Journal of
Mathematical
Inequalities
Volume 3, Number 3 (2009), 443–453

ON MONOTONE VARIATIONAL INEQUALITIES WITH RANDOM DATA

JOACHIM GWINNER AND FABIO RACITI

Dedicated to Professor Josip Pečarić
on the occasion of his 60^{th} birthday

Abstract. We study monotone variational inequalities with random data and give measurability, existence and uniqueness results in the general framework of a Hilbert space setting. Then we turn to the more structured case where a finite Karhunen-Loève expansion leads to a separation of the random and the deterministic variables. Here we present a discretization procedure with respect to the random variable based on averaging and truncation and on the approximation of the feasible random set. At last, we establish norm convergence of the approximation procedure.

1. Introduction

The theory of variational inequalities has been initiated by Stampacchia (e.g. [12]), in connection with partial differential equations (see also [4], [6]) and has become a powerful tool to study many problems in mechanics and physics. On the other hand, there has been an increasing interest in the study of variational inequalities describing equilibrium problems – mostly of finite dimension – arising in operations research and economic theory (e.g. [7], [8], [13]), which only in special cases admit an optimization formulation.

In this paper we study a class of variational inequalities with random data and extend some results obtained in [9] and [10] for the case of a bilinear form in a Hilbert space setting. Motivated by the need to cope with many nonlinear problems arising from various fields of applied sciences we carry out our study within the theory of monotone operators.

The paper is organized in four sections. In the following section we introduce a general class of random variational inequalities (RVI) defined by a random monotone operator on a random subset of a Hilbert space. We show that under suitable assumptions the (unique) solution of our RVI belongs to a certain Lebesgue space. Moreover, we provide an equivalent integral formulation of our RVI. In section 3 we turn to the more structured case where a finite Karhunen-Loève expansion ([5]) leads to a separation of the random and deterministic variables and formulate equivalent pointwise and integral formulations of the separated RVI. Then, in section 4, we put forth our approximation procedure in the random variable. By using the Mosco convergence result (Lemma 4.2) for the feasible random set we can prove our basic convergence theorem (Theorem 4.1) for the approximation procedure.

Mathematics subject classification (2000): 60H25, 49J40, 47H5, 49J55, 49M25.
Keywords and phrases: variational inequality, monotone operators, random constraint, measurability, mosco convergence, averaging, truncation.

2. The general problem

Let $(\Omega, \mathscr{A}, \mu)$ be a complete σ-finite measure space and $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ a separable Hilbert space. For all $\omega \in \Omega$, let $\mathscr{K}(\omega)$ be a closed, convex and nonempty subset of H. Consider a random vector δ in H and a Carathéodory function $\Phi : \Omega \times H \mapsto H$, i.e. for each fixed $x \in H$, $\Phi(\cdot, x)$ is measurable with respect to $\mathscr{A}$ and to the Borel algebra $\mathscr{B}(H)$, and for every $\omega \in \Omega$, $\Phi(\omega, \cdot)$ is continuous. Moreover, for each $\omega \in \Omega$, $\Phi(\omega, \cdot)$ is a monotone operator on H, i.e. $\langle \Phi(\omega,x) - \Phi(\omega,x'), x - x' \rangle \geqslant 0$, $\forall x, x' \in H$. Here let us simply write $\Phi(\omega) := \Phi(\omega, \cdot)$. With these data we consider the following

PROBLEM 1. For each $\omega \in \Omega$, find $x^*_\omega \in \mathscr{K}(\omega)$ such that

$$\langle \Phi(\omega, x^*_\omega), x - x^*_\omega \rangle \geqslant \langle \delta(\omega), x - x^*_\omega \rangle , \qquad \forall x \in \mathscr{K}(\omega). \tag{1}$$

Let us notice that under our assumptions Minty's Lemma holds (cfr. [12]) so that our problem is equivalent to

PROBLEM 2. For each $\omega \in \Omega$, find $x^*_\omega \in \mathscr{K}(\omega)$ such that

$$\langle \Phi(\omega, x), x - x^*_\omega \rangle \geqslant \langle \delta(\omega), x - x^*_\omega \rangle , \qquad \forall x \in \mathscr{K}(\omega). \tag{2}$$

Now we consider the set-valued map $\Sigma : \Omega \hookrightarrow H$ which, to each $\omega \in \Omega$, associates the solution set of (1). The measurability of Σ (with respect to the algebra $\mathscr{B}(H)$ of the Borel sets on H and to the σ-algebra $\mathscr{A}$ on Ω) has been proved in [10] for the case of a bilinear form. However, the proof given therein can be straightforwardly adapted to nonlinear operators.

