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FIXED POINTS AND GENERALIZED STABILITY FOR FUNCTIONAL EQUATIONS IN ABSTRACT SPACES

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. We use a fixed point method, initiated in [V. Radu, Fixed Point Theory 4(2003), No.1, 91-96], to prove the generalized Ulam-Hyers stability of functional equations in single variable for mappings with values in random normed spaces. This result is then used to obtain the stability for Cauchy, quadratic and monomial functional equations.

1. Introduction

D. H. Hyers [17] has given an affirmative answer to a question of Ulam by proving the stability of additive Cauchy equations in Banach spaces. Then T. Aoki [1] and Th. M. Rassias [26] considered the stability problem with unbounded Cauchy differences for additive and linear mappings, respectively. Their results include the following

THEOREM 1.1. (Hyers [17], Aoki [1], Gajda [12]). *Suppose that E is a real normed space, F is a real Banach space and $f:E\to F$ is a given function, such that the following condition holds*

$$\|f(x+y)-f(x)-f(y)\|_F \leqslant \theta(\|x\|_E^p+\|y\|_E^p),\ \forall x,y\in E, \tag{$\mathbf{1_p}$}$$

for some $p\in[0,\infty)\setminus\{1\}$. Then there exists a unique additive function $a:E\to F$ such that

$$\|f(x)-a(x)\|_F \leqslant \frac{2\theta}{|2-2^p|}\|x\|_E^p,\ \forall x\in E. \tag{$\mathbf{2_p}$}$$

This phenomenon is called *generalized Ulam-Hyers stability* or *Hyers-Ulam-Rassias stability* and has been extensively investigated for different functional equations. It is worth mentioning that almost all proofs used the idea conceived by D. H. Hyers. Namely, the additive function $a:E\to F$ is constructed, starting from the given function f, by the following formulae

$$a(x)=\lim_{n\to\infty}\frac{1}{2^n}f(2^nx),\ \text{if } p<1; \tag{$\mathbf{2_{p<1}}$}$$

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$$a(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right), \quad \text{if } p > 1. \qquad (\mathbf{2_{p>1}})$$

This method is called *the direct method* or *Hyers method*. It is often used to construct a solution of a given functional equation and is seen to be a powerful tool for studying the stability of many functional equations (see [10], [19], [13], [3], [18], for details).

On the other hand, in [25], [6] and [4] a *fixed point method* was proposed, by showing that many theorems concerning the stability of Cauchy and Jensen equations are consequences of the fixed point alternative. Subsequently, the method has been successfully used, e.g., in [5], [8] or [24].

Our aim is to highlight generalized Ulam-Hyers stability results for functional equations in a single variable for mappings with values in random normed spaces, obtained by using the fixed point alternative. The method introduces a metrical context and better clarifies the ideas of stability, which is seen to be plainly related to some fixed point of a suitable operator: our control conditions are responsible for the following fundamental facts: They ensure

1) the *contraction property* of a Schröder type operator J and

2) the first two successive approximations, f and Jf, to be at a *finite distance*.

Moreover,

3) they force the fixed point of J to be a *solution of the initial equation*.

Some illustrative applications to concrete (Cauchy, quadratic and monomial) functional equations are also given.

2. A general fixed point method

For the sake of convenience, we recall the following *alternative of fixed point* ([20], see also [27], chapter 5):

THEOREM 2.1. *Suppose we are given a complete generalized metric space $(\mathscr{E}, d)$ -i.e. one for which the metric d may assume infinite values- and a strictly contractive mapping $A : \mathscr{E} \to \mathscr{E}$, with the Lipschitz constant L. Then, for each given element $f \in \mathscr{E}$,*

either

(A_1) $d(A^n f, A^{n+1} f) = +\infty \quad , \ \forall n \geqslant 0,$

or

(A_2) *There exists a natural number n_0 such that*

(A_{20}) $d(A^n f, A^{n+1} f) < +\infty, \ \forall n \geqslant n_0$;

(A_{21}) *The sequence $(A^n f)$ is convergent to a fixed point f^* of A ;*

(A_{22}) *f^* is the unique fixed point of A in the set $\mathscr{E}^\star = \{g \in \mathscr{E},\ d(A^{n_0} f, g) < +\infty\}$;*

(A_{23}) $d(g, f^*) \leqslant \frac{1}{1-L} d(g, Ag), \forall g \in \mathscr{E}^\star$.

