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RESOLVENT OPERATOR METHOD FOR GENERAL VARIATIONAL INCLUSIONS

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. In this paper, we introduce a new class of variational inclusions involving three operator. Using the resolvent operator technique, we establish the equivalence between the general variational inclusions and the resolvent equations. We use this alternative equivalent formulation to suggest and analyze some iterative methods for solving the general variational inclusions. We also consider the criteria of these iterative methods under suitable conditions. Since the general variational inclusions include the variational inequalities and the related optimization problems as special cases, our results continue to hold for these problems.

1. Introduction

Variational inclusions involving three operators are useful and important extensions and generalizations of the general variational inequalities with a wide range of applications in industry, mathematical finance, economics, decision sciences, ecology, mathematical and engineering sciences, see [1–45] and the references therein. It is well known that the projection method and its variant forms including the Wiener-Hopf equations can not be extended and modified for solving the variational inclusions. These facts and comments have motivated to use the technique of the resolvent operators. This technique can lead to the development of very efficient and robust methods since one can treat each part of the original operator independently. A useful feature of these iterative methods for solving the general variational inclusion is that the resolvent step involves the maximal monotone operator only, while other parts facilitates the problem decomposition. Essentially using the resolvent technique, one can show that the variational inclusions are equivalent to the fixed point problems. This alternative equivalent formulation has played very crucial role in developing some very efficient methods for solving the variational inclusions and related optimization problems, see [15–38] and the references therein. Related to the variational inclusions, we have the problem of solving the resolvent equations, which are mainly due to Noor [20, 21, 23]. Essentially using the resolvent operator technique, we can establish the equivalence between the resolvent equations and the variational inclusions. This equivalence formulations is more general and flexible than the resolvent operator method. Resolvent

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equations technique has been used to suggest and analyze several iterative methods for solving variational inclusions and related problems, see [24–27, 32, 34–38] and the references therein.

Motivated and inspired by the recent research activities in these areas, we introduce some new classes of variational inclusions and resolvent equations. Essentially using the resolvent operator methods, we establish the equivalence between the resolvent equations and the general variational inclusions. This alternative equivalent formulation is used to suggest some iterative methods for solving the general variational inclusions. We study the convergence criteria of the new iterative method under some mild conditions. Since the variational inclusions include the mixed variational inequalities and related optimization problems as special cases, results proved in this paper continue to hold for these problems.

2. Basic Results

Let K be a nonempty closed and convex set in a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$ respectively. Let $T,A,g : H \longrightarrow H$ be three nonlinear operators.

Consider the problem of finding $u \in H$ such that

$$0 \in \rho Tu + u - g(u) + \rho A(u), \quad \rho > 0, \quad \text{a constant,} \tag{1}$$

which is known as the general variational inclusion. Problem (1) is also known as finding the zero of the sum of two (or more) monotone operators. Variational inclusions and related problems are being studied extensively by many authors and have important applications in operations research, optimization, mathematical finance, decision sciences and other several branches of pure and applied sciences, see [2–45] and the references therein.

If $A(.) \equiv \partial\varphi(.)$, where $\partial\varphi(.)$ is the subdifferential of a proper, convex and lower-semicontinuous function $\varphi : H \longrightarrow R \cup \{+\infty\}$, then the problem (1) reduces to finding $u \in H$ such that

$$0 \in \rho Tu + u - g(u) + \rho\partial\varphi(u),$$

or equivalently, finding $u \in H$ such that

$$\langle \rho Tu + u - g(u), g(v) - u\rangle + \rho\varphi(g(v)) - \rho\varphi(u) \geqslant 0, \quad \forall v \in H. \tag{2}$$

The inequality (2) is called the general mixed variational inequality or the general variational inequality of the second kind. It has been shown that a wide class of linear and nonlinear problems arising in various branches of pure and applied sciences can be studied in the unified framework of mixed variational inequalities, see [2–38].

We note that if φ is the indicator function of a closed convex set K in H, that is,

$$\varphi(u) \equiv I_K(u) = \begin{cases} 0, & \text{if} \quad u \in K \\ +\infty, & \text{otherwise,} \end{cases}$$

then the general mixed variational inequality (2) is equivalent to finding $u \in K$ such that

$$\langle \rho Tu + u - g(u), g(v) - u \rangle \geqslant 0, \quad \forall v \in H : g(v) \in K, \tag{3}$$

which is called the general variational inequality introduced and studied by Noor [29] in connection with nonconvex functions. See also Noor and Noor [31, 32] for more details.