To progress in our analysis we shall confine ourselves to the case of strongly monotone operators. The following definition will be used in the sequel.

DEFINITION 2.1. We call Φ uniformly strongly monotone, if there is some constant $c_0 > 0$ such that

$$\langle \Phi(\omega, x) - \Phi(\omega, x'), x - x' \rangle \geqslant c_0 \|x - x'\|^2 \quad \forall x, x' \in H, \forall \omega \in \Omega.$$

Under the additional strong monotonicity assumption, we can ensure the existence of a unique solution to (1) (cfr. [12]). Furthermore, with some $\zeta_0(\omega) \in \mathscr{K}(\omega)$ fixed, we get the preliminary estimate for the solution $\hat{X}(\omega) := x^*_\omega$

$$c_0 \|\zeta_0(\omega) - \hat{X}(\omega)\|^2 \leqslant \|\Phi(\omega, \zeta_0(\omega))\| \, \|\zeta_0(\omega) - \hat{X}(\omega)\| + \|\delta(\omega)\| \, \|\zeta_0(\omega) - \hat{X}(\omega)\| \tag{3}$$

whence,

$$\|\hat{X}(\omega)\| \leqslant (1/c_0) \|\Phi(\omega, \zeta_0(\omega))\| + (1/c_0) \|\delta(\omega)\| + \|\zeta_0(\omega)\| . \tag{4}$$

In many applications the modelling is often done with polynomial cost functions. We are then led to require the growth condition

$$\|\Phi(\omega, z)\| \leqslant a(\omega) + b(\omega) \|z\|^{p-1} \ \forall z \in H, \tag{5}$$

for some $p \geqslant 2$ which yields the following estimate for the solution:

$$c_0\|\hat{X}(\omega)\| \leqslant a(\omega) + b(\omega)\|\zeta_0(\omega)\|^{p-1} + \|\delta(\omega)\| + c_0\|\zeta_0(\omega)\| \,. \tag{6}$$

Thus, to obtain $\hat{X} \in L^q(\Omega,\mu,H)$, for some $q \geqslant 1$, we assume $a \in L^q(\Omega,\mu), b \in L^\infty(\Omega,\mu)$, $\delta \in L^q(\Omega,\mu)$ and that there exists some $\zeta_0(\omega) \in \mathscr{K}(\omega)$ such that $\zeta_0 \in L^{q(p-1)}(\Omega,\mu) \cap L^q(\Omega,\mu)$.

Since our final aim is to arrive at statistical quantities such as the mean value or the variance of the solution, we shall consider the case $q = 2$ or, in general, $q = p \geqslant 2$. Therefore we can state the following existence and uniqueness result.

THEOREM 2.1. *Let $(\Omega,\mathscr{A},\mu)$ be a complete σ-finite measure space, and let $\Phi(\omega,\cdot)$ be a strongly monotone operator on H for all $\omega \in \Omega$. Then the variational inequality (1) admits a unique solution $\hat{X} : \omega \in \Omega \mapsto \hat{X}(\omega) \in \mathscr{K}(\omega)$. Moreover, suppose that, Φ is uniformly strongly monotone, that the random vector δ belongs to $L^p(\Omega,\mu,H)$, that the growth condition (5) is satisfied and that there exists $\zeta_0 \in L^{(p-1)p}(\Omega,\mu,H) \cap L^p(\Omega,\mu,H)$ such that $\zeta_0(\omega)$ belongs to $\mathscr{K}(\omega)$. Then $\hat{X} \in L^p(\Omega,\mu,H)$.*

REMARK 2.1. Since Φ is Carathéodory, it is well known that the function $\omega \mapsto \Phi(\omega,u(\omega))$ is measurable, whenever $u : \Omega \mapsto H$ is measurable. Thus, the Carathéodory function Φ defines a mapping $N_\Phi : \mathscr{M} \mapsto \mathscr{M}$ on the space $\mathscr{M}$ of all the measurable functions on Ω, which is known as the Nemytskij operator (cfr. [1], [16]). Under our assumption, the Nemytskij operator N_Φ associated to Φ maps $L^p \mapsto L^{\frac{p}{p-1}}$. Moreover, if the measure μ is finite, the Nemytskij operator is also continuous and bounded. This hypothesis is clearly verified in the probability spaces we shall deal with in the sequel.

