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ON A FUNCTIONAL VOLTERRA–FREDHOLM INTEGRAL EQUATION, VIA PICARD OPERATORS

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Abstract. In this paper we present some results relative to existence, uniqueness, integral inequalities and data dependence for the solutions of the functional Volterra-Fredholm integral equation with deviating argument in a Banach space:

$$u(x,y) = g(x,y,h(u)(x,y)) + \int_0^x \int_0^y K(x,y,s,t,u(s,t))dsdt, \quad x,y \in \mathbb{R}_+$$

by Picard operators technique. This equation is a generalization of the equation (VF) from the paper: B.G. Pachpatte, On Volterra-Fredholm integral equation in two variables, Demonstratio Math., 40(2007), No. 4, 832-852.

1. Introduction

The present paper is motivated by a recent paper [5] by B.G. Pachpatte which studies a system of Volterra-Fredholm integral equations in two variables $x,y \in \mathbb{R}_+$. The aim of our paper is to study the following more general Volterra-Fredholm integral equation with deviating argument in a Banach space:

$$u(x,y) = g(x,y,h(u)(x,y)) + \int_0^x \int_0^y K(x,y,s,t,u(s,t))dsdt, \quad x,y \in \mathbb{R}_+. \tag{1.1}$$

Let $(E,|\cdot|)$ be a Banach space. Let $\tau > 0$ and

$$X_\tau := \{u \in C(\mathbb{R}_+^2, E) \mid \exists\, M(u) > 0 : \ |u(x,y)|e^{-\tau(x+y)} \leqslant M(u)\}.$$

On X_τ we consider Bielecki's norm

$$\|u\|_\tau := \sup_{x,y \in \mathbb{R}_+} (|u(x,y)|e^{-\tau(x+y)}).$$

It is clear that $(X_\tau, \|\cdot\|_\tau)$ is a Banach space.

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Relative to (1.1) we suppose that:

(c_1) $g \in C(\mathbb{R}_+^2 \times E, E)$, $K \in C(\mathbb{R}_+^4 \times E, E)$;

(c_2) $h : X_\tau \to X_\tau$ is such that

$$\exists\, l_h > 0 : \quad |h(u)(x,y) - h(v)(x,y)| \leqslant l_h \|u - v\|_\tau \cdot e^{\tau(x+y)},$$

$$\forall\, x,y \in \mathbb{R}_+,\ \forall\, u,v \in X_\tau;$$

(c_3) $\exists\, l_g > 0 : \quad |g(x,y,e_1) - g(x,y,e_2)| \leqslant l_g |e_1 - e_2|,$

$$\forall\, x,y \in \mathbb{R}_+,\ \forall\, e_1,e_2 \in E;$$

(c_4) $\exists\, l_K(x,y,s,t) : \quad |K(x,y,s,t,e_1) - K(x,y,s,t,e_2)|$

$$\leqslant l_K(x,y,s,t)|e_1 - e_2|,\ \forall\, x,y,s,t \in \mathbb{R}_+,\ e_1,e_2 \in E;$$

(c_5) $l_K \in C(\mathbb{R}_+^4, \mathbb{R}_+)$ and

$$\int_0^x \int_0^y l_K(x,y,s,t) e^{\tau(s+t)} ds dt \leqslant l e^{\tau(x+y)},\ \forall\, x,y \in \mathbb{R}_+;$$

(c_6) $l_h l_g + l < 1$.

We consider the operator

$$A : X_\tau \to X_\tau, \quad A(u)(x,y) := \text{second part of (1.1).}$$

First of all we shall prove that under the conditions $(c_1) - (c_6)$ the operator A is a Picard operator. So, we present some notions and results from Picard operators theory (see I.A. Rus [6]-[8]).

2. Picard operators

Let (X,d) be a metric space, $A : X \to X$ an operator and $c > 0$. We denote by F_A the fixed point set of A.

DEFINITION 2.1. (I. A. Rus [6]–[8]) A is a Picard operator (PO) if there exists $x_A^* \in X$ such that:

(i) $F_A = \{x_A^*\}$;

(ii) $A^n(x) \to x_A^*$ as $n \to \infty$, $\forall\, x \in X$.

DEFINITION 2.2. (I. A. Rus [8]) A is a c-Picard operator (c-PO) if A is PO and $d(x, x_A^*) \leqslant c d(x, A(x))$, $\forall\, x \in X$.

