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BOUNDS FOR THE NORMALIZED JENSEN–MERCER FUNCTIONAL

J. BARIĆ AND A. MATKOVIĆ

Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. We introduce the normalized Jensen-Mercer functional

$$\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{p}) = f(a)+f(b)-\sum_{i=1}^{n} p_i f(x_i) - f\left(a+b-\sum_{i=1}^{n} p_i x_i\right)$$

and establish the inequalities of type $M\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{q}) \geqslant \mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{p}) \geqslant m\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{q})$, where f is a convex function, $\boldsymbol{x}=(x_1,\dots,x_n)$ and m and M are real numbers satisfying certain conditions. We prove them for the case when $\boldsymbol{p}$ and $\boldsymbol{q}$ are nonnegative n-tuples and when $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfy the conditions for the Jensen-Steffensen inequality. We also give their integral versions in both cases.

1. Introduction

In paper [4] A. McD. Mercer proved the following variant of Jensen's inequality, to which we will refer as to the "Jensen-Mercer inequality".

THEOREM A. *Let $[a,b]$ be an interval in $\mathbb{R}$ and $x_1,\dots,x_n \in [a,b]$. Let $w_1,\dots,w_n$ be nonnegative real numbers such that $W_n=\sum_{i=1}^n w_i>0$. If $f:[a,b]\to\mathbb{R}$ is a convex function, then*

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leqslant f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i). \tag{1.1}$$

In paper [1] is proved that (1.1) remains valid even in the case when the condition "$\boldsymbol{w}=(w_1,\dots,w_n)$ is nonnegative n-tuple" is somewhat relaxed. More precisely the following is true.

THEOREM B. *Let $[a,b]$ be an interval in $\mathbb{R}$ and $\boldsymbol{x}=(x_1,\dots,x_n)\in[a,b]^n$ be a monotonic n-tuple. Let $\boldsymbol{w}=(w_1,\dots,w_n)$ be a real n-tuple such that*

$$0 \leqslant W_k \leqslant W_n \quad (k=1,\dots,n), \quad W_n>0, \tag{1.2}$$

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where $W_k = \sum_{i=1}^{k} w_i$ $(k = 1,2,\ldots,n)$. *If* $f:[a,b]\to\mathbb{R}$ *is a convex function, then* (1.1) *holds.*

Let $\mathscr{P}_n$ denotes the set of all nonnegative real n-tuples $(p_1,\ldots,p_n)$ with the property $\sum_{i=1}^{n} p_i = 1$. For any convex function $f:[a,b]\to\mathbb{R}$ and for any choice of n-tuples $\boldsymbol{x} = (x_1,\ldots,x_n)\in[a,b]^n$ and $\boldsymbol{p} = (p_1,\ldots,p_n)\in\mathscr{P}_n$ we define

$$\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{p}) := f(a)+f(b)-\sum_{i=1}^{n} p_i f(x_i) - f\left(a+b-\sum_{i=1}^{n} p_i x_i\right) \tag{1.3}$$

and we call it the normalized Jensen-Mercer functional. For a fixed function f and n-tuple $\boldsymbol{x}$, $\mathscr{M}_n(f,\boldsymbol{x},\cdot)$ can be observed as a function on $\mathscr{P}_n$. Note that $\mathscr{P}_n$ is obviously a convex subset in $\mathbb{R}^n$ and because of Theorem A, $\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{p})\geqslant 0$ for all $\boldsymbol{p}\in\mathscr{P}_n$.

In Section 2 we establish the inequalities of type $M\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{q})\geqslant\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{p})\geqslant m\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{q})$, where m and M are real constants satisfying certain conditions. We prove them for the case when $\boldsymbol{p}$ and $\boldsymbol{q}$ are nonnegative n-tuples and when $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfy the conditions for the Jensen-Steffensen inequality. In Section 4 we give the integral versions of all results from Section 2 and 3.

2. Bounds for the normalized Jensen-Mercer functional

We assume the notations from introduction.

