

## ALMOST EVERYWHERE CONVERGENCE OF A SUBSEQUENCE OF THE NÖRLUND LOGARITHMIC MEANS OF WALSH–KACZMARZ–FOURIER SERIES

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*Abstract.* The main aim of this paper is to prove that the maximal operator of a subsequence of the (one-dimensional) logarithmic means of Walsh-Kaczmarz-Fourier series is of weak type  $(1,1)$ . Moreover, we prove that the maximal operator of the logarithmic means of quadratical partial sums of double Walsh-Kaczmarz-Fourier series is of weak type  $(1,1)$ , provided that the supremum in the maximal operator is taken over special indices. The set of Walsh-Kaczmarz polynomials is dense in  $L^1$ , so by the well-known density argument the logarithmic means  $t_{2n}^K(f)$  converge a.e. to  $f$  for all integrable function  $f$ .

### 1. Introduction

The  $n$ -th Riesz's logarithmic means of a Fourier series is defined by

$$\frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k(f)}{k}, \quad l_n := \sum_{k=1}^{n-1} \frac{1}{k}.$$

The Riesz's logarithmic means with respect to the trigonometric system was studied by a lot of authors, e.g. Szász [17] and Yabuta [18], with respect to Walsh, Vilenkin system by Simon [13] and Gát [4].

Let  $\{q_k : k \geq 0\}$  be a sequence of nonnegative numbers, the  $n$ -th Nörlund means of an integrable function  $f$  is defined by

$$\frac{1}{Q_n} \sum_{k=1}^{n-1} q_{n-k} S_k(f),$$

where  $Q_n := \sum_{k=1}^{n-1} q_k$ . This Nörlund means of Walsh-Fourier series was investigated by Móricz and Siddiqi [10]. The case, when  $q_k = \frac{1}{k}$  is excluded, since the method of Móricz and Siddiqi does not work in this case.

If  $q_k := \frac{1}{k}$ , then we get the (Nörlund) logarithmic means:

$$t_n(f) := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k(f)}{n-k}.$$

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From now, we will write simply logarithmic means  $t_n(f)$ . Recently, Gát and Goginava [5] proved some convergence and divergence properties of this logarithmic means of functions in the class of continuous functions, and in the Lebesgue space. They proved that the maximal norm convergence function space of this logarithmic means is  $L \log^+ L$ .

The a.e. convergence of a subsequence of logarithmic means of Walsh-Fourier series of integrable functions was discussed by Gát and Goginava [8, 6]. More results on this logarithmic means with respect to unbounded Vilenkin system can be found in [2].

First, we give a brief introduction to the theory of dyadic analysis [12, 1].

Denote by  $\mathbf{Z}_2$  the discrete cyclic group of order 2, that is  $\mathbf{Z}_2 = \{0, 1\}$ , the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on  $\mathbf{Z}_2$  is given in the way that the measure of a singleton is  $1/2$ . Let

$$G := \prod_{k=0}^{\infty} \mathbf{Z}_2,$$

$G$  is called the Walsh group. The elements of  $G$  can be represented by a sequence  $x = (x_0, x_1, \dots, x_k, \dots)$  where  $x_k \in \{0, 1\}$  ( $k \in \mathbf{N}$ ) ( $\mathbf{N} := \{0, 1, \dots\}$ ,  $\mathbf{P} := \mathbf{N} \setminus \{0\}$ ).

The group operation on  $G$  is the coordinate-wise addition (denoted by  $+$ ), the measure (denoted by  $\mu$ ) and the topology are the product measure and topology. Consequently,  $G$  is a compact Abelian group. Dyadic intervals are defined by

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\}$$

for  $x \in G, n \in \mathbf{P}$ . They form a base for the neighborhoods of  $G$ . Let  $0 = (0 : i \in \mathbf{N}) \in G$  and  $I_n := I_n(0)$  for  $n \in \mathbf{N}$ .