LEMMA 2.1. (Abstract Gronwall lemma) ([8]) *Let $(X,d,\leqslant)$ be an ordered metric space and $A : X \to X$ an operator. We suppose that*

(i) A is PO;

(ii) A is increasing.

If we denote by x_A^* the unique fixed point of A, then

(a) $x \leqslant A(x) \ \Rightarrow\ x \leqslant x_A^*$;

(b) $x \geqslant A(x) \ \Rightarrow\ x \geqslant x_A^*$.

LEMMA 2.2. (I. A. Rus [8]) *Let* $(X,d,\leqslant)$ *be an ordered metric space and* $A,B,C: X \to X$ *be such that*

(i) $A \leqslant B \leqslant C$;

(ii) The operators A,B,C *are POs;*

(iii) The operator B *is increasing.*

Then

$$x_A^* \leqslant x_B^* \leqslant x_C^*.$$

3. Existence and uniqueness

We begin with

THEOREM 3.1. *Under the conditions* $(c_1)-(c_6)$ *the equation* (1.1) *has in* X_τ *a unique solution* u^* *and the sequence of successive approximations*

$$u_{n+1}(x,y) = g(x,y,h(u_n)(x,y)) + \int_0^x \int_0^y K(x,y,s,t,u_n(s,t))dsdt, \quad n \in \mathbb{N} \tag{3.1}$$

converges uniformly to u^*.

Proof. We have that the operator A is a contraction in X_τ with respect to $\|\cdot\|_\tau$. Indeed, for $u,v \in X_\tau$ from $(c_1)-(c_6)$ it follows

$$\begin{aligned} |A(u)(x,y) - A(v)(x,y)| &\leqslant |g(x,y,h(u)(x,y)) - g(x,y,h(v)(x,y))| \\ &\quad + l_g|h(u)(x,y) - h(v)(x,y)| \\ &\quad + \int_0^x \int_0^y l_K(x,y,s,t)|u(s,t) - v(s,t)|dsdt \\ &\leqslant l_g l_h \|u-v\|_\tau e^{\tau(x+y)} + l\|u-v\|_\tau e^{\tau(x+y)} \\ &\leqslant (l_g l_h + l)e^{\tau(x+y)}\|u-v\|_\tau. \end{aligned}$$

Hence

$$\|A(u) - A(v)\|_\tau \leqslant (l_g l_h + l)\|u-v\|_\tau,$$

for all $u,v \in X_\tau$.

From (c_6) we have that A is a contraction. So, from the contraction principle A is a c-Picard operator, where

$$c = \frac{1}{1 - l_g l_h - l}. \qquad \square$$

REMARK 3.1. If $E := \mathbb{R}^n$ and

$$g(x,y,h(u)(x,y)) := h(x,y) + \int_0^\infty \int_0^\infty G(x,y,s,t,u(s,t))dsdt \tag{3.2}$$

then we have Pachpatte's result.

REMARK 3.2. If $E := l^p(\mathbb{R})$, $1 \leqslant p < \infty$, then the equation (1.1) is an infinite system of functional-integral equations:

$$u_i(x,y) = g_i(x,y,h(u_1,\ldots,u_n,\ldots)(x,y) + \int_0^x \int_0^y K_i(x,y,s,t,u_1(s,t),\ldots,u_n(s,t),\ldots)dsdt$$

for all $x,y \in \mathbb{R}_+$ and $i \in \mathbb{N}^*$.

4. Integral inequalities

THEOREM 4.1. *Let* $(E,|\cdot|,\leqslant)$ *be an ordered Banach space. We suppose that:*
(i) the conditions $(c_1)-(c_6)$ *are satisfied;*
(ii) the operators

$$g(x,y,\cdot) : E \to E$$
$$h : E \to E$$
$$K(x,y,s,t,\cdot) : E \to E,$$

are increasing.