Let us now introduce a probability space $(\Omega,\mathscr{A},P)$ and the reflexive Banach space $L^p(\Omega,P,H)$ of random vectors V from Ω to H such that

$$E^P\|V\|^p = \int_\Omega \|V(\omega)\|^p dP(\omega) < \infty, \;\; p \geqslant 2 \tag{7}$$

and consider the convex and closed set

$$K := \{V \in L^p(\Omega,P,H) : V(\omega) \in \mathscr{K}(\omega), \text{P-almost sure}\}\,.$$

Under the growth condition (5) with $a \in L^p(\Omega,P), b \in L^\infty(\Omega,P)$, and assuming that $\delta \in L^p(\Omega,P,H)$ the integrals

$$\int_\Omega \langle \Phi(\omega,U(\omega)), V(\omega) - U(\omega)\,\rangle\, dP(\omega),\; \int_\Omega \langle \delta(\omega), V(\omega) - U(\omega)\rangle\, dP(\omega)$$

are well defined for all $U,V \in L^p(\Omega,P,H)$. Therefore, we can consider the following

PROBLEM 3. Find $U \in K$ such that, $\forall V \in K$,

$$\int_\Omega \langle \Phi(\omega,U(\omega)),\, V(\omega) - U(\omega)\rangle\, dP(\omega) \geqslant \int_\Omega \langle \delta(\omega), V(\omega) - U(\omega)\rangle\, dP(\omega) \tag{8}$$

or, in compact form using the expectation E^P,

$$E^P\{\langle N_\Phi(U), V-U\rangle\} \geqslant E^P\{\langle \delta, V-U\rangle\}. \tag{9}$$

Under our assumptions, (8) has a unique solution $U \in L^p(\Omega,P,H)$. Thus, by uniqueness, Problem 1 and Problem 3 are equivalent in the sense that from the integral formulation in Problem 3 we obtain a pointwise solution that is only defined P-a.e. on Ω and that coincides there with the pointwise solution of Problem 1.

3. The separated case – A probabilistic approach

Here and in the sequel we shall pose further assumptions on the structure of the random constraint set and on the operator. More precisely, with another Hilbert space G, a convex closed cone $C \subset G$ defining the order $0 \geqslant g \in G :\Leftrightarrow g \in C$, a linear operator $B : H \to G$ and a random vector γ in G being given, we consider the random set

$$M(\omega) := \{x \in H : Bx \leqslant \gamma(\omega)\},\ \omega \in \Omega.$$

Moreover, we assume that the deterministic and random variables are separated via a finite Karhunen-Loève expansion. In particular, we let the uniformly strongly monotone operator Φ defined by

$$\Phi(\omega,x) := A_0(x) + \sum_{j=1}^{k} S_j(\omega) A_j(x),$$

where all $S_j \in L^\infty(\Omega); j = 1,\dots,k$. The uniform strong monotonicity of Φ is ensured by the strong monotonicity of A_0 and all $\underline{s}_j A_j$, where $\underline{s}_j$ is a positive constant such that $S_j \geqslant \underline{s}_j$ P – a.s. (almost sure). We also require that Φ satisfies the growth condition (5).

Similarly let

$$\delta(\omega) := h_0 + \sum_{j=1}^{l} R_j(\omega) h_j,$$

where $h_0, h_1 \dots h_l \in H$ and all $R_j \in L^p(\Omega); j = 1,\dots,l$, and let

$$\gamma(\omega) := g_0 + \sum_{j=1}^{m} T_j(\omega) g_j,$$

where $g_0, g_1 \dots g_m \in G$ and all $T_j \in L^p(\Omega); j = 1,\dots,m$. To simplify the notation, we use the vector notation

$$S = (S_1,\dots,S_k)^T, \quad A = (A_1,\dots,A_k)^T, \quad R = (R_1,\dots,R_l)^T, \quad T = (T_1,\dots,T_m)^T,$$

$$h = (h_1,\dots,h_l)^T, \quad g = (g_1,\dots,g_m)^T.$$

Thus, we rewrite Problem 1 (ω-formulation) to that of finding $\hat{X} : \Omega \mapsto H$ such that $\hat{X}(\omega) \in M(\omega)$ (P-a.s.) and the following inequality holds for P-almost every elementary event $\omega \in \Omega$ and $\forall x \in M(\omega)$,

$$\langle (A_0 + S(\omega)^T A)[\hat{X}(\omega)], x - \hat{X}(\omega)\rangle \geqslant \langle h_0 + R(\omega)^T h, x - \hat{X}(\omega)\rangle . \tag{10}$$

Moreover, we can introduce the following closed convex nonvoid subset of $L^p(\Omega, P, H)$:

$$M^P := \{V \in L^p(\Omega, P, H) : BV(\omega) \leqslant g_0 + T(\omega)^T g \ , P - a.s.\}$$

and consider the following problem: Find $\hat{U} \in M^P$ such that, $\forall V \in M^P$,

$$\begin{aligned} &\int_\Omega \langle (A_0 + S(\omega)^T A)[\hat{U}(\omega)], V(\omega) - \hat{U}(\omega)\rangle \, dP(\omega) \\ &\qquad \geqslant \int_\Omega \langle h_0 + R(\omega)^T h, V(\omega) - \hat{U}(\omega)\rangle \, dP(\omega) . \end{aligned} \tag{11}$$

The r.h.s. of (11) defines a continuous linear form on $L^p(\Omega, P, H)$, while the l.h.s. defines a continuous form on $L^p(\Omega, P, H)$ which inherits strong monotonicity from the strong monotonicity of $A_0 + \underline{s}^T A$. Therefore, (see e.g. [12]), there exists a unique solution in M^P to problem (11). By uniqueness, problems (10) and (11) are equivalent.

