alpha (s-t)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\right\} \qquad (2.4)
\end{aligned}$$

The first integral over d$s$ in (2.4) can be written as

$$\begin{aligned}
\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s &= (t-a)^{\alpha+n-2}\int_a^t \left(\frac{s-a}{t-a}\right)^\alpha \left(\frac{t-s}{t-a}\right)^{n-2}\mathrm{d}s \\
&= \left[u = \frac{s-a}{t-a}\right] = (t-a)^{\alpha+n-1}\int_0^1 u^\alpha (1-u)^{n-2}\,\mathrm{d}u \\
&= (t-a)^{\alpha+n-1}B(\alpha+1, n-1).
\end{aligned}$$

Similarly can be done with other integrals in (2.4), so we obtain

$$\begin{aligned}
&\left|\int_a^b w(t)f(t)\,\mathrm{d}t - D(x) - t_n(x)\right| \\
&\quad \leqslant \frac{B(\alpha+1, n-1)}{(n-2)!}L\left\{\left|\frac{1}{2} - A(x)\right|\left[\int_a^b W(t)(t-a)^{\alpha+n-1}\,\mathrm{d}t\right.\right.
\end{aligned}$$

$$+\int_a^b (1-W(t))(b-t)^{\alpha+n-1}\,\mathrm{d}t\Big] \tag{2.5}$$
$$+|A(x)|\left[\int_a^b W(x,t)U_n(x,t)\,\mathrm{d}t+\int_a^b W(a+b-x,t)U_n(a+b-x,t)\,\mathrm{d}t\right]\Big\}.$$

Since we have

$$0\leqslant W(t)\leqslant 1,\quad t\in[a,b],$$

from (2.5) we obtain

$$\left|\frac{1}{2}-A(x)\right|\left[\int_a^b W(t)(t-a)^{\alpha+n-1}\,\mathrm{d}t+\int_a^b (1-W(t))(b-t)^{\alpha+n-1}\,\mathrm{d}t\right]$$
$$+|A(x)|\left[\int_a^b W(x,t)U_n(x,t)\,\mathrm{d}t+\int_a^b W(a+b-x,t)U_n(a+b-x,t)\,\mathrm{d}t\right]$$
$$\leqslant\frac{2}{\alpha+n}\left\{\left|\frac{1}{2}-A(x)\right|(b-a)^{\alpha+n}+|A(x)|\left[(x-a)^{\alpha+n}+(b-x)^{\alpha+n}\right]\right\},$$

which completes the proof. □

## 3. Nonweighted four-point quadrature formula and applications

Here we define

$$\widehat{t}_n(x)=\left(\frac{1}{2}-A(x)\right)\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b)-f^{(i+1)}(a)\right]\frac{(b-a)^{i+1}}{i!\,(i+2)}$$
$$+A(x)\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b)-f^{(i+1)}(a)\right]\frac{(x-a)^{i+2}+(b-x)^{i+2}}{i!\,(i+2)(b-a)},$$

and

$$\widehat{T}_n(x)=\left(\frac{1}{2}-A(x)\right)\left[\widehat{T}_n^a(b)+\widehat{T}_n^b(a)\right]$$
$$+A(x)\left[\widehat{T}_n^a(x)+\widehat{T}_n^b(x)+\widehat{T}_n^a(a+b-x)+\widehat{T}_n^b(a+b-x)\right],$$

where

$$\widehat{T}_n^a(x)=\frac{1}{(n-2)!\,(b-a)}\int_a^x (t-a)\left[\int_a^t \left(f^{(n)}(a)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t,$$
$$\widehat{T}_n^b(x)=\frac{1}{(n-2)!\,(b-a)}\int_x^b (b-t)\left[\int_t^b \left(f^{(n)}(b)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t. \tag{3.1}$$

COROLLARY 1 *Let $I$ be an open interval in $\mathbb{R}$, $[a,b]\subset I$, and let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geqslant 2$. Then for each $x\in\left(a,\frac{a+b}{2}\right]$ the following identity holds*

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t=D(x)+\widehat{t}_n(x)+\widehat{T}_n(x). \tag{3.2}$$

*Proof.* This is a special case of Theorem 1 for $w(t)=\frac{1}{b-a}$, $t\in[a,b]$. $\square$