THEOREM 1. *Let* $\boldsymbol{p} = (p_1,\ldots,p_n)$ *and* $\boldsymbol{q} = (q_1,\ldots,q_n)$ *be two* n*-tuples from* $\mathscr{P}_n$. *Let* m *and* M *be any real constants such that*

$$m\geqslant 0,\quad p_i - mq_i\geqslant 0,\quad Mq_i - p_i\geqslant 0\quad (i=1,\ldots,n). \tag{2.1}$$

If $f:[a,b]\to\mathbb{R}$ *is a convex function and* $\boldsymbol{x} = (x_1,\ldots,x_n)$ *is any* n*-tuple from* $[a,b]^n$, *then*

$$M\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{q})\geqslant\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{p})\geqslant m\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{q}). \tag{2.2}$$

Proof. Suppose that $\boldsymbol{p},\boldsymbol{q}\in\mathscr{P}_n$ and $m,M\in\mathbb{R}$ satisfy (2.1). From $p_i - mq_i\geqslant 0$ $(i=1,\ldots,n)$ follows that $1-m=\sum_{i=1}^{n}(p_i - mq_i)\geqslant 0$ i.e., $m\leqslant 1$, and from $Mq_i - p_i\geqslant 0$ $(i=1,\ldots,n)$ follows that $M-1=\sum_{i=1}^{n}(Mq_i - p_i)\geqslant 0$ i.e., $M\geqslant 1$. If $m=1$ or $M=1$, then $\boldsymbol{p}=\boldsymbol{q}$ and (2.2) obviously holds. Hence, it remains to consider the case when $m<1$ and $M>1$.

Applying Theorem A with $w_i := p_i - mq_i$ and using the convexity of the function f we obtain the right inequality in (2.2).

Similarly, applying Theorem A with $w_i := Mq_i - p_i$ and using the convexity of the function f we obtain the left inequality in (2.2). $\square$

COROLLARY 1. *Let* $\boldsymbol{p} = (p_1,\ldots,p_n)$ *and* $\boldsymbol{q} = (q_1,\ldots,q_n)$ *be two* n*-tuples from* $\mathscr{P}_n$ *such that* $q_i > 0$ $(i=1,\ldots,n)$. *Let*

$$m = m(\boldsymbol{p},\boldsymbol{q}) := \min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\},\quad M = M(\boldsymbol{p},\boldsymbol{q}) := \max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\}. \tag{2.3}$$

If $f:[a,b]\to\mathbb{R}$ is a convex function and $\boldsymbol{x}=(x_1,\ldots,x_n)$ is any n-tuple from $[a,b]^n$, then (2.2) *holds.*

Proof. Obviously $m\geqslant 0$ and

$$\frac{p_i}{q_i}-m\geqslant 0,\quad M-\frac{p_i}{q_i}\geqslant 0\quad (i=1,\ldots,n),$$

which implies

$$p_i-mq_i\geqslant 0,\quad Mq_i-p_i\geqslant 0\quad (i=1,\ldots,n).$$

Hence, m and M satisfy the conditions of Theorem 1. □

3. Bounds for the normalized Jensen-Mercer functional under the Jensen-Steffensen conditions

Let $\tilde{\mathscr{P}}_n$ denotes the set of all real n-tuples $\boldsymbol{p}=(p_1,\ldots,p_n)$ satisfying the following Jensen-Steffensen conditions

$$0\leqslant P_k\leqslant 1\quad (k=1,\ldots,n-1),\quad P_n=1, \tag{3.1}$$

where $P_k:=\sum_{i=1}^k p_i\quad (k=1,\ldots,n)$. Since any n-tuple $\boldsymbol{p}$ from $\mathscr{P}_n$ obviously satisfies (3.1), $\mathscr{P}_n\subseteq\tilde{\mathscr{P}}_n$. Notice that $\tilde{\mathscr{P}}_n$ is also a convex subset of $\mathbb{R}^n$.

Let $f:[a,b]\to\mathbb{R}$ be a convex function, $\boldsymbol{x}=(x_1,\ldots,x_n)\in[a,b]^n$ any monotonic n-tuple and $\boldsymbol{p}=(p_1,\ldots,p_n)\in\tilde{\mathscr{P}}_n$. Then $\sum_{i=1}^n p_ix_i\in[a,b]$ (see for example [1]) and $\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{p})$ is well defined. Also, because of Theorem B, $\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{p})\geqslant 0$ for all $\boldsymbol{p}\in\tilde{\mathscr{P}}_n$.