Furthermore, let  $L^p(G)$  denote the usual Lebesgue spaces on  $G$  (with the corresponding norm  $\|\cdot\|_p$ ),  $\mathcal{A}_n$  the  $\sigma$ -algebra generated by the sets  $I_n(x) (x \in G)$  and  $E_n$  the conditional expectation operator with respect to  $\mathcal{A}_n (n \in \mathbf{N})$ . The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N}).$$

Each natural number  $n$  can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0, 1\} \quad (i \in \mathbf{N}),$$

where only a finite number of  $n_i$ 's different from zero. Let the order of  $n > 0$  be denoted by  $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$ . That is,  $|n|$  is the integral part of the binary logarithm of  $n$  and  $2^{|n|} \leq n < 2^{|n|+1}$ .

Define the Walsh-Paley functions by

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k}.$$

Let the Walsh-Kaczmarz functions [16] be defined by  $\kappa_0 = 1$  and for  $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The Walsh-Paley system is  $\omega := (\omega_n : n \in \mathbf{N})$  and the Walsh-Kaczmarz system is  $\kappa := (\kappa_n : n \in \mathbf{N})$ . It is well known that

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{\omega_n : 2^k \leq n < 2^{k+1}\}$$

for all  $k \in \mathbf{N}$  and  $\kappa_0 = \omega_0$ .

A relation between Walsh-Kaczmarz functions and Walsh-Paley functions was given by V. A. Skvortsov in the following way (see [15]). Let the transformation  $\tau_A : G \rightarrow G$  be defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \dots, x_1, x_0, x_A, x_{A+1}, \dots)$$

for  $A \in \mathbf{N}$ . We have that

$$\kappa_n(x) = r_{|n|}(x) \omega_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G).$$

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, the Fejér kernels, the logarithmic means and logarithmic kernels:

$$\begin{aligned} \hat{f}^\alpha(n) &:= \int_G f \bar{\alpha}_n, \quad S_n^\alpha f := \sum_{k=0}^{n-1} \hat{f}^\alpha(k) \alpha_k, \quad D_n^\alpha := \sum_{k=0}^{n-1} \alpha_k, \\ \sigma_n^\alpha f &:= \frac{1}{n} \sum_{k=0}^n S_k^\alpha f, \quad K_n^\alpha := \frac{1}{n} \sum_{k=0}^n D_k^\alpha, \\ t_n^\alpha(f) &:= \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k^\alpha f}{n-k}, \quad F_n^\alpha := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{D_k^\alpha}{n-k}, \end{aligned}$$

where  $\alpha_n = \omega_n$  or  $\kappa_n$  ( $n \in \mathbf{P}$ ).  $D_0^\alpha := 0$ .

It is known [12] that

$$D_{2^n}(x) = \begin{cases} 2^n, & x \in I_n, \\ 0, & \text{otherwise } (n \in \mathbf{N}) \end{cases} \quad (1)$$

and  $E_n f = S_{2^n}(f)$ . The maximal operator  $\sigma^{K,*}$  is defined by  $\sigma^{K,*} f := \sup_{n \in \mathbf{P}} |\sigma_n^K f|$  for  $f \in L^1(G)$ . The maximal operator  $\sigma^{K,*}$  was investigated by G. Gát in [3].

Next, we introduce some notation with respect to the theory of two-dimensional system. Let the two-dimensional Walsh group be  $G \times G$  and the two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, Dirichlet kernels, the Marcinkiewicz means and Marcinkiewicz kernels be defined as:

$$\begin{aligned} \hat{f}^\alpha(n_1, n_2) &:= \int_{G \times G} f \alpha_{n_1} \alpha_{n_2} d\mu, \\ S_{n_1, n_2}^\alpha f(x^1, x^2) &:= \sum_{k=0}^{n_1-1} \sum_{l=0}^{n_2-1} \hat{f}^\alpha(k, l) \alpha_k(x^1) \alpha_l(x^2), \\ D_{n_1, n_2}^\alpha(x^1, x^2) &:= D_{n_1}^\alpha(x^1) D_{n_2}^\alpha(x^2), \\ \mathcal{M}_n^\alpha f &:= \frac{1}{n} \sum_{k=0}^n S_{k,k}^\alpha f, \quad \mathcal{K}_n^\alpha := \frac{1}{n} \sum_{k=0}^n D_{k,k}^\alpha. \end{aligned}$$

where  $\alpha_n =$  either  $\omega_n$  or  $\kappa_n$  ( $n \in \mathbf{P}$ ).