In order to get rid of the abstract sample space Ω, we consider the joint distribution $\mathbb{P}$ of the random vector (R, S, T) and work with the special probability space $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \mathbb{P})$, where the dimension $d := k + l + m$. To simplify our analysis we shall suppose that R, S and T are independent random vectors. Let $r = R(\omega)$, $s = S(\omega)$, $t = T(\omega)$. For each $y = (r, s, t) \in \mathbb{R}^d$, consider the set

$$M(y) := \{x \in H : Bx \leqslant g_0 + t^T g\} .$$

Then the pointwise version of our problem now reads: Find $\hat{x} = \hat{x}(y)$ such that $\hat{x}(y) \in M(y)$, $\mathbb{P}$ - a.s., and the following inequality holds for $\mathbb{P}$ - almost every $y \in \mathbb{R}^d$ and $\forall x \in M(y)$,

$$\langle (A_0 + s^T A)[\hat{x}(y)], x - \hat{x}(y)\rangle \geqslant \langle h_0 + r^T h, x - \hat{x}(y)\rangle . \tag{12}$$

In order to obtain the integral formulation of (12), consider the space $L^p(\mathbb{R}^d, \mathbb{P}, H)$ and introduce the closed convex nonvoid set

$$M_{\mathbb{P}} := \{v \in L^p(\mathbb{R}^d, \mathbb{P}, H) : Bv(r, s, t) \leqslant g_0 + t^T g, \ \mathbb{P} - a.s.\} .$$

This leads to the problem: Find $\hat{u} \in M_{\mathbb{P}}$ such that, $\forall v \in M_{\mathbb{P}}$,

$$\int_{\mathbb{R}^d} \langle (A_0 + s^T A)[\hat{u}(y)], v - \hat{u}\rangle \, d\mathbb{P}(y) \geqslant \int_{\mathbb{R}^d} \langle h_0 + r^T h, v(y) - \hat{u}(y)\rangle \, d\mathbb{P}(y) . \tag{13}$$

By using the same arguments as in section 2, problems (12) and (13) are equivalent.

4. An Approximation Procedure

Without loss of generality, we can suppose that $R \in L^p_l(\Omega, P) = [L^p(\Omega, P)]^l$ and $T \in L^p_m(\Omega, P)$ are nonnegative (otherwise we can use the standard decomposition in the positive part and the negative part). Moreover, we assume that the support (the set of possible outcomes) of $S \in L^\infty_k(\Omega, P)$ is the interval $[\underline{s}, \overline{s}] \subset (0, \infty)^k$. Furthermore we assume that the distributions P_R, P_S, P_T are continuous with respect to the Lebesgue measure λ, so that according to the theorem of Radon-Nikodym, they have the probability densities $\varphi_{R_j}, \varphi_{S_j}, \varphi_{T_j}$. Hence, $\mathbb{P} = P_R \otimes P_S \otimes P_T$, $dP_R(r) = \varphi_R(r)\, dr$, $dP_S(s) = \varphi_S(s)\, ds$ and $dP_T(t) = \varphi_T(t)\, dt$, where shortly $\varphi_R(r) := \prod_j \varphi_{R_j}(r_j)$, $\varphi_S(s) := \prod_j \varphi_{S_j}(s_j)$, $\varphi_T(t) := \prod_j \varphi_{T_j}(t_j)$. Let us note that $v \in L^p(\mathbb{R}^d, \mathbb{P}, H)$ means that $(r,s,t) \mapsto \varphi_R(r)\varphi_S(s)\varphi_T(t) v(r,s,t)$ belongs to $L^p(\mathbb{R}^d, \lambda, H)$.