THEOREM 2. *Let $\boldsymbol{p}=(p_1,\ldots,p_n)$ and $\boldsymbol{q}=(q_1,\ldots,q_n)$ be two n-tuples from $\tilde{\mathscr{P}}_n$. Let m and M be any real constants such that*

$$m\geqslant 0,\quad P_k-mQ_k\geqslant 0,\quad (1-P_k)-m(1-Q_k)\geqslant 0\quad (k=1,\ldots,n-1) \tag{3.2}$$

and

$$MQ_k-P_k\geqslant 0,\quad M(1-Q_k)-(1-P_k)\geqslant 0\quad (k=1,\ldots,n-1), \tag{3.3}$$

where $P_k=\sum_{i=1}^k p_i$, $Q_k=\sum_{i=1}^k q_i$. If $f:[a,b]\to\mathbb{R}$ is a convex function and $\boldsymbol{x}=(x_1,\ldots,x_n)\in[a,b]^n$ is any monotonic n-tuple, then

$$M\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{q})\geqslant\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{p})\geqslant m\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{q}). \tag{3.4}$$

Proof. Assume that $\boldsymbol{p},\boldsymbol{q}\in\tilde{\mathscr{P}}_n$ and $m,M\in\mathbb{R}$ satisfy (3.2) and (3.3). From (3.2) follows that it has to be $m\leqslant 1$, and from (3.3) follows that it has to be $M\geqslant 1$. If $m=1$ or $M=1$, then $\boldsymbol{p}=\boldsymbol{q}$ and (3.4) obviously holds. Hence, it remains to consider the case when $m<1$ and $M>1$.

To prove the right inequality in (3.4) we consider the n-tuple $\boldsymbol{w}=(w_1,\ldots,w_n)$ defined by $w_i:=p_i-mq_i$ $(i=1,\ldots,n)$. From (3.2) follows that $\boldsymbol{w}$ satisfies conditions

(1.2). Now, we follow our proof of Theorem 1, but instead of using Theorem A we use Theorem B.

To prove the left inequality in (3.4) we consider the n-tuple $\boldsymbol{w} = (w_1, \dots, w_n)$ defined by $w_i := Mq_i - p_i$, $(i = 1, \dots, n)$. From (3.3) follows that $\boldsymbol{w}$ satisfies conditions (1.2). Again, we follow our proof of Theorem 1, but using Theorem B instead of Theorem A. □

COROLLARY 2. *Let $\boldsymbol{p} = (p_1, \dots, p_n)$ and $\boldsymbol{q} = (q_1, \dots, q_n)$ be two n-tuples from $\tilde{\mathscr{P}}_n$. For $k \in \{1, \dots, n\}$ denote $P_k := \sum_{i=1}^k p_i$, $Q_k := \sum_{i=1}^k q_i$. Assume that $0 < Q_k < 1$ for all $k \in \{1, \dots, n-1\}$ and define*

$$\tilde{m} = \tilde{m}(\boldsymbol{p}, \boldsymbol{q}) := \min\left\{ \tfrac{P_k}{Q_k}, \tfrac{1-P_k}{1-Q_k} : k = 1, \dots, n-1 \right\}, \tag{3.5}$$

$$\tilde{M} = \tilde{M}(\boldsymbol{p}, \boldsymbol{q}) := \max\left\{ \tfrac{P_k}{Q_k}, \tfrac{1-P_k}{1-Q_k} : k = 1, \dots, n-1 \right\}. \tag{3.6}$$

If $f : [a,b] \to \mathbb{R}$ is a convex function and if $\boldsymbol{x} = (x_1, \dots, x_n) \in [a,b]^n$ is any monotonic n-tuple, then

$$\tilde{M}\mathscr{M}_n(f, \boldsymbol{x}, \boldsymbol{q}) \geqslant \mathscr{M}_n(f, \boldsymbol{x}, \boldsymbol{p}) \geqslant \tilde{m}\mathscr{M}_n(f, \boldsymbol{x}, \boldsymbol{q}). \tag{3.7}$$

Proof. Since $0 < Q_k < 1$ for all $k \in \{1, \dots, n-1\}$, $\tilde{m}$ and $\tilde{M}$ are well defined and obviously (3.2) and (3.3) are satisfied for $m = \tilde{m}$ and $M = \tilde{M}$. Therefore we can apply Theorem 2 to obtain (3.7). □

We can consider the uniform distribution $\boldsymbol{u} = (\frac{1}{n}, \dots, \frac{1}{n})$ and the corresponding nonweighted Jensen-Mercer functional

$$\mathscr{M}_n(f, \boldsymbol{x}) := \mathscr{M}_n(f, \boldsymbol{x}, \boldsymbol{u}) = f(a) + f(b) - \frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(a + b - \frac{1}{n}\sum_{i=1}^n x_i\right).$$

Then we can state the following special case of Corollary 2.

