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LOG–CONVEXITY OF COMBINATORIAL SEQUENCES FROM THEIR CONVEXITY

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*Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday*

Abstract. A sequence $(x_n)_{n\geqslant 0}$ of positive real numbers is log-convex if the inequality $x_n^2 \leqslant x_{n-1}x_{n+1}$ is valid for all $n \geqslant 1$. We show here how the problem of establishing the log-convexity of a given combinatorial sequence can be reduced to examining the ordinary convexity of related sequences. The new method is then used to prove that the sequence of Motzkin numbers is log-convex.

1. Introduction

The problem of determining the logarithmic convexity/concavity of combinatorial sequences has attracted a considerable attention over the course of last decade. This resulted in extension of the classical repertoire [4, 5, 6, 7, 14, 15, 16] by several new methods based on matrix techniques [1], calculus [10, 11], interlacing [9], finite differences [8, 12] and sequence transformations [3, 18]. Those methods have been successfully applied to a wide range of combinatorial sequences given by two- or more-term recurrences with polynomial coefficients. The main reason of their effectiveness lies in their conceptual simplicity and (almost) automaticity. In a typical case, one starts from the recurrence defining a combinatorial sequence, passes to the recurrence for the sequence of quotients of successive elements and then shows that this sequence is increasing by checking some simple conditions. The procedure can be performed almost mechanically and does not rely on ingenuity and imagination necessary for constructing intricate injections characteristic for combinatorial proofs [5, 7, 14]. However, the conceptual simplicity does not always translate to simplicity of technical implementation. Our experience with calculus, interlacing, and finite difference methods points to the nonlinearity of the quotient recurrences as the main source of technical difficulties. In this paper we present a method that does not rely on nonlinear recurrences. This is achieved by adapting a classical result from the theory of functions that characterizes the log-convexity in terms of ordinary convexity. However, there is a price to be paid: instead of dealing with one (nonlinear) recurrence, we will have to examine an uncountable family of linear recurrences. Fortunately, it turns out that the whole family can be

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treated in a uniform way. The final result is a method that has all the applicability and robustness of the calculus/finite difference methods, while being free of most of their technical difficulties. The application of the method is illustrated by proving that the sequence of Motzkin numbers is log-convex.

2. Preliminaries

We start by recalling the definitions of convexity and log-convexity of sequences and functions.

A real sequence $(x_n)_{n\geqslant 0}$ is *convex* if $x_n \leqslant \frac{1}{2}(x_{n-1}+x_{n+1})$ for $n \geqslant 0$. A sequence $(x_n)_{n\geqslant 0}$ of positive real numbers is *log-convex* if $x_n^2 \leqslant x_{n-1}x_{n+1}$ is valid for $n \geqslant 1$. Obviously, the log-convexity of a sequence is equivalent to the ordinary convexity of the sequence of its logarithms. This suggests a way of reducing the log-convexity to the convexity. However, this approach is not suitable for combinatorial sequences given by two- or more-term recurrences, since those recurrences do not translate nicely into recurrences for their logarithms.

An alternative way of defining log-convexity of a sequence of positive real numbers is via its quotient sequence. For a given positive sequence $(x_n)_{n\geqslant 0}$ its quotient sequence $(q_n)_{n\geqslant 1}$ is defined by $q_n = \frac{x_n}{x_{n-1}}$. A sequence (x_n) is log-convex if and only if its quotient sequence q_n is (weakly) increasing. It is easy to see that if a combinatorial sequence (x_n) is given by a recurrence $x_n = R(n)x_{n-1} + S(n)x_{n-2}$ for $n \geqslant 2$ and some initial conditions x_0 and x_1, its quotient sequence (q_n) satisfies the recurrence $q_n = R(n) + \frac{S(n)}{q_{n-1}}$ for $n \geqslant 1$ with the initial condition $q_1 = \frac{x_1}{x_0}$. The last recurrence is nonlinear, and this nonlinearity leads to to the complications mentioned in introduction.

Let us get back to reducing the log-convexity to the convexity. It turns out that there is a way of achieving this goal while preserving linearity of the recurrences defining (a_n).

A function $f: I \to \mathbb{R}$ is *convex* if $f((1-t)a+tb) \leqslant (1-t)f(a)+tf(b)$ for all $t \in [0,1]$ and all $[a,b] \subseteq I$. A positive function $f: I \to \mathbb{R}$ is *log-convex* if its logarithm $\log f(x)$ is a convex function. Formally this can be written as $f((1-t)a+tb) \leqslant f(a)^{(1-t)}f(b)^t$ for all $t \in [0,1]$ and all $[a,b] \subseteq I$.

The log-convexity has a geometrical interpretation that parallels the familiar "below the secant" geometrical interpretation of ordinary convexity: The portion of the graph of a log-convex function f over a segment $[a,b]$ lies below the graph of the exponential function $\alpha e^{\beta x}$ whose values at a and b coincide with the corresponding values of f [2].