REMARK 2.1. The fixed points of A, if any, need not be uniquely determined *in the whole space $\mathscr{E}$* and do depend on the initial guess f.

We recall (see, e.g. Schweizer&Sklar, [28]) that a *distribution function* F is a mapping from $[0, \infty)$ into $[0, 1]$, nondecreasing and left-continuous, with $F(0) = 0$.

The class of all distribution functions F, with $\lim_{x\to\infty} F(x) = 1$, is denoted by D_+. An element of D_+ is $\varepsilon_0 = \begin{cases} 0, & \text{if } t = 0, \\ 1, & \text{if } t > 0. \end{cases}$.

Suppose that X is a real vector space, F is a mapping from X into D_+ (for any x in X, $F(x)$ is denoted by F_x) and T is a t-norm. The triple (X,F,T) is called a *random normed space* (briefly RN-space) iff the following conditions are satisfied:

(RN-1): $F_x = \varepsilon_0$ iff $x = \theta$, the null vector;

(RN-2): $F_{ax}(t) = F_x\left(\frac{t}{|a|}\right), \forall t > 0, \forall a \in \mathbb{R}, a \neq 0$;

(RN-3): $F_{x+y}(t_1+t_2) \geqslant T(F_x(t_1), F_y(t_2)), \forall x,y \in X$ and $t_1,t_2 > 0$.

Recall that a *triangular norm (t-norm)* is a mapping $T : [0,1]\times[0,1] \to [0,1]$, which is associative, commutative and increasing in each variable, with $T(a,1) = 1, \forall a \in [0,1]$. The most important t-norms are $T_M(a,b) = \min\{a,b\}$, $T_1(a,b) = \max\{a+b-1,0\}$, $\mathrm{Prod}(a,b) = a\cdot b$.

Every normed space $(X,||\cdot||)$ defines a random normed space (X,F,T_M), with

$$F_x(t) = \frac{t}{t+||x||},\ \forall t > 0.$$

If the $t-$norm T is such that $\sup_{0<a<1} T(a,a) = 1$, then every random normed space (X,F,T) is a *metrizable linear topological space* with the $(\varepsilon,\lambda)-$topology, induced by the base of neighborhoods $U(\varepsilon,\lambda) = \{x \in X, F_x(\varepsilon) > 1-\lambda\}$. If $T = T_M$, then (X,F,T) is locally convex.

A sequence (x_n) in a random normed space (X,F,T) *converges* to $x \in X$ in the $(\varepsilon,\lambda)-$topology if $\lim_{n\to\infty} F_{x_n-x}(t) = 1, \forall t > 0$. A sequence (x_n) is called *Cauchy sequence* if $\lim_{m,n\to\infty} F_{x_n-x_m}(t) = 1, \forall t > 0$. The random normed space (X,F,T) is *complete* if every Cauchy sequence in X is convergent.

In the following theorem we prove a generalized Ulam-Hyers stability result (compare with [2], [11] and [9]) for the single variable functional equation

$$(w\circ g\circ\eta)(x) = g(x),\ \forall x \in X, \tag{2.1}$$

where

1. the unknown is a mapping g from the Abelian group X into a *complete random normed space* (Y,F,T_M)
2. η is a self-mapping of the Abelian group X;
3. w is a Lipschitzian self-mapping (with Lipschitz constant ℓ_w) of the complete random normed space Y.