Now the probabilistic integral formulation of our problem reads: Find $\hat{u} \in M_{\mathbb{P}}$ such that, $\forall v \in M_{\mathbb{P}}$,

$$\int_{\mathbb{R}^l_+} \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} \langle (A_0 + s^T A)(\hat{u}), v - \hat{u} \rangle\, \varphi_R(r)\varphi_S(s)\varphi_T(t)\, dy \geqslant$$
$$\int_{\mathbb{R}^l_+} \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} \langle h_0 + r^T h, v - \hat{u} \rangle\, \varphi_R(r)\varphi_S(s)\varphi_T(t)\, dy.$$

In order to give an approximation procedure for the solution $\hat{u}$, let us start with a discretization of the space $X := L^p(\mathbb{R}^d, \mathbb{P}, H)$ and introduce a sequence $\{\pi_n\}_n$ of partitions of the support $\Upsilon := \mathbb{R}^l_+ \times [\underline{s}, \overline{s}] \times \mathbb{R}^m_+$ of the probability measure $\mathbb{P}$ induced by the random elements R, S, T. To be precise, let $\pi_n = (\pi_n^R, \pi_n^S, \pi_n^T)$; $\pi_n^R = (\pi_n^{R_1}, \dots, \pi_n^{R_l})$, $\pi_n^S = (\pi_n^{S_1}, \dots, \pi_n^{S_k})$, $\pi_n^T = (\pi_n^{T_1}, \dots, \pi_n^{T_m})$, where

$$\pi_n^{R_j} := (r^0_{n,j}, \dots, r^{N_n^{R_j}}_{n,j}), \quad \pi_n^{S_j} := (s^0_{n,j}, \dots, s^{N_n^{S_j}}_{n,j}), \quad \pi_n^{T_j} := (t^0_{n,j}, \dots, t^{N_n^{T_j}}_{n,j}),$$
$$0 = r^0_{n,j} < r^1_{n,j} < \dots r^{N_n^{R_j}}_{n,j} = n \ (j = 1, \dots, l),$$
$$\underline{s}_j = s^0_{n,j} < s^1_{n,j} < \dots s^{N_n^{S_j}}_{n,j} = \overline{s}_j \ (j = 1, \dots, k),$$
$$0 = t^0_{n,j} < t^1_{n,j} < \dots t^{N_n^{T_j}}_{n,j} = n \ (j = 1, \dots, m).$$

We impose that
$$|\pi_n| := \max\{|\pi_n^R|, |\pi_n^S|, \pi_n^T|\} \to 0 \quad (n \to \infty),$$
where
$$|\pi_n^R| := \max\{|\pi_n^{R_j}| : (j = 1, \dots, l)\},$$
$$|\pi_n^{R_j}| := \max\{r^1_{n,j} - r^0_{n,j}, \dots, r^{N_n^{R_j}}_{n,j} - r^{N_n^{R_j}-1}_{n,j}\},$$
$$|\pi_n^S| := \max\{|\pi_n^{S_j}| : (j = 1, \dots, k)\},$$
$$|\pi_n^{S_j}| := \max\{s^1_{n,j} - s^0_{n,j}, \dots, s^{N_n^{S_j}}_{n,j} - s^{N_n^{S_j}-1}_{n,j}\},$$

$$|\pi_n^T| := \max\{|\pi_n^{T_j}| : (j = 1,\dots,m)\},$$
$$|\pi_n^{T_j}| := \max\{t_{n,j}^1 - t_{n,j}^0,\dots,t_{n,j}^{N_n^{T_j}} - t_{n,j}^{N_n^{T_j}-1}\}.$$

These partitions give rise to the exhausting sequence $\{\Upsilon_n\}$ of subsets of Υ, where each Υ_n is given by the finite disjoint union of the intervals:

$$I_h^n := I_{h_R}^n \times I_{h_S}^n \times I_{h_T}^n,$$

where we use the multiindices $h = (h_R, h_S, h_T)$,

$$h_R = (h_1^R,\cdots,h_l^R),\, h_S = (h_1^S,\cdots,h_k^S),\, h_T = (h_1^T,\cdots,h_m^T)$$

and

$$I_{h_R}^n := \Pi_{j=1}^l [r_{n,j}^{h_j^R-1}, r_{n,j}^{h_j^R}),\quad I_{h_S}^n := \Pi_{j=1}^k [s_{n,j}^{h_j^S-1}, s_{n,j}^{h_j^S}),\quad I_{h_T}^n := \Pi_{j=1}^m [t_{n,j}^{h_j^T-1}, t_{n,j}^{h_j^T}).$$

For each $n \in \mathbb{N}$ let us consider the space of the Z-valued simple functions on Υ_n, extended by 0 outside of Υ_n:

$$\Pi_0^n(Z) := \{v_n : v_n(y) = \sum_h v_h^n 1_{I_h^n}(y), v_h^n \in Z\},$$

where Z is any separable Hilbert space (here $Z = H$ or $Z = G$) or a subset thereof and 1_I denotes the $\{0,1\}$-valued characteristic function of a subset I.