COROLLARY 2 *Let $I$ be an open interval in $\mathbb{R}$, $[a,b]\subset I$, and let $f:I\to\mathbb{R}$ be such that that for some $n\geqslant 2$, $L>0$ and $\alpha\in(0,1]$ the derivative $f^{(n-1)}$ is absolutely continuous and $f^{(n)}:[a,b]\to\mathbb{R}$ is an $\alpha$-$L$-Hölder type function. Then for each $x\in\left(a,\frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-D(x)-\widehat{t}_n(x)\right|\\&\quad\leqslant\frac{2B(\alpha+1,n-1)}{(b-a)(\alpha+n+1)(n-2)!}\\&\qquad\times L\left\{\left|\frac{1}{2}-A(x)\right|(b-a)^{\alpha+n+1}+|A(x)|\left[(x-a)^{\alpha+n+1}+(b-x)^{\alpha+n+1}\right]\right\}.\end{aligned}$$

*Proof.* This is a special case of Theorem 2 for $w(t)=\frac{1}{b-a}$, $t\in[a,b]$. $\square$

The next step is setting

$$A(x)=\frac{(b-a)^2}{12(x-a)(b-x)}.$$

This special choice of the function $A$ enables us to establish our generalizations of the well known Simpson's $3/8$ formula (1.1) and Lobatt's formula (1.2). We will also show how to apply the results of Section 2 to obtain some error estimates for these quadrature rules if they involve $\alpha$-$L$-Hölder type functions.

### 3.1. $x=\frac{2a+b}{3}$

Suppose that all the assumptions of Corollary 1 hold. Then our generalization of Simpson's $3/8$ formula states

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t=D\left(\frac{2a+b}{3}\right)+\widehat{t}_n\left(\frac{2a+b}{3}\right)+\widehat{T}_n\left(\frac{2a+b}{3}\right),$$

where

$$D\left(\frac{2a+b}{3}\right)=\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right],$$

$$\widehat{t}_n\left(\frac{2a+b}{3}\right)=\frac{1}{8}\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b)-f^{(i+1)}(a)\right]\frac{\left(3^{i+1}+2^{i+2}+1\right)(b-a)^{i+1}}{3^{i+1}i!(i+2)}$$

and

$$\begin{aligned}\widehat{T}_n\left(\frac{2a+b}{3}\right)=\frac{1}{8}&\left[\widehat{T}_n^a(b)+3\widehat{T}_n^a\left(\frac{2a+b}{3}\right)+3\widehat{T}_n^b\left(\frac{2a+b}{3}\right)\right.\\&\left.+3\widehat{T}_n^a\left(\frac{a+2b}{3}\right)+3\widehat{T}_n^b\left(\frac{a+2b}{3}\right)+\widehat{T}_n^b(a)\right].\end{aligned}$$

Here $\widehat{T}_n^a(x)$ and $\widehat{T}_n^b(x)$ are as in (3.1 ).

COROLLARY 3 *Suppose that all the assumptions of Corollary 2 hold. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D\left(\frac{2a+b}{3}\right) - \widehat{t}_n\left(\frac{2a+b}{3}\right)\right| \leqslant \frac{B(\alpha+1,n-1)\left(3^{\alpha+n}+2^{\alpha+n+1}+1\right)(b-a)^{\alpha+n}}{4\cdot 3^{\alpha+n}(\alpha+n+1)(n-2)!}L.$$

*Proof.* This is a special case of Corollary 2 for $x=\frac{2a+b}{3}$. □

EXAMPLE 1 *Let us consider the special case $n=2$ in Corollary 3 (that is if $f'$ is absolutely continuous and $f''$ is of $\alpha$-$L$-Hölder type). We have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D\left(\frac{2a+b}{3}\right) - \widehat{t}_2\left(\frac{2a+b}{3}\right)\right| \leqslant \frac{\left(3^{\alpha+2}+2^{\alpha+3}+1\right)(b-a)^{\alpha+2}}{4\cdot 3^{\alpha+2}(\alpha+1)(\alpha+3)}L,$$

*where*

$$D\left(\frac{2a+b}{3}\right) = \frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right],$$

$$\widehat{t}_2\left(\frac{2a+b}{3}\right) = \frac{b-a}{12}\left\{2\left[f'(b)-f'(a)\right]-\left[f''(b)+f''(a)\right](b-a)\right\}.$$