THEOREM 2.2. *Suppose that* $f : X \to Y$ *satisfies*

$$F_{(w\circ f\circ\eta)(x)-f(x)} \geqslant \psi(x),\ \forall x \in X, \tag{$\mathbf{C}_\psi$}$$

with some fixed mapping $\psi : X \to D_+$. *If there exists* $L < 1$ *such that*

$$\psi(\eta(x))(Lt) \geqslant \psi(x)(\ell_w t),\ \forall x \in X, \forall t > 0 \tag{$\mathbf{H}_\psi$}$$

then there exists a ***unique mapping*** $c : X \to Y$ *which satisfies both*
the equation

$$(w \circ c \circ \eta)(x) = c(x), \forall x \in X \qquad (\mathbf{E}_{w,\eta})$$

and ***the estimation***

$$F_{c(x)-f(x)}(t) \geqslant \psi(x)((1-L)t), \forall x \in X, \qquad \left(\mathbf{Est}_{\psi}\right)$$

for almost all $t > 0$. *Namely, the solution mapping* c *can be acquired through the Hyers method:*

$$c(x) = \lim_{n\to\infty} \left(w^n \circ f \circ \eta^n\right)(x), \forall x \in X.$$

Proof. Let us consider the mapping $G(x,t) := \psi(x)(t)$, and the set $\mathscr{E} := \{g|\ g : X \to Y \text{ is a function}\}$. We introduce a generalized metric on $\mathscr{E}$ (as usual, $inf\ \emptyset = \infty$):

$$d(g,h) = d_G(g,h) = \inf\left\{K \in \mathbb{R}_+, F_{g(x)-h(x)}(Kt) \geqslant G(x,t),\ \forall x \in X, \forall t > 0\right\}.$$

The proof of the fact that $(\mathscr{E}, d_G)$ is a *complete generalized metric space* can be found e.g. in (Hadžić & Pap & Radu, [16]) or (Miheţ & Radu, [21]).

Now, define the mapping

$$J : \mathscr{E} \to \mathscr{E}, Jg(x) := (w \circ g \circ \eta)(x). \qquad (\mathbf{OP})$$

Step I. Using our hypotheses, it follows that J *is strictly contractive* on $\mathscr{E}$. Indeed, for any $g,h \in \mathscr{E}$ we have:

$$d(g,h) < K \Longrightarrow F_{g(x)-h(x)}(Kt) \geqslant G(x,t), \forall x \in X, \forall t > 0$$

and

$$\begin{aligned} F_{Jg(x)-Jh(x)}(LKt) &= F_{w(g(\eta(x)))-w(h(\eta(x)))}(LKt) \geqslant F_{\ell_w(g(\eta(x))-h(\eta(x)))}(LKt) \\ &= F_{(g(\eta(x)))-h(\eta(x))}\left(\frac{LKt}{\ell_w}\right) \geqslant G\left(\eta(x), \frac{Lt}{\ell_w}\right) \\ &= \psi(\eta(x))\left(\frac{Lt}{\ell_w}\right) \geqslant \psi(x)(t) = G(x,t). \end{aligned}$$

Therefore $d(Jg,Jh) \leqslant LK$, which implies

$$d(Jg,Jh) \leqslant Ld(g,h), \forall g,h \in \mathscr{E}. \qquad (\mathbf{CC_L})$$

This says that J is a *strictly contractive* self-mapping of $\mathscr{E}$, with the Lipschitz constant $L < 1$.

Step II. Obviously, $d(f,Jf) < \infty$. In fact, using the relation $(\mathbf{C}_{\psi})$ it results that $d(f,Jf) \leqslant 1$.