The cubical Nörlund logarithmic means and kernels are defined by

$$t_n^\alpha(f) := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_{k,k}^\alpha f}{n-k}, \quad \mathbf{F}_n^\alpha := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{D_{k,k}^\alpha}{n-k}.$$

Let  $\mathcal{A}_{n,n}$  denote the  $\sigma$ -algebra generated by the sets  $I_n(x) \times I_n(y)$  ( $x, y \in G$ ) and  $E_{n,n}$  the conditional expectation operator with respect to  $\mathcal{A}_{n,n}$  ( $n \in \mathbf{N}$ ). Define the maximal operator of the Marcinkiewicz means and the maximal function of a function  $f \in L^1(G \times G)$  by

$$\mathcal{M}^{K^*} f := \sup_{n \in \mathbf{P}} |\mathcal{M}_n^K f|, \quad f^* := \sup_{n \in \mathbf{N}} |E_{n,n} f|.$$

The maximal operator  $\mathcal{M}^{K^*}$  was investigated by the author in [11].

For two-dimensional variable  $(x, y) \in G \times G$  we use the notations

$$\alpha_n^1(x, y) = \alpha_n(x), \quad D_n^{\alpha,1}(x, y) = D_n^\alpha(x), \quad K_n^{\alpha,1}(x, y) = K_n^\alpha(x),$$

$$\alpha_n^2(x, y) = \alpha_n(y), \quad D_n^{\alpha,2}(x, y) = D_n^\alpha(y), \quad K_n^{\alpha,2}(x, y) = K_n^\alpha(y),$$

for any  $n \in \mathbf{N}$ .

## 2. The a.e. convergence of a subsequence of one-variable logarithmic means

**THEOREM 1.** *Let  $\{m_n : n \geq 1\}$  be a sequence of positive integers wich satisfies*

$$\sum_{n=1}^{\infty} \frac{\log^2(m_n - 2^{|m_n|} + 1)}{\log m_n} < \infty.$$

*Then the operator  $t^{K^*}(f) := \sup_{n \geq 1} |t_{m_n}^K(f)|$  is of weak type  $(1, 1)$ .*

Analogue of this result on Walsh-Fourier logarithmic means was given by Goginava [8].

**COROLLARY 1.** *Let  $\{m_n : n \geq 1\}$  be a sequence of positive integers which satisfies the condition of Theorem 1 and let  $f \in L^1(G)$ , then*

$$t_{m_n}^K(f, x) \rightarrow f(x) \text{ a.e. as } n \rightarrow \infty.$$

**COROLLARY 2.** *Let  $f \in L^1(G)$ , then*

$$t_{2^n}^K(f, x) \rightarrow f(x) \text{ a.e. as } n \rightarrow \infty.$$

The basis of the proof of Theorem 1 are the following lemmas.

LEMMA 1. Let  $2^A \leq m < 2^{A+1}$ , then

$$\begin{aligned} l_m F_m^K(x) &= l_{m-2^{A-1}+1} D_{2^A}(x) \\ &\quad - \omega_{2^{A-1}}(x) \sum_{j=1}^{2^{A-1}-1} \left( \frac{1}{m-2^A+j} - \frac{1}{m-2^A+j+1} \right) j K_j^\omega(\tau_{A-1}(x)) \\ &\quad - \omega_{2^{A-1}}(x) \frac{2^{A-1}}{m-2^{A-1}} K_{2^{A-1}}^\omega(\tau_{A-1}(x)) + r_A(x) l_{m-2^A} F_{m-2^A}^\omega(\tau_A(x)) \\ &\quad + \sum_{s=0}^{2^{A-1}-2} \left( \frac{1}{m-s} - \frac{1}{m-s+1} \right) s K_s^K(x) \\ &\quad + \frac{2^{A-1}-1}{m-2^{A-1}+1} K_{2^{A-1}-1}^K(x). \end{aligned}$$

*Proof.* During the proof of Lemma 1 we will use the following equations:

$$D_{2^A+j}^K(x) = D_{2^A}(x) + r_A(x) D_j^\omega(\tau_A(x)), \quad j = 0, 1, \dots, 2^A - 1 \quad (2)$$

and

$$D_{2^A-j}^K(x) = D_{2^A}(x) - \omega_{2^{A-1}}(x) D_j^\omega(\tau_{A-1}(x)), \quad j = 0, 1, \dots, 2^{A-1}. \quad (3)$$