If $g \equiv I$, the identity operator, then problem (3) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geqslant 0, \quad \forall v \in K, \tag{4}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [44] in 1964. For the recent trends and developments in variational inclusions and inequalities, see [2–45] and the references therein.

We also need the following well known concepts and results.

DEFINITION 2.1. [5] If A is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with A is defined by

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \text{for all } u \in H,$$

where I is the identity operator. It is well known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. In addition, the resolvent operator is a single-valued and nonexpansive, that is, for all $u, v \in H$,

$$\|J_A(u) - J_A(v)\| \leqslant \|u - v\|.$$

REMARK 2.1. It is well known that the subdifferential $\partial\varphi$ of a proper, convex and lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$ is a maximal monotone operator, we denote by

$$J_\varphi(u) = (I + \rho\partial\varphi)^{-1}(u), \quad \text{for all } u \in H,$$

the resolvent operator associated with $\partial\varphi$, which is defined everywhere on H. In particular, the resolvent operator J_φ has the following interesting characterization.

LEMMA 2.1. [5] *For a given $z \in H$, $u \in H$ satisfies the inequality*

$$\langle u - z, v - u \rangle + \rho\varphi(v) - \rho\varphi(u) \geqslant 0, \quad \textit{for all } v \in H,$$

if and only if

$$u = J_\varphi z,$$

where $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator.

This property of the resolvent operator J_φ plays an important part in developing the numerical methods for solving the mixed variational inequalities.

If the function $\varphi(.)$ is the indicator function of a closed convex set K in H, then it is well known that $J_\varphi = P_K$, the projection operator of H onto the closed convex set K.

Using the definition of the resolvent operator J_A, one can easily prove the following well known result. For the sake of completeness and to convey an idea, we include its proof.

LEMMA 2.2. *The function $u \in H$ is a solution of the variational inclusion (1) if and only if $u \in H$ satisfies the relation*

$$u = J_A[g(u) - \rho Tu],$$

where $\rho > 0$ is a constant and $J_A = (I + \rho A)^{-1}$ is the resolvent operator associated with the maximal monotone operator.

Proof. Let $u \in H$ be a solution of(1). Then

$$\begin{aligned} 0 &\in \rho Tu + u - g(u) + \rho A(u) \\ &\Longleftrightarrow -(g(u) - \rho Tu) + (I + \rho A)(u) \\ &\Longleftrightarrow u = (I + \rho A)^{-1}[g(u) - \rho Tu] = J_A[g(u) - \rho Tu], \end{aligned}$$

the required result. $\square$

It is clear from Lemma 2.2 that general variational inclusion (1) and the fixed point problems are equivalent. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

We now recall some well known concepts and notions.

DEFINITION 2.2. A mapping $T : H \to H$ is called β-Lipschitz if for all $x, y \in H$, there exists a constant $\beta > 0$, such that

$$||Tx - Ty|| \leqslant \beta ||x - y||.$$

DEFINITION 2.3. A mapping $T : H \to H$ is called α-strongly monotone if for all $x, y \in K$, there exists a constant $\alpha > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geqslant \alpha ||x - y||^2.$$

3. Main Results

In this section, we use the general resolvent operator technique to suggest and analyze some iterative methods for solving the general variational inclusion (1).

ALGORITHM 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes:

$$u_{n+1} = (1 - a_n)u_n + a_n J_A[g(u_n) - \rho Tu_n]. \tag{5}$$

where $a_n \in [0, 1]$ for all $n \geqslant 0$. Algorithm 3.1 is also known as Mann iteration.

We now discuss some special cases of Algorithm 3.1 for solving the mixed general variational inequalities (2).

I. If $A(.) \equiv \partial\varphi(.)$, the subdifferential of a proper lower-semicontinuous and convex function φ, then $J_A = J_\varphi = (I + \rho\partial\varphi)^{-1}$ and consequently Algorithm 3.1 collapses to:

ALGORITHM 3.2. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = (1 - a_n)u_n + a_n J_{\partial\varphi}[g(u_n) - \rho T u_n],$$

where $a_n \in [0,1]$ for all $n \geqslant 0$. Algorithm 3.2 is called one-step method for solving the general mixed variational inequalities (2) and appears to be a new one.