Step III. Now we can apply the fixed point alternative to obtain the existence of a mapping $c: G \to Y$ such that:

- c is a fixed point of J, that is

$$(w \circ c \circ \eta)(x) = c(x) \ , \forall x \in X. \qquad (\mathbf{E}_{w,\eta})$$

The mapping c is the unique fixed point of J in the set

$$\mathscr{F} = \{g \in \mathscr{E},\ d_G(f,g) < \infty\}.$$

This says that c is the unique mapping verifying *both* $(\mathbf{E}_{w,\eta})$ and (2.2), where

$$\exists K < \infty \text{ such that } F_{c(x)-f(x)}(Kt) \geqslant \psi(x)(t), \forall x \in X, \forall t > 0. \qquad (2.2)$$

Moreover,

- $d(J^n f, c) \xrightarrow[n\to\infty]{} 0$, which implies $\lim\limits_{n\to\infty} F_{c(x)-J^n f(x)}(t) = 1, \forall t > 0$ and $\forall x \in X$, whence

$$c(x) = \lim_{n\to\infty} (w^n \circ c \circ \eta^n)(x), \forall x \in X.$$

- $d(f,c) \leqslant \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f,c) \leqslant \frac{1}{1-L},$$

hence

$$F_{c(x)-f(x)}\left(\frac{t}{1-L}\right) \geqslant G(x,t), \ \forall x \in X,$$

for almost all $t > 0$. It results that

$$F_{c(x)-f(x)}(t) \geqslant G(x,(1-L)t) = \psi(x)((1-L)t), \ \forall x \in X,$$

for almost all $t > 0$, that is $(\mathbf{Est}_\psi)$ is seen to be true. $\square$

3. Applications to Cauchy functional equations and to quadratic functional equations

We will consider the additive Cauchy functional equation

$$f(x+y) = f(x) + f(y), \quad \text{for all } x, y \in X, \qquad (3.1)$$

where the "unknowns" are functions $f: X \to Y$, between two vector spaces. By using Theorem 2.2 we prove a generalized Ulam-Hyers stability result for (3.1) (compare with [22]).

Recall that a 2-divisible group is an Abelian group $(X,+)$ such that for any $x \in X$ there exists a unique $a \in X$ with the property $x = 2a$; this unique element a is denoted by $\frac{x}{2}$.

THEOREM 3.1. *Let X be a 2-divisible group, (Y,F,T_M) a complete random normed space, and a fixed $i \in \{0,1\}$. Suppose that the mapping $f: X \to Y$ satisfies an inequality of the form*

$$F_{f(x+y)-f(x)-f(y)} \geqslant \varphi(x,y),\ \forall x,y \in X, \tag{$\mathbf{Add}_\varphi$}$$

for some function $\varphi : X \times X \to D_+$.

If there exists $L = L(i) < 1$ such that

$$\varphi(2^{1-i}x, 2^{1-i}y)\left(\frac{Lt}{2^{i-1}}\right) \geqslant \varphi(2^i x, 2^i y)(2^i t), \forall x \in X, \forall t > 0, \tag{$\varphi_\mathbf{i}$}$$

then there exists a ***unique additive*** *mapping $a : X \to Y$ which satisfies the inequality*

$$F_{a(x)-f(x)}(t) \geqslant \varphi\left(\frac{x}{2^i}, \frac{x}{2^i}\right)\left(\frac{1-L}{2^{i-1}}t\right), \forall x \in X, \tag{$\mathbf{Est}_i$}$$

for almost all $t > 0$.

Proof. In case $i = 0$ we set $x = y$ in $(\mathbf{Add}_\varphi)$ and we see that

$$F_{f(2x)-2f(x)}(t) \geqslant \varphi(x,x)(t), \forall x \in X, \forall t > 0$$

hence

$$F_{\frac{f(2x)}{2}-f(x)}(t) \geqslant \varphi(x,x)(2t), \forall x \in X, \forall t > 0. \tag{$\mathbf{A}_{\varphi,\mathbf{0}}$}$$

In case $i = 1$ we replace both x and y by $\frac{x}{2}$ in $(\mathbf{Add}_\varphi)$ and we obtain

$$F_{2f\left(\frac{x}{2}\right)-f(x)}(t) \geqslant \varphi\left(\frac{x}{2}, \frac{x}{2}\right)(t), \forall x \in X, \forall t > 0. \tag{$\mathbf{A}_{\varphi,\mathbf{1}}$}$$