If $u^* \in X_\tau$ *is the unique solution of the equation (*1.1*) and* $u \in X_\tau$ *is a solution of the following inequality*

$$u(x,y) \leqslant g(x,y,h(u)(x,y)) + \int_0^x \int_0^y K(x,y,s,t,u(s,t))dsdt,\ \forall\, x,y \in \mathbb{R}_+ \tag{4.1}$$

then

$$u(x,y) \leqslant u^*(x,y).$$

Proof. We consider the operator

$$A : X_\tau \to X_\tau, \quad A(u)(x,y) := \text{second part of (1.1)}.$$

The operator A is a Picard operator. From the condition (ii) A is increasing. Then the proof follows from Lemma 2.1. □

EXAMPLE 4.1. Consider:

$$g(x,y,h(u)(x,y)) \leqslant h(x,y) \tag{4.2}$$

and

$$K(x,y,s,t,u(s,t)) \leqslant b(x,y)L(s,t,u(s,t)),$$

where $h,b : \mathbb{R}_+^2 \to \mathbb{R}_+$ are continuous functions and $L : \mathbb{R}_+^3 \to \mathbb{R}_+$ is a continuous function which satisfies the condition

$$0 \leqslant L(x,y,v) - L(x,y,w) \leqslant M(x,y,w)(v-w)$$

for $x,y \in \mathbb{R}_+$ and $v \geqslant w \geqslant 0$. M is a nonnegative continuous function defined on $\mathbb{R}_+^3$.

Then

$$u(x,y) \leqslant h(x,y) + b(x,y) \int_0^x \int_0^y L(s,t,u(s,t))dsdt. \tag{4.3}$$

Consider the operator $B : X_\tau \to X_\tau$, $B(u)(x,y) :=$ last part of (4.3).
It is clear that the operator B is PO on X_τ and is increasing.
Let u_B^* be the unique fixed point of B. Thus, from Lemma 2.2, we have

$$u^*(x,y) = A(u^*) \leqslant B(u_B^*) = u_B^*.$$

From the papers [1]+[2] (S.S. Dragomir and N.M. Ionescu), we have

$$u_B^*(x,y) = h(x,y) + b(x,y)\left[\exp\left(\int_0^x \int_0^y P(s,t)dsdt\right) - 1\right] \tag{4.4}$$

where

$$P(x,y) = [L^2(x,y,h(x,y)) + M^2(x,y,h(x,y))b^2(x,y)]^{1/2},$$

for $x,y \in \mathbb{R}_+$.

Then

$$u(x,y) \leqslant u_B^*(x,y).$$

5. Data dependence: Monotony

Consider the following integral equations

$$u_i(x,y) = g_i(x,y,h(u)(x,y)) + \int_0^x \int_0^y K_i(x,y,s,t,u(s,t))dsdt, \quad i = 1,2,3. \tag{5.1i}$$

THEOREM 5.1. *We suppose that*
(i) $g_i, h, K_i,\ i = 1,2,3$ *satisfy the conditions* $(c_1)-(c_6)$*;*
(ii) $g_1 \leqslant g_2 \leqslant g_3,\ K_1 \leqslant K_2 \leqslant K_3$*;*
(iii) the operators:

$$g_2(x,y,\cdot) : E \to E,$$
$$h : E \to E,$$
$$K_i(x,y,s,t,\cdot) : E \to E,\ i = 1,2,3$$

are increasing.

Then the equation (5.1i) *has a unique solution* u_i^* *and*

$$u_1^* \leqslant u_2^* \leqslant u_3^*. \tag{5.2}$$

Proof. Consider the operators A_i, $i = 1,2,3$, $A_i : X_\tau \to X_\tau$

$$A_i(u)(x,y) := \text{second part of (5.1i)}, \quad i = 1,2,3.$$

These operators are as in Lemma 2.2. We remark that $u_i^* \in X_\tau$, $i = 1,2,3$. The proof follows from Lemma 2.2. □

6. Data dependence: Continuity

In this section we shall use the following notations:

$$\widetilde{g}(u)(x,y) := g(x,y,h(u)(x,y))$$

and

$$\widetilde{K}(u)(x,y) := \int_0^x \int_0^y K(x,y,s,t,u(s,t))dsdt.$$

Let us consider the following equations

$$u_i(x,y) = g_i(x,y,h(u)(x,y)) + \int_0^x \int_0^y K_i(x,y,s,t,u(s,t))dsdt, \quad i = 1,2. \tag{6.1i}$$