A sum of convex functions is a convex function. This is reflected in the fact that a product of log-convex functions is a log-convex function. A bit less obvious is the fact that also a sum of log-convex functions is itself log-convex. This has been known for long time - there are proofs from the first half of the last century. A new and elegant proof appeared recently in Russian literature [2], based on the following characterization of log-convex functions.

THEOREM A. *A function $f(x)$ is log-convex if and only if the function $e^{\alpha x}f(x)$ is convex for all $\alpha \in \mathbb{R}$.*

The result is attributed to P. Montel, who gave a proof in his 1928 paper on subharmonic functions [13].

By translating Theorem A into context of sequences, by considering restrictions of functions to the positive integers, we obtain a characterization of log-convex sequences in terms of convexity.

COROLLARY 1. *A sequence* $(x_n)_{n\geqslant 0}$ *of positive real numbers is log-convex if and only if the sequence* $(a^n x_n)_{n\geqslant 0}$ *is convex for all* $a > 0$.

3. Combinatorial sequences

The above characterization can be further simplified if we restrict our attention to the class of combinatorial sequences. We call a sequence $(x_n)_{n\geqslant 0}$ *combinatorial* if it is increasing without bounds and all its elements are positive integers.

COROLLARY 2. *A combinatorial sequence* $(x_n)_{n\geqslant 0}$ *is log-convex if and only if the sequence* $(a^n x_n)_{n\geqslant 0}$ *is convex for all* $0 < a \leqslant 1$.

Proof. For all $a > 1$ the sequence (a^n) is strictly increasing and convex. If a combinatorial sequence is convex, then the sequence $(a^n x_n)$ for $a > 1$ will be convex, as a product of two increasing convex sequences. Hence, it suffices to check that $(a^n x_n)$ is convex for all $0 < a \leqslant 1$.

Now take an $a \in (0,1]$ and form the sequence $(a^n x_n)$. This sequence is convex if and only if $a^{n+1}x_{n+1} + a^{n-1}x_{n-1} \geqslant 2a^n x_n$, or, equivalently, $ax_{n+1} + \frac{1}{a}x_{n-1} \geqslant 2x_n$. The AGM inequality now implies that the left-hand side is at least $2\sqrt{x_{n+1}x_{n-1}}$, and this, in turn, leads to $x_{n+1}x_{n-1} \geqslant x_n^2$. $\square$

Corollary 2 is the main result of the present paper. Its strength is most evident on those combinatorial sequences that are defined by linear recurrences, and it follows from the fact that the sequences $(a^n x_n)_{n\geqslant 0}$ satisfy linear recurrences closely related to the one that defines (x_n). In the next section we show how Corollary 2 is used to establish the log-convexity of a combinatorial sequence defined by a two-term linear recurrence with polynomial coefficients.

4. A case study: Motzkin numbers

The sequence (M_n) of Motzkin numbers has many combinatorial interpretations; we refer the reader to [17] for a non-exhaustive list. It satisfies a linear two-term recurrence:

$$M_n = \frac{2n+1}{n+2}M_{n-1} + 3\frac{n-1}{n+2}M_{n-2}, \quad n \geqslant 2, \quad M_0 = M_1 = 1.$$

Now, take a $0 < a \leqslant 1$ and form a sequence (b_n) by setting $b_n = a^n M_n$. By expressing M_n via b_n and substituting into the defining recurrence, we obtain the recurrence for b_n:

$$b_n = a\frac{2n+1}{n+2}b_{n-1} + 3a^2\frac{n-1}{n+2}b_{n-2}.$$

By substituting this expression into the condition $b_n + b_{n-2} \geqslant 2b_{n-1}$ we obtain the inequality for the quotient $\frac{b_{n-1}}{b_{n-2}}$:

$$\frac{b_{n-1}}{b_{n-2}} \leqslant \frac{(3a^2+1)n+2-3a^2}{2(1-a)n+4-a}.$$

Hence, the sequence (b_n) is convex if and only if the quotient $\frac{b_n}{b_{n-1}}$ is bounded from above by an upper bound of the form $g(n,a) = \frac{(3a^2+1)(n+1)+2-3a^2}{2(1-a)(n+1)+4-a}$. This upper bound can be established by induction on n from the defining recurrence.