Now we can apply Theorem 2.2 with $w : X \to Y$, $\eta : X \to X$ and $\psi : X \to D_+$ defined by

$$w(x) := \frac{x}{2^{1-2i}},\ \eta(x) := 2^{1-2i}x,\ \psi(x)(t) := \varphi\left(\frac{x}{2^i}, \frac{x}{2^i}\right)(2^{1-i}t), i \in \{0,1\}.$$

Clearly, $\ell_w = \frac{1}{2^{1-2i}}$ and, by using $(\mathbf{A}_{\varphi,\mathbf{i}})$ and the hypothesis $(\varphi_\mathbf{i})$, we obtain that $(\mathbf{C}_\psi)$ and $(\mathbf{H}_\psi)$ hold true.

Then there exists a unique mapping $a : X \to Y$,

$$a(x) := \lim_{n\to\infty}(w^n \circ a \circ \eta^n)(x) = \lim_{n\to\infty}\frac{a\left(2^{n(1-2i)}x\right)}{2^{n(1-2i)}}, \forall x \in X, \tag{3.2}$$

which satisfies the following equation

$$(w \circ a \circ \eta)(x) = a(x) \Leftrightarrow a(2x) = 2a(x),\ \forall x \in X$$

and the inequality

$$F_{f(x)-a(x)}(t) \geqslant \psi(x)((1-L)t) = \varphi\left(\frac{x}{2^i}, \frac{x}{2^i}\right)(2^{1-i}(1-L)t), \forall x \in X,$$

for almost $t > 0$.

The additivity of a can be proven by using the fact that the continuity of T_M implies continuity of the mapping $z \to F_z$. Actually, by relation $(\varphi_{\mathbf{i}})$, we will obtain

$$\begin{aligned}
F_{a(x+y)-a(x)-a(y)}(t) &= \lim_{n\to\infty} F_{\frac{a\left(2^{n(1-2i)}(x+y)\right)}{2^{n(1-2i)}} - \frac{a\left(2^{n(1-2i)}x\right)}{2^{n(1-2i)}} - \frac{a\left(2^{n(1-2i)}y\right)}{2^{n(1-2i)}}}(t) \\
&= \lim_{n\to\infty} F_{a\left(2^{n(1-2i)}(x+y)\right)-a\left(2^{n(1-2i)}x\right)-a\left(2^{n(1-2i)}y\right)}\left(2^{n(1-2i)}t\right) \\
&\geqslant \lim_{n\to\infty} \varphi\left(2^{n(1-2i)}x, 2^{n(1-2i)}y\right)(2^{n(1-2i)}t) \\
&\geqslant \lim_{n\to\infty} \varphi(x,y)\left(\frac{t}{L^n}\right) = 1, \forall x,y \in X,
\end{aligned}$$

for almost all $t > 0$. Therefore

$$F_{a(x+y)-a(x)-a(y)}(t) = 1, \forall x,y \in X,$$

for almost all $t > 0$, whence $a(x+y) - a(x) - a(y) = 0$, for all $x,y \in X$. □

Now we will consider the following quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \qquad \text{for all } x,y \in X, \tag{3.3}$$

where $f : X \to Y$, is a mapping between two vector spaces. As an application of Theorem 2.2 we prove a generalized Ulam-Hyers stability result for the equations of type (3.3) and mappings with values in random normed spaces (compare with Mirmostafaee & Moslehian, [23], Theorem 3.3).

THEOREM 3.2. *Let us consider a 2-divisible group X, a complete random normed space (Y,F,T_M), and a fixed $i \in \{0,1\}$. Suppose that the mapping $f : X \to Y$, with $f(0) = 0$, satisfies an inequality of the form*

$$F_{f(x+y)+f(x-y)-2f(x)-2f(y)} \geqslant \varphi(x,y), \forall x,y \in X, \tag{$\mathbf{Quad}_\varphi$}$$

where $\varphi : X \times X \to D_+$ is a given function.