We have

THEOREM 6.1. *We suppose that:*
(i) g_i, h, K_i, $i = 1,2$, satisfy the conditions $(c_1)-(c_6)$;
(ii) there exist $\eta_i \in \mathbb{R}_+$, $i = 1,2$, such that

$$\|\widetilde{g}_1(u) - \widetilde{g}_2(u)\|_\tau \leqslant \eta_1 \quad \text{and} \quad \|\widetilde{K}_1(u) - \widetilde{K}_2(u)\|_\tau \leqslant \eta_2, \ \forall\, u \in X_\tau.$$

Then, if u_i^* is the unique solution of (6.1i), then:

$$\|u_1^* - u_2^*\| \leqslant \eta_1 + \eta_2.$$

Proof. The proof follows from Theorem 7.1.1 in I.A. Rus [7]. □

7. Data dependence: Differentiability

In this section we take $E = \mathbb{R}$.

To study the differentiability of the fixed point with respect to parameters we need the fiber contraction theorem.

THEOREM 7.1. (Fiber contraction theorem [3], [4], [7]) *Let (X,d) be a metric space and (Y,ρ) be a complete metric space.*

Let $B : X \times Y \to X \times Y$ be a continuous operator and $A : X \to X$, $\mathscr{D} : X \times Y \to Y$ two operators. We suppose that:

(i) $B(u,w) = (A(u), \mathscr{D}(u,w))$ for all $u \in X$, $w \in Y$,
(ii) A is a Picard operator;
(iii) there exists $q \in (0,1)$ such that

$$\rho(\mathscr{D}(u,w), \mathscr{D}(u,\overline{w})) \leqslant q\rho(w,\overline{w})$$

for all $u \in X$ and $w, \overline{w} \in Y$.

Then B is a Picard operator.

In what follows we consider the equation

$$u(x,y,\lambda) = g(x,y,h(u)(x,y),\lambda) + \int_0^x \int_0^y K(x,y,s,t,u(s,t,\lambda),\lambda)dsdt, \ \forall\, x,y \in \mathbb{R}_+,\ \lambda \in I. \tag{7.1}$$

THEOREM 7.2. *We suppose that*
(i) the conditions $(c_1)-(c_6)$ *and the Theorem 3.1 are satisfied;*

(ii) $K \in C(\mathbb{R}_+^4 \times I \times \mathbb{R}, \mathbb{R})$, *has continuous derivative* $\dfrac{\partial K}{\partial u}$ *and there exists* $q \in (0,1)$ *such that*

$$\int_0^x \int_0^y \left| \frac{\partial K}{\partial u} \right| e^{\tau(s+t)} dsdt \leqslant q e^{\tau(x+y)},\ \forall\, x,y \in \mathbb{R}_+. \tag{7.2}$$

Then the solution of the equation (7.1) *is differentiable with respect to* λ *and* $\dfrac{\partial u}{\partial \lambda} \in C(\mathbb{R}_+^2 \times I, \mathbb{R})$.

Proof. In what follows we use the fiber contraction theorem (Theorem 7.1).
Let $X := C(\mathbb{R}_+^2 \times I, \mathbb{R})$ and the operator $A : X \to X$ defined by

$$\begin{aligned} &A(u)(x,y,\lambda) := \text{the second part of (6.1i)} \qquad (7.3)\\ &A(u)(x,y,\lambda) = g(x,y,h(u)(x,y),\lambda)\\ &\qquad + \int_0^x \int_0^y K(x,y,s,t,u(s,t,\lambda),\lambda)dsdt,\ \forall\, x,y \in \mathbb{R}_+,\ \lambda \in I. \end{aligned}$$

Let $u(x,y,\lambda)$ be the unique fixed point of A. We suppose that there exists $\dfrac{\partial u}{\partial \lambda}$. Then from (7.1) it follows that

$$\begin{aligned} \frac{\partial u(x,y,\lambda)}{\partial \lambda} &= \frac{\partial g(x,y,h(u)(x,y),\lambda)}{\partial \lambda} \qquad (7.4)\\ &+ \int_0^x \int_0^y \frac{\partial K(x,y,s,t,u(s,t,\lambda),\lambda)}{\partial u} \cdot \frac{\partial u(s,t,\lambda)}{\partial \lambda} dsdt\\ &+ \int_0^x \int_0^y \frac{\partial K(x,y,s,t,u(s,t,\lambda),\lambda)}{\partial \lambda} dsdt \end{aligned}$$