To prove (3), we write for  $j \leq 2^{A-1}$

$$\begin{aligned} D_{2^A-j}^K &= D_{2^A} - \sum_{k=2^A-j}^{2^A-1} \kappa_k = D_{2^A} - \sum_{l=0}^{j-1} \kappa_{2^A-l-1} \\ &= D_{2^A} - r_{A-1} \sum_{l=0}^{j-1} \omega_{2^{A-1}-l-1} \circ \tau_{A-1}. \end{aligned}$$

For  $0 \leq l < j \leq 2^{A-1}$  we have  $\omega_{2^A-1-l}(x) = \omega_{2^{A-1}}(x) \omega_l(x)$  and  $\omega_{2^A-1-l}(\tau_{A-1}(x)) = \omega_{2^{A-1}-1}(x)$ . These imply

$$\begin{aligned} D_{2^A-j}^K &= D_{2^A} - r_{A-1} \omega_{2^A-1-l} \circ \tau_{A-1} \sum_{l=0}^{j-1} \omega_l \circ \tau_{A-1} \\ &= D_{2^A} - r_{A-1} \omega_{2^A-1-l} D_j^\omega \circ \tau_{A-1} \end{aligned}$$

and (3) is complete.

Let  $|m| = A$ , then

$$l_m F_m^K(x) = \sum_{j=1}^{2^A} \frac{D_j^K(x)}{m-j} + \sum_{j=2^{A+1}}^{m-1} \frac{D_j^K(x)}{m-j} =: I + II.$$

First, we discuss II by the help of (2).

$$\begin{aligned} II &= \sum_{j=1}^{m-2^A-1} \frac{D_{2^A+j}^K(x)}{m-2^A-j} = l_{m-2^A} D_{2^A}(x) + r_A(x) \sum_{j=1}^{m-2^A-1} \frac{D_j^\omega(\tau_A(x))}{m-2^A-j} \\ &= l_{m-2^A} D_{2^A}(x) + r_A(x) l_{m-2^A} F_{m-2^A}^\omega(\tau_A(x)). \end{aligned}$$

Now, we investigate I.

$$I = \sum_{j=0}^{2^A-1} \frac{D_{2^A-j}^K(x)}{m-2^A+j} = \sum_{j=0}^{2^A-1} \frac{D_{2^A-j}^K(x)}{m-2^A+j} + \sum_{j=2^{A-1}+1}^{2^A-1} \frac{D_{2^A-j}^K(x)}{m-2^A+j} =: I_1 + I_2.$$

By the help of (3) and Abel’s transformation we could write

$$\begin{aligned} I_1 &= D_{2^A}(x) \sum_{j=0}^{2^A-1} \frac{1}{m-2^A+j} - \omega_{2^A-1}(x) \sum_{j=1}^{2^A-1} \frac{D_j^\omega(\tau_{A-1}(x))}{m-2^A+j} \\ &= (l_{m-2^{A-1}+1} - l_{m-2^A}) D_{2^A}(x) \\ &\quad - \omega_{2^A-1}(x) \sum_{j=1}^{2^A-1-1} \left( \frac{1}{m-2^A+j} - \frac{1}{m-2^A+j+1} \right) j K_j^\omega(\tau_{A-1}(x)) \\ &\quad - \omega_{2^A-1}(x) \frac{2^{A-1}}{m-2^{A-1}} K_{2^{A-1}}^\omega(\tau_{A-1}(x)). \end{aligned}$$

At last, we discuss  $I_2$ . We set  $s := 2^A - j$  and use Abel’s transformation for  $I_2$ .

$$\begin{aligned} I_2 &= \sum_{s=1}^{2^A-1-1} \frac{D_s^K(x)}{m-s} = \sum_{s=0}^{2^A-1-1} \frac{D_s^K(x)}{m-s} \\ &= \sum_{s=0}^{2^A-1-2} \left( \frac{1}{m-s} - \frac{1}{m-s+1} \right) s K_s^K(x) + \frac{2^A-1}{m-2^{A-1}+1} K_{2^A-1}^K(x). \end{aligned}$$