To approximate an arbitrary function $w \in L^p(\mathbb{R}^d,\mathbb{P},Z)$ we employ the mean value truncation operator μ_0^n associated to the partition π_n given by

$$\mu_0^n w := \sum_h (\mu_h^n w)\, 1_{I^n}, \tag{14}$$

where

$$\mu_h^n w := \begin{cases} \dfrac{1}{\mathbb{P}(I_h)} \displaystyle\int_{I_h^n} w(y)\, d\mathbb{P}(y) & \text{if } \mathbb{P}(I_h^n) > 0; \\ 0 & \text{otherwise.} \end{cases}$$

From Lemma 2.5 in ([9]) (and the remarks therein) we obtain the following result.

LEMMA 4.1. *The linear operator $\mu_0^n : L^p(\mathbb{R}^d,\mathbb{P},Z) \to L^p(\mathbb{R}^d,\mathbb{P},Z)$ is bounded with $\|\mu_0^n\| = 1$, and for $n \to \infty$, μ_0^n converges pointwise in $L^p(\mathbb{R}^d,\mathbb{P},Z)$ to the identity.*

This lemma reflects the well-known density of the class of the simple functions in L^p space. It shows that the mean value truncation operator μ_0^n acts as a projector on $L^p(\mathbb{R}^d,\mathbb{P},H)$, and thus our approximation method is a projection method according to the terminology of [14].

In order to construct approximations for

$$M_{\mathbb{P}} = \{v \in L^p(\mathbb{R}^d,\mathbb{P},H) : Bv(r,s,t) \leqslant t,\ \mathbb{P} - a.s.\}$$

we introduce the orthogonal projector $\tau : (r,s,t) \in \mathbb{R}^d \mapsto t \in \mathbb{R}^m$ and let, for each elementary quadrangle I_h^n,

$$\tau_h^n = \mu_h^n \tau \in \mathbb{R}^m, \qquad \mu_0^n \tau = \sum_h \tau_h^n 1_{I_h^n} \in \Pi_0^n(\mathbb{R}^m).$$

Thus we arrive at the following sequence of convex, closed sets

$$M_{\mathbb{P}}^n := \{v \in \Pi_0^n(H) : Bv_h^n \leqslant \tau_h^n, \forall h\}.$$

Note that the set $M_{\mathbb{P}}^n$ is of polyhedral type, if H is of finite dimension.

Also we have to approximate the random variables R and S and introduce

$$\rho_n = \Big(\sum_{\nu=1}^{N_n^{R_j}} r_{n,j}^{\nu-1} 1_{[r_{n,j}^{\nu-1}, r_{n,j}^{\nu})}\Big)_{j=1,\dots,l} \in \Pi_0^n(\mathbb{R}^l),$$
$$\sigma_n = \Big(\sum_{\nu=1}^{N_n^{S_j}} s_{n,j}^{\nu-1} 1_{[s_{n,j}^{\nu-1}, s_{n,j}^{\nu})}\Big)_{j=1,\dots,k} \in \Pi_0^n(\mathbb{R}^k).$$

For later use we observe that $\sigma_n \to \sigma(r,s,t) = s$ in $L^\infty(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$, while, as a consequence of the Chebyshev inequality (see e.g. [2]), $\rho_n \to \rho(r,s,t) = r$ in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l)$.

Thus for each $n \in \mathbb{N}$ we are led to the following approximate problem: Find $\hat{u}_n \in M_{\mathbb{P}}^n$ such that, $\forall v_n \in M_{\mathbb{P}}^n$,

$$\int_{\mathbb{R}^d} \langle (A_0 + \sigma_n(y)^T A)[\hat{u}_n(y)], v_n(y) - \hat{u}_n(y)\rangle \, d\mathbb{P}(y) \qquad (15)$$
$$\geqslant \int_{\mathbb{R}^d} \langle h_0 + \rho_n(y)^T h, v_n(y) - \hat{u}_n(y)\rangle \, d\mathbb{P}(y).$$

The crucial step towards to the convergence of the approximation procedure described above is the following set convergence result.

LEMMA 4.2. *The set $M_{\mathbb{P}}^n$ converges to $M_{\mathbb{P}}$ in the sense of Mosco (*[3], [15]*), i.e.*

$$\textit{weak-limsup}_{n\to\infty} M_{\mathbb{P}}^n \subset M_{\mathbb{P}} \subset \textit{strong-liminf}_{n\to\infty} M_{\mathbb{P}}^n.$$

Proof. We can refer to [10] where we already remarked that the proof detailed there for lemma 4.3 extends to the more general present situation, provided we can show that the positive operator μ_0^n commutes with the linear operator B. On the other hand, the operator μ_0^n can be understood as a conditional expectation operator.