**3.2.** $[a,b]=[-1,1],\, x=-\frac{\sqrt{5}}{5}$

Suppose that all the assumptions of Corollary 1 hold. Then our generalization of Lobatt's formula states

$$\frac{1}{2}\int_{-1}^{1} f(t)\,\mathrm{d}t = D\left(-\frac{\sqrt{5}}{5}\right)+\widehat{t}_n\left(-\frac{\sqrt{5}}{5}\right)+\widehat{T}_n\left(-\frac{\sqrt{5}}{5}\right),$$

where

$$D\left(-\frac{\sqrt{5}}{5}\right) = \frac{1}{12}\left[f(-1)+5f\left(-\frac{\sqrt{5}}{5}\right)+5f\left(\frac{\sqrt{5}}{5}\right)+f(1)\right],$$

$$\widehat{t}_n\left(-\frac{\sqrt{5}}{5}\right) = \frac{1}{12}\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(1)-f^{(i+1)}(-1)\right] \times \frac{\left[2^{i+2}\cdot 5^{i+1}+\left(5-\sqrt{5}\right)^{i+2}+\left(5+\sqrt{5}\right)^{i+2}\right]}{2\cdot 5^{i+1} i!\,(i+2)}$$

and

$$\widehat{T}_n\left(-\frac{\sqrt{5}}{5}\right)=\frac{1}{12}\left[\widehat{T}_n^{-1}(1)+5\widehat{T}_n^{-1}\left(-\frac{\sqrt{5}}{5}\right)+5\widehat{T}_n^{1}\left(-\frac{\sqrt{5}}{5}\right)\right.$$
$$\left.+5\widehat{T}_n^{-1}\left(\frac{\sqrt{5}}{5}\right)+5\widehat{T}_n^{1}\left(\frac{\sqrt{5}}{5}\right)+\widehat{T}_n^{1}(-1)\right].$$

Here $\widehat{T}_n^a(x)$ and $\widehat{T}_n^b(x)$ are again as in (3.1 ).

COROLLARY 4 *Suppose that all the assumptions of Corollary* 2 *hold. Then we have*

$$\left|\frac{1}{2}\int_{-1}^{1}f(t)\,\mathrm{d}t-D\left(-\frac{\sqrt{5}}{5}\right)-\widehat{t}_n\left(-\frac{\sqrt{5}}{5}\right)\right|$$
$$\leqslant\frac{B(\alpha+1,n-1)\left(2^{\alpha+n+1}\cdot 5^{\alpha+n}+\left(5-\sqrt{5}\right)^{\alpha+n+1}+\left(5+\sqrt{5}\right)^{\alpha+n+1}\right)}{12\cdot 5^{\alpha+n}(\alpha+n+1)(n-2)!}L.$$

*Proof.* This is a special case of Corollary 2 for $[a,b]=[-1,1]$ and $x=-\frac{\sqrt{5}}{5}$. $\square$

EXAMPLE 2 *Let us consider again the special case* $n=2$ *in Corollary* 4 *(that is if* $f'$ *is absolutely continuous and* $f''$ *is of* $\alpha$*-L-Hölder type). We have*

$$\left|\frac{1}{2}\int_{-1}^{1}f(t)\,\mathrm{d}t-D\left(-\frac{\sqrt{5}}{5}\right)-\widehat{t}_2\left(-\frac{\sqrt{5}}{5}\right)\right|$$
$$\leqslant\frac{2^{\alpha+3}\cdot 5^{\alpha+2}+\left(5-\sqrt{5}\right)^{\alpha+3}+\left(5+\sqrt{5}\right)^{\alpha+3}}{12\cdot 5^{\alpha+2}(\alpha+1)(\alpha+3)}L.$$

*where*

$$D\left(-\frac{\sqrt{5}}{5}\right)=\frac{1}{12}\left[f(-1)+5f\left(-\frac{\sqrt{5}}{5}\right)+5f\left(\frac{\sqrt{5}}{5}\right)+f(1)\right],$$

$$\widehat{t}_2\left(-\frac{\sqrt{5}}{5}\right)=\frac{1}{3}\left[f'(1)-f'(-1)-f''(1)-f''(-1)\right].$$

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