This completes the proof of Lemma 1.  $\square$

LEMMA 2. Let  $\overline{\lim}_{n \rightarrow \infty} \frac{\log^2(m_n - 2^{\lfloor m_n \rfloor + 1})}{\log m_n} < \infty$ , then

$$\|F_{m_n}^K\|_1 \leq c < \infty, \quad n = 1, 2, \dots$$

*Proof.* We have

$$\|K_j^\omega \circ \tau_A\|_1 = \|K_j^\omega\|_1 \leq c < \infty, \quad j, A = 1, 2, \dots$$

and

$$\|K_j^K\|_1 \leq c < \infty, \quad j = 1, 2, \dots$$

(See [14]). Moreover,

$$\|F_m^K\|_1 \leq \frac{1}{l_m} \sum_{j=1}^{m-1} \frac{\|D_j^K\|_1}{m-j} \leq \frac{1}{l_m} \sum_{j=1}^{m-1} \frac{\ln j + 1}{m-j} = O(l_m).$$

In the same way  $\|F_m^\omega\|_1 \leq O(l_m)$  (See [8]). Using Lemma 1, we immediately have

$$\begin{aligned} \|F_{m_n}^\kappa\|_1 &\leq 1 + \frac{1}{l_{m_n}} \sum_{j=1}^{2^{|m_n|}-1} \frac{\|K_j^\omega \circ \tau_{|m_n|-1}\|_1}{j} + \frac{1}{l_{m_n}} \|K_{2^{|m_n|-1}}^\omega \circ \tau_{|m_n|-1}\|_1 \\ &\quad + \frac{l_{m_n-2^{|m_n|}}}{l_{m_n}} \|F_{m_n-2^{|m_n|}}^\omega \circ \tau_{|m_n|}\|_1 \\ &\quad + \frac{1}{l_{m_n}} \sum_{s=0}^{2^{|m_n|}-2} \frac{\|K_s^\kappa\|_1}{m_n-s} + \frac{1}{l_{m_n}} \|K_{2^{|m_n|-1}-1}^\kappa\|_1 \\ &= O\left(\frac{\log^2(m_n-2^{|m_n|}+1)}{\log m_n}\right) = O(1). \end{aligned}$$

This completes the proof of Lemma 2.  $\square$

*Proof of Theorem 1.* The maximal function  $f^* := \sup_{n \in \mathbf{N}} |f * D_{2^n}|$  is of weak type  $(1, 1)$  [12]. In the article [3] Gát introduced the operators  $L, M$  defined by

$$Lf := \sup_{A \in \mathbf{N}} |f * r_A K_{2^A}^\omega \circ \tau_A| \quad \text{and} \quad Mf := \sup_{\substack{n, A \in \mathbf{N} \\ |n| \leq A}} |f * r_A K_n^\omega \circ \tau_A|$$

and he showed that the operators  $L, M$  and  $\sigma^{\kappa,*}$  are of weak type  $(1, 1)$ . Now, we define the modified kernels  $\tilde{K}_n \circ \tau_A$  by  $\tilde{K}_n \circ \tau_A := \omega_{2^A-1} K_n \circ \tau_A$  for  $n \in \mathbf{P}, |n| = A$  and the operator  $\tilde{L}, \tilde{M}$  by

$$\tilde{L}f := \sup_{A \in \mathbf{N}} |f * r_A \tilde{K}_{2^A}^\omega \circ \tau_A| \quad \text{and} \quad \tilde{M}f := \sup_{\substack{n, A \in \mathbf{N} \\ |n| \leq A}} |f * r_A \tilde{K}_n^\omega \circ \tau_A|.$$

The method of Gát in [3] gives that the operators  $\tilde{L}, \tilde{M}$  are of weak type  $(1, 1)$ .

At last, let  $f \in L^1(G)$ ,  $\text{supp } f \subset I_k$  and  $\int_{I_k} f = 0$ . Set  $n(k) := \min\{n : |m_n| \geq k\}$ .

If  $n \leq n(k)$  then

$$t_{m_n}^\kappa(f, x) = \int_G f(y) F_{m_n}^\kappa(x+y) d\mu(y) = F_{m_n}^\kappa(x) \int_G f(y) d\mu(y) = 0.$$

Consequently, set  $n > n(k)$ .