COROLLARY 3. *Let $\boldsymbol{p} = (p_1, \dots, p_n)$ be n-tuple from $\tilde{\mathscr{P}}_n$. For $k \in \{1, \dots, n\}$ denote $P_k := \sum_{i=1}^k p_i$ and define*

$$\tilde{m}_0 := n \cdot \min\left\{ \tfrac{P_k}{k}, \tfrac{1-P_k}{n-k} : k = 1, \dots, n-1 \right\},$$

$$\tilde{M}_0 := n \cdot \max\left\{ \tfrac{P_k}{k}, \tfrac{1-P_k}{n-k} : k = 1, \dots, n-1 \right\}.$$

If $f : [a,b] \to \mathbb{R}$ is a convex function and if $\boldsymbol{x} = (x_1, \dots, x_n) \in [a,b]^n$ is any monotonic n-tuple, then

$$\tilde{M}_0\mathscr{M}_n(f, \boldsymbol{x}) \geqslant \mathscr{M}_n(f, \boldsymbol{x}, \boldsymbol{p}) \geqslant \tilde{m}_0\mathscr{M}_n(f, \boldsymbol{x}).$$

Next, we show that Theorem 2 in some way provides an improvement of Corollary 1. Denote by Π_n the set off all permutations of $(1, 2, \dots, n)$. Suppose that $\pi = (\pi(1), \pi(2), \dots, \pi(n)) \in \Pi_n$ and denote $\boldsymbol{a}_\pi := (a_{\pi(1)}, a_{\pi(2)}, \dots, a_{\pi(n)})$ for any n-tuple $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$. First we prove one simple auxiliary result.

LEMMA 1. *Let* $\boldsymbol{p} = (p_1,\dots,p_n)$ *and* $\boldsymbol{q} = (q_1,\dots,q_n)$ *be two nonnegative* n*-tuples from* $\mathscr{P}_n$. *If* $q_i > 0$ *for all* $i \in \{1,\dots,n\}$, *then* $\tilde{m}(\boldsymbol{p},\boldsymbol{q})$ *and* $\tilde{M}(\boldsymbol{p},\boldsymbol{q})$ *are well defined by* (3.5) *and* (3.6) *and*

$$\max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} \geqslant \tilde{M}(\boldsymbol{p},\boldsymbol{q}), \quad \tilde{m}(\boldsymbol{p},\boldsymbol{q}) \geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\}. \tag{3.8}$$

Proof. Since $q_i > 0$ for all $i \in \{1,\dots,n\}$, it is obvious that $0 < Q_k < 1$ for all $k \in \{1,\dots,n-1\}$, so that $\tilde{m}(\boldsymbol{p},\boldsymbol{q})$ and $\tilde{M}(\boldsymbol{p},\boldsymbol{q})$ are well defined by (3.5) and (3.6). Also, for any $k \in \{1,\dots,n-1\}$ we can write

$$P_k := \sum_{i=1}^{k} p_i = \sum_{i=1}^{k} \frac{p_i}{q_i} q_i, \quad 1 - P_k = \sum_{i=k+1}^{n} p_i = \sum_{i=k+1}^{n} \frac{p_i}{q_i} q_i.$$

Now,

$$\max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} Q_k = \max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} \sum_{i=1}^{k} q_i \geqslant P_k \geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} \sum_{i=1}^{k} q_i = \min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} Q_k,$$

i.e.,

$$\max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} \geqslant \frac{P_k}{Q_k} \geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\}.$$

Similarly,

$$\max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} \geqslant \frac{1-P_k}{1-Q_k} \geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} \quad \text{for all } k \in \{1,\dots,n-1\},$$

and (3.8) immediately follows. □

REMARK 1. It is clear that inequalities stated in Lemma 1 can be strict. For example, if $n = 5$, $\boldsymbol{p} = \left(\frac{1}{6},\frac{1}{3},\frac{1}{9},\frac{1}{6},\frac{2}{9}\right)$ and $\boldsymbol{q} = \left(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right)$, then

$$\max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} = \frac{5}{3} > \tilde{M}(\boldsymbol{p},\boldsymbol{q}) = \frac{5}{4}, \quad \tilde{m}(\boldsymbol{p},\boldsymbol{q}) = \frac{5}{6} > \min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} = \frac{5}{9}.$$

It is not hard to see that generally

$$\max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} = \max_{\pi\in\Pi_n} \tilde{M}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi), \quad \min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} = \min_{\pi\in\Pi_n} \tilde{m}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi).$$