In what follows we denote

$$w(x,y,\lambda) = \frac{\partial u(x,y,\lambda)}{\partial \lambda},$$

then we have

$$\begin{aligned} w(x,g,\lambda) &= \frac{\partial g(x,y,h(u)(x,y),\lambda)}{\partial \lambda} \qquad (7.5)\\ &+ \int_0^x \int_0^y \frac{\partial K(x,y,s,t,u(s,t,\lambda),\lambda)}{\partial u} w(s,t,\lambda) dsdt\\ &+ \int_0^x \int_0^y \frac{\partial K(x,y,s,t,u(s,t,\lambda),\lambda)}{\partial \lambda} dsdt. \end{aligned}$$

This relation suggests to consider the following operator

$$\mathscr{D} : C(\mathbb{R}_+^2 \times I, \mathbb{R}) \times C(\mathbb{R}_+^2 \times I, \mathbb{R}) \to C(\mathbb{R}_+^2 \times I, \mathbb{R})$$

where

$$\mathscr{D}(u,w)(x,y,\lambda) := \text{second part of (7.5)}$$

for all $x,y \in \mathbb{R}_+$ and $\lambda \in I$.

The operator $\mathscr{D}$ is a contraction with respect to w, indeed

$$\begin{aligned}|\mathscr{D}(u,w)(x,y,\lambda) - \mathscr{D}(u,\overline{w})(x,y,\lambda)| &\leqslant \int_0^x \int_0^y \left| \frac{\partial K(x,y,s,t,u(s,t,\lambda),\lambda)}{\partial u} w(s,t,\lambda) \right. \\ &\quad \left. - \frac{\partial K(x,y,s,t,u(s,t,\lambda),\lambda)}{\partial u} \overline{w}(s,t,\lambda) \right| dsdt \\ &\leqslant \int_0^x \int_0^y \left| \frac{\partial K(x,y,s,t,u(s,t,\lambda),\lambda)}{\partial u} \right| |w(s,t,\lambda) - \overline{w}(s,t,\lambda)| dsdt \\ &\leqslant \int_0^x \int_0^y \left| \frac{\partial K(x,y,s,t,u(s,t,\lambda),\lambda)}{\partial u} \right| |w(s,t,\lambda) - \overline{w}(s,t,\lambda)| e^{-\tau(s+t)} e^{\tau(s+t)} dsdt \\ &\leqslant \|w(s,t,\lambda) - \overline{w}(s,t,\lambda)\| q e^{\tau(x+y)}.\end{aligned}$$

Then we have

$$\|\mathscr{D}(u,w)(x,y,\lambda) - \mathscr{D}(u,\overline{w})(x,y,\lambda)\| \leqslant q\|w(x,y,\lambda) - \overline{w}(x,y,\lambda)\|, \quad q \in (0,1). \quad (7.6)$$

Then the operator $\mathscr{D}$ is a contraction and has a unique fixed point w, $\mathscr{D}(u,w) = w$. If we take the operator:

$$B : C(\mathbb{R}_+^2 \times I, \mathbb{R}) \times C(\mathbb{R}_+^2 \times I, \mathbb{R}) \to C(\mathbb{R}_+^2 \times I, \mathbb{R}) \times C(\mathbb{R}_+^2 \times I, \mathbb{R})$$

$$B(u,w) = (A(u), \mathscr{D}(u,w)),$$

from Theorem 7.1, we have that B is a Picard operator. This implies that the sequences

$$\begin{aligned}&u_{n+1} := A(u_n) \\ &w_{n+1} := \mathscr{D}(u_n, w_n),\ n \in \mathbb{N},\end{aligned}$$

converge uniformly to $(u^*, w^*) \in F_B$, for all $u_0, w_0 \in C(\mathbb{R}_+^2 \times I, \mathbb{R})$.

Let $u_0(x,y,\cdot) \in C^1(I)$ and $w_0 := \dfrac{\partial u_0}{\partial \lambda}$. Then by induction we prove that

$$w_n = \frac{\partial u_n}{\partial \lambda}.$$

Therefore, (u_n) converges uniformly to u^* and $\left(\dfrac{\partial u_n}{\partial \lambda}\right)_{n\in\mathbb{N}}$ converges uniformly to w^*. From the above converges it follows that there exists $\dfrac{\partial u}{\partial \lambda}$ and $\dfrac{\partial u}{\partial \lambda} = w$. $\square$

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