Define the operator  $N$  by

$$Nf := \sup_{n \geq 1} |f * r_{|m_n|} \frac{l_{m_n-2^{|m_n|}}}{l_{m_n}} F_{m_n-2^{|m_n|}}^\omega \circ \tau_{|m_n|}|.$$

We have

$$\begin{aligned} & \int_{I_k} \sup_{n \geq n(k)} \frac{l_{m_n - 2^{|m_n|}}}{l_{m_n}} |F_{m_n - 2^{|m_n|}}^\omega(\tau_{|m_n|}(x))| d\mu(x) \\ & \leq \sum_{n=1}^\infty \frac{\log(m_n - 2^{|m_n|} + 1)}{\log m_n} \|F_{m_n - 2^{|m_n|}}^\omega \circ \tau_{|m_n|}\|_1 \\ & \leq \sum_{n=1}^\infty \frac{\log(m_n - 2^{|m_n|} + 1)}{\log m_n} \|F_{m_n - 2^{|m_n|}}^\omega\|_1 \\ & \leq \sum_{n=1}^\infty \frac{\log^2(m_n - 2^{|m_n|} + 1)}{\log m_n} \leq c. \end{aligned}$$

This implies

$$\begin{aligned} \int_{I_k} Nf(x) d\mu(x) & \leq \int_{I_k} |f(y)| \left( \int_{I_k} \sup_{n \geq n(k)} |F_{m_n - 2^{|m_n|}}^\omega(\tau_{|m_n|}(x+y))| d\mu(x) \right) d\mu(y) \\ & \leq c \|f\|_1. \end{aligned}$$

From Lemma 2 the operator  $N$  and  $t^{K,*}$  is of type  $(\infty, \infty)$ . The operator  $N$  is sublinear and quasi-local, this gives by standard argument [12] that the operator  $N$  is of weak type  $(1, 1)$ .

Lemma 1 and

$$\begin{aligned} t^{K,*}(f) & \leq cf^* + \sup_{n \geq 1} \frac{1}{l_{m_n}} \sum_{j=1}^{2^{|m_n|}-1} \frac{\tilde{M}f}{j} + c\tilde{L}f + Nf + \sup_{n \geq 1} \frac{1}{l_{m_n}} \sum_{s=0}^{2^{|m_n|}-2} \frac{\sigma^{K,*}f}{m_n - s} + \frac{1}{l_{m_n}} \sigma^{K,*}f \\ & \leq cf^* + c\tilde{M}f + c\tilde{L}f + cNf + c\sigma^{K,*}f \end{aligned}$$

complete the proof of Theorem 1.  $\square$

COROLLARY 3. *The operator  $t^{K,*}$  is of type  $(p, p)$  for all  $1 < p \leq \infty$ .*

### 3. The a.e. convergence of a subsequence of logarithmic means of quadratical partial sums

Define the two-dimensional maximal operator  $\mathbf{t}_{\natural}^K$  by

$$\mathbf{t}_{\natural}^K f(x^1, x^2) := \sup_{n \in \mathbf{P}} |\mathbf{t}_{2^n}^K(f, x^1, x^2)|.$$

We will use that the maximal function  $f^*$  is of weak type  $(1, 1)$  and of type  $(p, p)$  for all  $1 < p \leq \infty$  [9] and the maximal operator  $\mathcal{M}^{K,*}$  has the same property [11].



THEOREM 2. The operator  $\mathbf{t}_{\natural}^K$  is of weak type  $(1, 1)$ . and of type  $(p, p)$  for all  $1 < p \leq \infty$ .

By standard argument we have

COROLLARY 4. Let  $f \in L^1(G \times G)$ , then

$$\mathbf{t}_{2^n}^K(f, x^1, x^2) \rightarrow f(x^1, x^2) \text{ a.e. as } n \rightarrow \infty.$$

The analogue of this result with respect to Walsh-Fourier logarithmic means was given by Gát and Goginava [6].