Therefore it is enough to show for any $u \in L^p(\mathbb{R}^d, \mathbb{P}, H)$, the following identity for Bochner integrals:

$$B\Big(\int v(y)\, d\mathbb{P}(y)\Big) = \int B(u(y))\, d\mathbb{P}(y).$$

To verify this, take any η in the dual G^* of G and use the duality relation

$$\begin{aligned}\langle \int B(u(y))\, d\mathbb{P}(y), \eta\rangle_{G\times G^*} &= \int \langle B(u(y)), \eta\rangle_{G\times G^*}\, d\mathbb{P}(y)\\ &= \int \langle u(y), B^*\eta\rangle_{H\times H^*}\, d\mathbb{P}(y) = \langle \int u(y)\, d\mathbb{P}(y), B^*\eta\rangle_{H\times H^*}\\ &= \langle B(\int u(y)\, d\mathbb{P}(y)), \eta\rangle_{G\times G^*}\,.\end{aligned}$$

Now, we can establish our basic convergence result. □

THEOREM 4.1. *The sequence $\hat{u}_n$ generated by the approximate problems in (*15*) converges strongly in $L^p(\mathbb{R}^d,\mathbb{P},H)$ to the unique solution $\hat{u}$ of (*13*).*

Proof.

1) We show that $\{\hat{u}_n\}$ is norm-bounded.

To this end let us fix arbitrarily an element v of $M_{\mathbb{P}}$. By Lemma 4.2 we can find $v_n \in M^n_{\mathbb{P}}$ such that $\lim_n v_n = v$ in $L^p(\mathbb{R}^d,\mathbb{P},H)$. Hence, it is sufficient to prove that $\{\hat{u}_n - v_n\}$ is bounded in $L^p(\mathbb{R}^d,\mathbb{P},H)$. But this follows from the a priori bound (3) for the problems (15) (with $\zeta_0 := v_n$) and from the boundedness of the convergent sequence v_n.

2) We show that any weak limit point u of $\{\hat{u}_n\}$ – which exists due to part 1) – is a solution of (13).

First, by the l.h.s. of Mosco convergence in Lemma 4.2, any such weak limit point u of $\{\hat{u}_n\}$ is feasible, i.e. $u \in M_{\mathbb{P}}$. Then, thanks to Minty's Lemma, it is sufficient to prove that u satisfies for all $v \in M_{\mathbb{P}}$

$$\int_{\mathbb{R}^d} \langle (A_0 + s^T A)(v), v-u\rangle\, d\mathbb{P}(y) \geqslant \int_{\mathbb{R}^d} \langle h_0 + r^T h, v-u\rangle\, d\mathbb{P}(y)\,. \tag{16}$$

If $v \in M_{\mathbb{P}}$ is arbitrarily chosen, let $v_n \in M^n_{\mathbb{P}}$ such that $\lim_n v_n = v$ strongly in $L^p(\mathbb{R}^d,\mathbb{P},H)$. Such a sequence $\{v_n\}$ exists thanks to $M_{\mathbb{P}} \subset$ strong-$\liminf_n M^n_{\mathbb{P}}$ according to Lemma 4.2. By the definition of $\hat{u}_n$ in (15) and by Minty's Lemma, we have for all $n \in \mathbb{N}$

$$\int_{\mathbb{R}^d} \langle (A_0 + \sigma_n(y)^T A)(v_n), v_n - \hat{u}_n\rangle\, d\mathbb{P}(y) \geqslant \int_{\mathbb{R}^d} \langle h_0 + \rho_n(y)^T h, v_n - \hat{u}_n\rangle\, d\mathbb{P}(y)\,.$$

Let us observe that since $v_n \to v$ strongly in $L^p(\mathbb{R}^d,\mathbb{P})$, $A_0 v_n \to A_0 v$ and $A v_n \to Av$, strongly in $L^{\frac{p}{p-1}}$, thanks to the continuity of the Nemystskij operators (see remark (2.1)). Since $\sigma_n \to \sigma$ strongly in $L^\infty(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k)$, $\rho_n \to \rho$ strongly in $L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^l)$, $v_n \to v$ strongly in $L^p(\mathbb{R}^d,\mathbb{P},H)$, and $\hat{u}_n - v_n \rightharpoonup u - v$ weakly in $L^p(\mathbb{R}^d,\mathbb{P},H)$, the left and right hand side converge, respectively, to the left and right hand side of (16). Moreover, since the solution $\hat{u}$ is unique, all the sequence $\{\hat{u}_n\}$ converges weakly to $\hat{u}$.