An upper bound on $\frac{b_n}{b_{n-1}}$ can be obtained from the recurrence

$$\frac{b_n}{b_{n-1}} = a\frac{2n+1}{n+2} + 3a^2\frac{n-1}{n+2}\frac{b_{n-2}}{b_{n-1}}$$

by replacing $\frac{b_{n-2}}{b_{n-1}}$ by some greater quantity, i.e., by an upper bound on $\frac{b_{n-2}}{b_{n-1}}$. But such an upper bound is then a lower bound for $\frac{b_{n-1}}{b_{n-2}}$. Hence, we will need a lower bound, $d(n,a)$, that will drive the recurrence in the inductive proof of $\frac{b_n}{b_{n-1}} \leqslant g(n,a)$. The easiest way to find such lower bound is to solve the inequality

$$a\frac{2n+1}{n+2} + 3a^2\frac{n-1}{n+2}\frac{1}{d(n-1,a)} \leqslant g(n,a)$$

for the unknown quantity $d(n-1,a)$. By doing so, we obtain $d(n-1,a)$, and from there $d(n,a)$ in the form

$$d(n,a) = \frac{3a^2n[2(1-a)n+8-5a]}{(7a^2-4a+1)n^2+(28a^2-22a+7)n+24a^2-24a+12}.$$

Now we verify the conjectured bounds by induction on n.

LEMMA 3. *For all $n \in \mathbb{N}$ and $a \in (0,1]$ we have*

$$d(n,a) \leqslant \frac{b_n}{b_{n-1}} \leqslant g(n,a).$$

Proof. The base of induction follows from the inequalities

$$d(1,a) = 3a^2\frac{10-7a}{59a^2-50a+20} \leqslant \frac{b_1}{b_0} = a \leqslant g(1,a) = \frac{3a^2+4}{8-5a}.$$

Now assume that $d(k,a) \leqslant \frac{b_k}{b_{k-1}} \leqslant g(k,a)$ is valid for all $1 \leqslant k \leqslant n-1$. The inequality

$$\frac{b_n}{b_{n-1}} \leqslant a\frac{2n+1}{n+2} + 3a^2\frac{n-1}{n+2}\frac{1}{d(n-1,a)} \leqslant g(n,a)$$

is trivially valid, since $d(n-1,a)$ was chosen so as to satisfy it. The only remaining problem is to verify the other inequality.

By plugging the expression for $g(n,a)$ into the recurrence, we obtain

$$\frac{b_n}{b_{n-1}} \geqslant a\frac{2n+1}{n+2}+3a^2\frac{n-1}{n+2}\frac{1}{g(n-1,a)} = \frac{a[(6a+2)n+1-6a]}{(3a^2+1)n+2-3a^2} = r(n,a).$$

The claim will follow if $r(n,a) \geqslant d(n,a)$, i.e., if $r(n,a)-d(n,a) \geqslant 0$. It easy to see that both denominators in $r(n,a)$ and $d(n,a)$ are positive for $a \in (0,1]$ and $n \in \mathbb{N}$. Hence, the common denominator of the difference $r(n,a)-d(n,a)$ is positive. The numerator of $r(n,a)-d(n,a)$ is given by

$$\begin{gathered} a[12(1-8a+14a^2-12a^3)+(31-88a+94a^2+48a^3-45a^4)n \\ +3(a+1)(3a-1)(3a^2+6a-5)n^2+2(a+1)^2(3a-1)^2n^3]. \end{gathered}$$

Let us denote the term in square brackets by $s(n,a)$. We would like to show that $s(n,a) \geqslant 0$ for all $n \in \mathbb{N}$ and $a \in (0,1]$. By computing the partial derivative $\frac{\partial s(n,a)}{\partial n}$ and equating it to zero, we obtain a quadratic equation for n:

$$\begin{gathered} 6(a+1)^2(3a-1)^2n^2+6(a+1)(3a-1)(3a^2+6a-5)n \\ +(31-88a+94a^2+48a^3-45a^4)=0. \end{gathered}$$

Its discriminant $D(a) = 117a^4+12a^3-170a^2-4a+13$ has four real zeros; one of them, $\alpha = \frac{1}{78}\left(-2-20\sqrt{3}+39\sqrt{\frac{3232}{1521}+\frac{80}{507\sqrt{3}}}\right) \approx 0.274558$, falls into $(0,1]$. From $D(0)>0$ it follows that the equation $\frac{\partial s(n,a)}{\partial n}=0$ has no real roots for $a>\alpha$, and it is easy to see that both real roots of this equation for $0<a\leqslant\alpha$ are negative. Hence, $\frac{\partial s(n,a)}{\partial n}$ does not change its sign for $n \in \mathbb{N}$ and $a \in (0,1]$. Now the nonnegativity of $s(n,a)$ follows by checking that both $s(1,a) = 60(2a-1)^2$ and $\frac{\partial s(n,a)}{\partial n}|_{n=1} = 67-208a+46a^2+264a^3+63a^4$ are nonnegative on $(0,1]$, and that, indeed, is the case. □

Hence, the quotient $\frac{b_n}{b_{n-1}}$ is bounded from below and from above by $d(n,a)$ and $g(n,a)$, respectively, and then the sequence (b_n) is convex. By invoking Corollary 2, we arrive to our final result.

THEOREM 4. *The sequence (M_n) of Motzkin numbers is log-convex.*

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