*Proof of Theorem 2.* First, we decompose the  $2^n$ -th logarithmic kernels.

$$\begin{aligned} l_{2^n} \mathbf{F}_{2^n}^K(x^1, x^2) &= \sum_{j=1}^{2^n-1} \frac{D_{2^n-j}^K(x^1) D_{2^n-j}^K(x^2)}{j} + \sum_{j=2^{n-1}+1}^{2^n-1} \frac{D_{2^n-j}^K(x^1) D_{2^n-j}^K(x^2)}{j} \\ &=: I + II. \end{aligned}$$

In  $I$  we use (3)

$$\begin{aligned} I &= \sum_{j=1}^{2^n-1} \frac{D_{2^n}(x^1) D_{2^n}(x^2)}{j} - D_{2^n}(x^1) \omega_{2^n-1}(x^2) \sum_{j=1}^{2^n-1} \frac{D_j^\omega(\tau_{n-1}(x^2))}{j} \\ &\quad - D_{2^n}(x^2) \omega_{2^n-1}(x^1) \sum_{j=1}^{2^n-1} \frac{D_j^\omega(\tau_{n-1}(x^1))}{j} \\ &\quad + \omega_{2^n-1}(x^1) \omega_{2^n-1}(x^2) \sum_{j=1}^{2^n-1} \frac{D_j^\omega(\tau_{n-1}(x^1)) D_j^\omega(\tau_{n-1}(x^2))}{j} \\ &=: l_{2^n} \sum_{n=1}^4 \mathbf{F}_{2^n}^{1,i}(x^1, x^2). \end{aligned}$$

Since  $\mathbf{F}_{2^n}^{1,1}(x^1, x^2) = \frac{l_{2^{n-1}+1}}{l_{2^n}} D_{2^n}(x^1) D_{2^n}(x^2)$  we have

$$\mathbf{t}_{\natural}^{1,1} f := \sup_{n \in \mathbf{P}} |f * \mathbf{F}_{2^n}^{1,1}| \leq c f^*.$$

To discuss  $\mathbf{F}_{2^n}^{1,2}$  we will use Abel's transformation ( $\mathbf{F}_{2^n}^{1,3}$  goes in the same way)

$$\sum_{j=1}^{2^n-1} \frac{D_j^\omega}{j} = \sum_{j=1}^{2^n-1-1} \left( \frac{1}{j} - \frac{1}{j+1} \right) j K_j^\omega + K_{2^n-1}^\omega = \sum_{j=1}^{2^n-1} \frac{K_j^\omega}{j+1} + K_{2^n-1}^\omega.$$

Define the operator  $O$  by

$$O f := \sup_{\substack{n, A \in \mathbf{N} \\ |n| \leq A}} |f * D_{2^A}^1 r_A^2 K_n^{\omega, 2} \circ \tau_A|,$$

in [11] (Lemma 10) it was proved that the operator  $O$  is of weak type  $(1, 1)$  and of type  $(p, p)$  for all  $1 < p \leq \infty$ . Define the modified kernel  $\tilde{K}_n^\omega \circ \tau_A := \omega_{2^{A-1}} r_A^\omega \circ \tau_A$  and the modified operator  $\tilde{O}$  by

$$\tilde{O}f := \sup_{\substack{n, A \in \mathbf{N} \\ |n| \leq A}} |f * D_{2^{A+1}}^1 r_A^2 \tilde{K}_n^{\omega, 2} \circ \tau_A|.$$

The method of [11] gives that the operator  $\tilde{O}$  is of weak type  $(1, 1)$  and of type  $(p, p)$  for all  $1 < p \leq \infty$ . These imply

$$\mathbf{t}_{\natural}^{1,2} f := \sup_{n \in \mathbf{P}} |f * \mathbf{F}_{2^n}^{1,2}| \leq \sup_{n \in \mathbf{P}} \frac{1}{l_{2^n}} \sum_{j=1}^{2^{n-1}-1} \frac{1}{j+1} \tilde{O}f + c\tilde{O}f \leq c\tilde{O}f.$$

Now, we discuss  $\mathbf{F}_{2^n}^{1,4}$ . Abel’s transformation gives that

$$\sum_{j=1}^{2^{n-1}-1} \frac{D_j^{\omega, 1} D_j^{\omega, 2}}{j} = \sum_{j=1}^{2^{n-1}-1} \frac{\mathcal{K}_j^\omega}{j+1} + \mathcal{K}_{2^{n-1}}^\omega$$

and

$$\mathbf{F}_{2^n}^{1,4} = \frac{1}{l_{2^n}} \omega_{2^{n-1}}^1 \omega_{2^{n-1}}^2 \left( \sum_{j=1}^{2^{n-1}-1} \frac{\mathcal{K}_j^\omega \circ (\tau_{n-1} \times \tau_{n-1})}{j+1} + \mathcal{K}_{2^{n-1}}^\omega \circ (\tau_{n-1} \times \tau_{n-1}) \right).$$