3) In order to show the claimed norm convergence, let us fix $u_n \in M^n_{\mathbb{P}}$ such that $\lim_n u_n = \hat{u}$ strongly in $L^p(\mathbb{R}^d, \mathbb{P}, H)$ according to Lemma 4.2. Then by (15) we can estimate, thanks to Minty's Lemma and uniform monotonicity,

$$\begin{aligned} 0 &\leqslant c_0 \|u_n - \hat{u}_n\|_2^2 \\ &\leqslant \int_{\mathbb{R}^d} \langle (A_0 + \sigma_n(y)^T A)(u_n) - (A_0 + \sigma_n(y)^T A)(\hat{u}_n), u_n - \hat{u}_n \rangle \, d\mathbb{P}(y) \\ &\leqslant \int_{\mathbb{R}^d} \langle (A_0 + \sigma_n(y)^T A)(u_n), u_n - \hat{u}_n \rangle \, d\mathbb{P}(y) \\ &\quad - \int_{\mathbb{R}^d} \langle h_0 + \rho_n(y)^T h, u_n - \hat{u}_n \rangle \, d\mathbb{P}(y) . \end{aligned}$$

Finally from $\sigma_n \to \sigma$ strongly in $L^\infty(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$, $\rho_n \to \rho$ strongly in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l)$, and $\hat{u}_n \rightharpoonup \hat{u}$ weakly in $L^p(\mathbb{R}^d, \mathbb{P}, H)$ we conclude the proof. □

In future work we shall apply our theoretical findings to a class of nonlinear traffic equilibria models on networks with random data, that parallels previous work [11] in the case of a linear cost model, but needs extra considerations.

Acknowledgements. One of the authors, F. R., has been supported by the University of Catania (Contributo per l'aggiornamento all'estero dei docenti) during the period that he spent at the University of the Federal Armed Forces in Munich.

REFERENCES

[1] APPEL, J., ZABREJKO, P. P., *Nonlinear Superposition Operators*, Cambridge University, 1990.

[2] ASH, R. B., *Real Analysis and Probability*, Academic Press, New York, 1972.

[3] ATTOUCH, H., *Variational Convergence for Functions and Operators*, Pitman, Boston, 1984.

[4] BAIOCCHI, C., CAPELO, A., *Variational and Quasivariational Inequalities*, Wiley, Chichster, New York, 1984.

[5] DUER, A., *On the optimality of the discrete Karhunen-Loeve expansion*, SIAM J. Control Optimization, **36** (1998), 1937–1939.

[6] DUVAUT, G., LIONS J. L., *Inequalities in Mechanics and Physics*, Springer, Berlin, 1976.

[7] FACCHINEI, F., PANG, J.-S., *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Two Volumes. Springer, New York, 2003.

[8] GIANNESSI, F., MAUGERI, A. (EDS.), *Variational Inequalities and Network Equilibrium Problems*, Erice, 1994; Plenum Press, New York, 1995.

[9] GWINNER, J., *A class of random variational inequalities and simple random unilateral boundary value problems: Existence, discretization, finite element approximation*, Stochastic Anal. Appl., **18** (2000), 967–993.

[10] GWINNER J., RACITI F., *On a Class of Random Variational Inequalities on Random Sets*, Numerical Functional Analysis and Optimization, **27** (2006), 619–636.

[11] GWINNER J., RACITI F., *Random equilibrium problems on networks*, Mathematical and Computer Modelling, **43** (2006), 880–891.

[12] KINDERLEHRER, D., STAMPACCHIA, G., *An Introduction to Variational Inequalities and their Applications*, Academic Press, New York, 1980.

[13] KONNOV I., *Equilibrium Models and Variational Inequalities*, North-Holland, Amsterdam, 2007.

[14] LEPP, R., *Projection and discretization methods in stochastic programming*, J. Comput. Appl. Math., **56** (1994), 55–64.

[15] MOSCO, U., *Convergence of convex sets and of solutions of variational inequalities*, Advances in Math., **3** (1969), 510–585.

[16] ZEIDLER E., *Nonlinear Functional Analysis and its Applications*, Vol. II/B, Nonlinear Monotone Operators, Springer Verlag, New York, 1990.

(Received November 15, 2008)

Joachim Gwinner
Institut für Mathematik
Fakultät für Luft- und Raumfahrtechnik
Universität der Bundeswehr München
D-85577 Neubiberg
Germany
e-mail: Joachim.Gwinner@unibw-muenchen.de

Fabio Raciti
Facoltà di Ingegneria dell'Università di Catania
Dipartimento di Matematica e Informatica dell'Università di Catania
V.le A. Doria, 6
95125-Catania
Italy
e-mail: fraciti@dmi.unict.it