II. If φ is the indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto the closed convex set K. In this case Algorithm 3.2 reduces to the following method.

ALGORITHM 3.3. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = (1 - a_n)u_n + a_n P_K[g(u_n) - \rho T u_n],$$

where $a_n \in [0,1]$ for all $n \geqslant 0$. Algorithm 3.3 is a one-step method for solving the general variational inequalities (3). Noor [29] has studied the convergence analysis of Algorithm 3.3 and its various special cases.

In brief, Algorithm 3.1 is quite general and includes several iterative methods for solving general mixed variational inequalities and related optimization problems as special cases.

We now study those conditions under which the approximate solution obtained from Algorithm 3.1 to a solution of the variational inclusion (1).

THEOREM 3.1. *Let T be α-strongly monotone with constant $\alpha > 0$ and β-Lipschitz with constant $\beta > 0$ and let g be σ-strongly monotone with constant $\sigma > 0$ and δ-Lipschitz with constant $\delta > 0$. If*

$$\left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - \beta^2(2k - k^2)}}{\beta^2}, \tag{6}$$

$$\alpha > \beta\sqrt{k(2-k)}, \quad k < 1, \tag{7}$$

where

$$k = \sqrt{1 - 2\sigma + \delta^2}, \tag{8}$$

and $a_n \in [0,1]$, $\sum_{n=0}^{\infty} a_n = \infty$, then the approximate solution u_{n+1} obtained from Algorithm 3.1 converges to a solution u of the general variational inclusion (1).

Proof. Let $u \in H$ be a solution of (1). Then, from Lemma 2.1, we have

$$u = (1 - a_n)u + a_n J_A\{g(u) - \rho T u\} \tag{9}$$

where $a_n \in [0,1]$ and $u \in H$ is a solution of (1). To prove the result, we need first to evaluate $||u_{n+1} - u||$ for all $n \geqslant 0$. From (5) and (10), and the nonexpansivity of J_A, we have

$$\begin{aligned}||u_{n+1}-u|| &= ||(1-a_n)u_n + a_nJ_A\{g(u_n)-\rho Tu_n\} - (1-a_n)u - a_nJ_A\{g(u)-\rho Tu\}|| \\ &\leqslant (1-a_n)||u_n-u|| + a_n||g(u_n)-g(u)-\rho(Tu_n-Tu)|| \\ &\leqslant (1-a_n)\|u_n-u\| + a_n\|u_n-u-(g(u_n)-g(u))\| \\ &\quad a_n\|u_n-u-\rho(Tu_n-Tu)\|. \end{aligned} \tag{10}$$

From the α-strongly monotonicity and the β-Lipschitz of the operator $T,$ we have

$$\begin{aligned}||u_n-u-\rho(Tu_n-Tu)||^2 &= ||u_n-u||^2 - 2\rho\langle Tu_n-Tu, u_n-u\rangle + \rho^2||Tu_n-Tu||^2 \\ &\leqslant ||u_n-u||^2 - 2\rho\|u_n-u\|^2 + \beta^2\rho^2||u_n-u||^2 \\ &= \sqrt{1-2\rho\alpha+\rho^2\beta^2}\|u_n-u\|^2, \end{aligned} \tag{11}$$

In a similar way, we have

$$\begin{aligned}\|u_n-u-(g(u_n)-g(u))\|^2 &\leqslant [1-2\sigma+\delta^2]\|u_n-u\|^2 \\ &= k^2\|u_n-u\|^2, \end{aligned} \tag{12}$$

where k is defined by (8).

Combining (10), (11) and (12), we have

$$\begin{aligned}\|u_{n+1}-u\| &\leqslant (1-a_n)\|u_n-u\| + a_n\theta\|u_n-u\| \\ &\leqslant [1-a_n(1-\theta)]\,\|u_n-u\|, \end{aligned}$$

where $\theta = \sqrt{1-2\rho\alpha+\rho^2\beta^2}+k.$

Since $\sum_{n=0}^{\infty} a_n = \infty$ and $1-\theta > 0,$ we have , $\lim_{n\to\infty}||u_{n+1}-u|| = 0,$ which completing the proof. □

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