If there exists $L = L(i) < 1$ such that

$$\varphi(2^{1-i}x, 2^{1-i}y)\left(\frac{Lt}{2^{2(i-1)}}\right) \geqslant \varphi(2^i x, 2^i y)(2^{2i}t), \forall x \in X, \forall t > 0, \tag{$\varphi_{\mathbf{i}}$}$$

then there exists a unique quadratic mapping $q : X \to Y$ which satisfies the inequality

$$F_{q(x)-f(x)}(t) \geqslant \varphi\left(\frac{x}{2^i}, \frac{x}{2^i}\right)\left(\frac{1-L}{2^{2(i-1)}}t\right), \forall x \in X, \tag{$\mathbf{Est}_i$}$$

for almost all $t > 0$.

Proof. It is similar to that of Theorem 3.1. We apply Theorem 2.2 for $w: X \to Y$, $\eta: X \to X$ and $\psi: X \to D_+$ given by

$$w(x) := \frac{x}{2^{2(1-2i)}},\ \eta(x) := 2^{1-2i}x,\ \psi(x)(t) := \varphi\left(\frac{x}{2^i}, \frac{x}{2^i}\right)(2^{2(1-i)}t). \qquad \square$$

4. The general case of the monomial functional equation

For an Abelian group X and a vector space Y consider the difference operators defined, for $y \in X$ and mappings $f: X \to Y$, in the following manner:

$$\Delta_y^1 f(x) := f(x+y) - f(x), \text{ for all } x \in X,$$

and, inductively,

$$\Delta_y^{n+1} = \Delta_y^1 \circ \Delta_y^n, \text{ for all } n \geqslant 1.$$

A mapping $f: X \to Y$ is called a *monomial function of degree N* if it is a solution of the *monomial functional equation.*

$$\Delta_y^N f(x) - N!f(y) = 0,\ \forall\, x, y \in X. \tag{4.1}$$

Notice that the monomial equation of degree 1 is exactly the additive Cauchy equation, while for $N = 2$ the monomial equation has the form $f(x+2y) - 2f(x+y) + f(x) - 2f(y) = 0$, which is equivalent to the well-known quadratic functional equation.

In the sequel, the positive integer N will be fixed.

THEOREM 4.1. *Let X be a linear space and (Y, F, T_M) be a complete random normed space. Suppose that the mapping $f: X \to Y$, with $f(0) = 0$, satisfies an inequality of the form*

$$F_{\Delta_y^N f(x) - N!f(y)} \geqslant \varphi(x,y), \forall x, y \in X, \tag{$\mathbf{Mon}_\varphi$}$$

where $\varphi: X \times X \to D_+$ is a given function.

If there exists $L = L(i) < 1$ such that

$$\varphi(2x, 2y)\left(2^N Lt\right) \geqslant \varphi(x,y)(t), \forall x \in X, \forall t > 0, \tag{$\varphi_\mathbf{i}$}$$

then there exists a unique monomial mapping $M: X \to Y$ which satisfies the inequality

$$F_{M(x)-f(x)}(t) \geqslant \operatorname{Min}\left\{\varphi(0,2x)\left(\frac{N!\cdot 2^N}{N+1}t\right); \left\{\varphi(ix,x)\left(\frac{N!\cdot 2^N}{(N+1)\cdot\binom{N}{N-i}}t\right)\right\}_{i=\overline{0,N}}\right\}, \tag{$\mathbf{Est}_i$}$$

$\forall x \in X$, for almost all $t > 0$.