Define the modified kernel  $\tilde{\mathcal{K}}_j^\omega \circ (\tau_{n-1} \times \tau_{n-1}) := \omega_{2^{n-1}-1}^1 \omega_{2^{n-1}-1}^2 \mathcal{K}_j^\omega \circ (\tau_{n-1} \times \tau_{n-1})$  for  $j \leq 2^{n-1}$  and the operators  $M, \tilde{M}$  by

$$Mf := \sup_{\substack{n, A \in \mathbf{N} \\ |n| \leq A}} |f * r_A^1 r_A^2 \mathcal{K}_n^\omega \circ (\tau_A \times \tau_A)|, \quad \tilde{M}f := \sup_{\substack{n, A \in \mathbf{N} \\ |n| \leq A}} |f * r_A^1 r_A^2 \tilde{\mathcal{K}}_n^\omega \circ (\tau_A \times \tau_A)|.$$

The operator  $M$  is of weak type  $(1, 1)$  and of type  $(p, p)$  for all  $1 < p \leq \infty$  ([11], Lemma 8). The method of [11] gives that the operator  $\tilde{M}$  has the same property. We note that during the proof of Lemma 8 in [11] we used the fact that  $r_A^1 r_A^2$  is  $\mathcal{A}_{A+1, A+1}$ -measurable and  $\mathcal{K}_n^\omega \circ (\tau_A \times \tau_A)$  (and  $\tilde{\mathcal{K}}_n^\omega \circ (\tau_A \times \tau_A) = \omega_{2^{A-1}-1}^1 \omega_{2^{A-1}-1}^2 \mathcal{K}_n^\omega \circ (\tau_A \times \tau_A)$  too) is  $\mathcal{A}_{A, A}$ -measurable function for  $|n| \leq A$ .

These imply

$$\mathbf{t}_{\natural}^{1,4} f := \sup_{n \in \mathbf{P}} |f * \mathbf{F}_{2^n}^{1,4}| \leq \sup_{n \in \mathbf{P}} \frac{1}{l_{2^n}} \sum_{j=1}^{2^{n-1}-1} \frac{1}{j+1} \tilde{M}f + c\tilde{M}f \leq c\tilde{M}f.$$

At last, we have to see  $II$ . In  $II$  let be  $s := 2^n - j$  and use Abel’s transformation! Thus,

$$II = \sum_{s=0}^{2^{n-1}-1} \frac{D_s^{K, 1} D_s^{K, 2}}{2^n - s} = \sum_{s=0}^{2^{n-1}-2} \left( \frac{1}{2^n - s} - \frac{1}{2^n - s + 1} \right) s \mathcal{K}_s^K + \frac{2^{n-1} - 1}{2^{n-1} + 1} \mathcal{K}_{2^{n-1}-1}^K.$$

Define  $\mathbf{F}_{2^n}^2$  and  $\mathbf{t}_{\natural}^2$  by

$$\mathbf{F}_{2^n}^2 := \frac{1}{l_{2^n}} \sum_{s=0}^{2^{n-1}-2} \left( \frac{1}{2^n-s} - \frac{1}{2^n-s+1} \right) s \mathcal{K}_s^K + \frac{2^{n-1}-1}{l_{2^n}(2^{n-1}+1)} \mathcal{K}_{2^{n-1}-1}^K$$

and

$$\begin{aligned} \mathbf{t}_{\natural}^2 f &:= \sup_{n \in \mathbf{P}} |f * \mathbf{F}_{2^n}^2| \leq \sup_{n \in \mathbf{P}} \frac{1}{l_{2^n}} \sum_{s=0}^{2^{n-1}-2} \frac{1}{2^n-s} \mathcal{M}^{K,*} f + c \mathcal{M}^{K,*} f \\ &\leq c \mathcal{M}^{K,*} f. \end{aligned}$$

This completes the proof of Theorem 2.  $\square$

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