For the proof of our theorem, we need the following fundamental lemma:

LEMMA 4.2. *Let us consider an Abelian group X, a random normed space (Y,F,T_M) and a mapping $\varphi : X\times X \to [0,\infty)$. If the function $f : X \to Y$ satisfies $(\mathbf{Mon}_\varphi)$ then, for all $x\in X$ and $t>0$, we have*

$$F_{\frac{f(2x)}{2^N}-f(x)}(t) \geqslant \mathrm{Min}\left\{\varphi(0,2x)\left(\frac{N!\cdot 2^N}{N+1}t\right);\left\{\varphi(ix,x)\left(\frac{N!\cdot 2^N}{(N+1)\cdot\binom{N}{N-i}}t\right)\right\}_{i=\overline{0,N}}\right\} \tag{4.2}$$

Proof. As in [15], we define the functions $T_i : G\to Y$, $i=0,1,...,N$, by

$$T_i(x) := \Delta_x^N f(ix)-(N!)f(x)\ , \quad \forall x\in X.$$

By using the relation $(\mathbf{Mon}_\varphi)$ we see that

$$F_{T_i(x)} \geqslant \varphi(ix,x)\ ,\ \forall x\in X \tag{4.3}$$

and

$$F_{T_0(2x)} \geqslant \varphi(0,2x)\ ,\ \forall x\in X. \tag{4.4}$$

Following the ideas from (Gilanyi, [14], Lemma 2.2.) or (Cădariu & Radu, [7], Lemma 2.2.) it results the identity

$$K_0T_0(x)+K_1T_1(x)+...+K_NT_N(x)+2^N(N!)f(x) = T_0(2x)+(N!)f(2x)$$

for all $x\in X$, where $K_i=\binom{N}{N-i}$, $i=0,1,...,N$. Using (4.3) and (4.4) we get, for all $x\in X$ and $t>0$,

$$\begin{aligned}
&F_{\frac{f(2x)}{2^N}-f(x)}(t) = F_{\frac{1}{N!\cdot 2^N}(K_0T_0(x)+K_1T_1(x)+...+K_NT_N(x)-T_0(2x))}(t)\\
&= F_{K_0T_0(x)+K_1T_1(x)+...+K_NT_N(x)-T_0(2x)}(N!\cdot 2^N t)\\
&\geqslant T_M\left(F_{T_0(2x)}\left(\frac{N!\cdot 2^N}{N+1}t\right), F_{K_0T_0(x)+K_1T_1(x)+...+K_NT_N(x)}\left(\frac{N!\cdot 2^N}{N+1}Nt\right)\right)\\
&\geqslant T_M\left(F_{T_0(2x)}\left(\frac{N!\cdot 2^N t}{N+1}\right), T_M\left(F_{\sum_{j=0}^{N-1}K_jT_j(x)}\left(\frac{N!\cdot 2^N(N-1)t}{N+1}\right), F_{K_NT_N(x)}\left(\frac{N!\cdot 2^N t}{N+1}\right)\right)\right)\\
&\geqslant \mathrm{Min}\left\{\varphi(0,2x)\left(\frac{N!\cdot 2^N}{N+1}t\right);\left\{\varphi(ix,x)\left(\frac{N!\cdot 2^N}{(N+1)\cdot\binom{N}{N-i}}t\right)\right\}_{i=\overline{0,N}}\right\} \qquad \square
\end{aligned}$$

Proof of Theorem 4.1. We apply Theorem 2.2 for $w: X\to Y$, $\eta : X\to X$ and $\psi : X\to D_+$ defined by $w(x) := \frac{x}{2^N}$, with $\ell_w = \frac{1}{2^N}$, $\eta(x) := 2x$, and

$$\psi(x)(t) := \mathrm{Min}\left\{\varphi(0,2x)\left(\frac{N!\cdot 2^N}{N+1}t\right);\left\{\varphi(ix,x)\left(\frac{N!\cdot 2^N}{(N+1)\cdot\binom{N}{N-i}}t\right)\right\}_{i=\overline{0,N}}\right\}.$$

We only notice that

$$M(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^{nN}}$$

is the fixed point of the mapping given by $Jg(x) := \frac{g(2x)}{2^N}$ and the fact that M is a monomial function can be proven in a similar way as the additivity of a (in Theorem 3.1). □

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REFERENCES

[1] T. AOKI, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64–66.

[2] J. A. BAKER, *The stability of certain functional equations*, Proc. AMS, **112**, 3 (1991), 729–732.

[3] D. G. BOURGIN, *Classes of transformations and bordering transformations*, Bull. AMS, **57** (1951), 223–237.

[4] L. CĂDARIU, V. RADU, *Fixed points and the stability of Jensen's functional equation*, JIPAM, J. Inequal. Pure Appl. Math., **4**, 1 (2003), Art.4 (http://jipam.vu.edu.au).

[5] L. CĂDARIU, V. RADU, *Fixed points and the stability of quadratic functional equations*, An. Univ. Timişoara, Ser. Mat.-Inform., **41**, 1 (2003), 25–48.

[6] L. CĂDARIU, V. RADU, *On the stability of the Cauchy functional equation: a fixed points approach*, Iteration Theory (ECIT 02) (J. Sousa Ramos, D. Gronau, C. Mira, L. Reich, A.N. Sharkovsky - Eds.), Grazer Math. Ber., Bericht, **346** (2004), 43–52.

[7] L. CĂDARIU, V. RADU, *Stability properties for monomial functional equations*, An. Univ. Timişoara, Ser. Mat.-Inform., **43**, 1 (2005), 23–38.

[8] L. CĂDARIU, V. RADU, *Fixed points in generalized metric spaces and the stability of a cubic functional equation*, Y. J. Cho, J. K. Kim & S. M. Kang - Eds., Fixed Point Theory and Applications **7**, Nova Science Publ., 2007, 53–68.

[9] L. CĂDARIU, V. RADU, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory and Applications, Vol. **2008** (2008), Article ID 749392, 15 pages.

[10] G. L. FORTI, *Hyers-Ulam stability of functional equations in several variables*, Aeq. Math., **50** (1995), 143–90.

[11] G. L. FORTI, *Elementary remarks on Ulam-Hyers stability of linear functional equations*, J. Math. Anal. Appl., **328**, 1 (2007), 109–118.

[12] Z. GAJDA, *On stability of additive mappings*, Internat. J. Math. Math. Sci., **14** (1991), 431–434.

[13] P. GĂVRUŢĂ, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431–436.

[14] A. GILANYI, *A characterization of monomial functions*, Aequationes Math., **54**, 3 (1997), 289–307.

[15] A. GILANYI, *On the stability of monomial functional equations*, Publ. Math., **56**, 1-2 (2000), 201–212.

[16] O. HADŽIĆ, E. PAP, V. RADU, *Generalized contraction mapping principles in probabilistic metric spaces*, Acta Math. Hungar., **101** (2003), 131–148.

[17] D. H. HYERS, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224.

[18] D. H. HYERS, *Transformations with bounded m-th differences*, Pacific J. Math., **11** (1961), 591–602.

[19] D. H. HYERS, G. ISAC, TH. M. RASSIAS, *Stability of Functional Equations in Several Variables*, Basel, 1998.

[20] B. MARGOLIS, J. B. DIAZ, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **74** (1968), 305–309.

[21] D. MIHEŢ, V. RADU, *Generalized pseudo-metrics and fixed points in probabilistic metric spaces*, Carpathian Journal of Mathematics, **23**, 1-2 (2007), 126–132.

[22] D. MIHEŢ, V. RADU, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl., **343**, 1 (2008), 567–572.

[23] A. K. MIRMOSTAFAEE, M. S. MOSLEHIAN, *Fuzzy almost quadratic functions*, Results in Math., **52** (2008), 161–177.

[24] M. MIRZAVAZIRI, M. S. MOSLEHIAN, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc., **37**, 3 (2006), 361–376.

[25] V. RADU, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, **4**, 1 (2003), 91–96.

[26] TH. M. RASSIAS, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300.

[27] I. A. RUS, *Principii şi Aplicaţii ale Teoriei Punctului Fix*, Ed. Dacia, Cluj-Napoca, 1979.

[28] B. SCHWEIZER, A. SKLAR, *Probabilistic metric spaces*, North Holland, Series in Probability and Applied Mathematics 1983, Second Ed. Dover Publications 2005.

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