THEOREM 3. *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function and* $\boldsymbol{x} = (x_1,\dots,x_n) \in [a,b]^n$ *be any* n*-tuple. Let* $\pi = (\pi(1),\pi(2),\dots,\pi(n))$ *be a permutation of* $(1,2,\dots,n)$ *such that* $\boldsymbol{x}_\pi$ *is monotonic (nondecreasing or nonincreasing). If* $\boldsymbol{p} = (p_1,\dots,p_n)$ *and* $\boldsymbol{q} = (q_1,\dots,q_n)$ *are two* n*-tuples from* $\mathscr{P}_n$ *such that* $q_i > 0$ *for all* $i \in \{1,\dots,n\}$, *then*

$$\begin{aligned}\max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} \mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{q}) &\geqslant \tilde{M}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{q}) \geqslant \mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{p}) \\ &\geqslant \tilde{m}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{q}) \geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} \mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{q}),\end{aligned} \tag{3.9}$$

where $\tilde{m}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)$ *and* $\tilde{M}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)$ *are defined as in* (3.5) *and* (3.6). *The first and the last inequality can be strict.*

Proof. Since π is chosen so that $\boldsymbol{x}_\pi$ is monotonic we can apply Corollary 2 and Lemma 1 to the n-tuples $\boldsymbol{p}_\pi$ and $\boldsymbol{q}_\pi$ to get

$$\begin{aligned}\max_{1\leqslant i\leqslant n}\left\{\frac{p_{\pi(i)}}{q_{\pi(i)}}\right\}\mathscr{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi) &\geqslant \tilde{M}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)\mathscr{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi)\geqslant \mathscr{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{p}_\pi)\\ &\geqslant \tilde{m}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)\mathscr{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi)\\ &\geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{p_{\pi(i)}}{q_{\pi(i)}}\right\}\mathscr{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi).\end{aligned}$$

Since $\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{p})$ doesn't change if we simultaneously permute the components of $\boldsymbol{x}$ and $\boldsymbol{p}$, we have $\mathscr{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{p}_\pi)=\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{p})$ and $\mathscr{M}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi)=\mathscr{M}_n(f,\boldsymbol{x},\boldsymbol{q})$. Also it is obvious that $\max_{1\leqslant i\leqslant n}\left\{\frac{p_{\pi(i)}}{q_{\pi(i)}}\right\}=\max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\}$ and $\min_{1\leqslant i\leqslant n}\left\{\frac{p_{\pi(i)}}{q_{\pi(i)}}\right\}=\min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\}$. Therefore, sequence of inequalities (3.9) holds. By Remark 1 the first and the last inequality in the sequence can be strict. □

4. Integral versions

In [3] the following theorem is proved.

THEOREM C. *Let $(\Omega,\mathscr{A},\mu)$ be a probability measure space, and let $x:\Omega\to[a,b]$ $(-\infty<a<b<\infty)$ be a measurable function. Then for any continuous convex function $f:[a,b]\to\mathbb{R}$,*

$$f\left(a+b-\int_\Omega x\,\mathrm{d}\mu\right)\leqslant f(a)+f(b)-\int_\Omega f(x)\,\mathrm{d}\mu$$

holds.

It can analogously be proved that for a measure space $(\Omega,\mathscr{A},\mu)$ with $0<\mu(\Omega)<\infty$ the integral version of the Jensen-Mercer inequality

$$f\left(a+b-\frac{1}{\mu(\Omega)}\int_\Omega x\,\mathrm{d}\mu\right)\leqslant f(a)+f(b)-\frac{1}{\mu(\Omega)}\int_\Omega f(x)\,\mathrm{d}\mu \tag{4.1}$$

holds. In a special case when $\Omega=[\alpha,\beta]$, where $-\infty<\alpha<\beta<\infty$ and $\lambda:[\alpha,\beta]\to\mathbb{R}$ is any nondecreasing function such that $\lambda(\beta)\neq\lambda(\alpha)$ inequality (4.1) becomes

$$f\left(a+b-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta x(t)\mathrm{d}\lambda(t)\right)\leqslant f(a)+f(b)-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f(x(t))\mathrm{d}\lambda(t). \tag{4.2}$$

Also, we can prove that (4.2) remains valid even in the case when the condition "λ is nondecreasing function" is somewhat relaxed. We use the following integral variant of the Jensen-Steffensen inequality given by R. P. Boas [2] (see also [5, p. 59]):

THEOREM D. *Let* $x:[\alpha,\beta]\to(a,b)$ *be a continuous and monotonic function (either nondecreasing or nonincreasing), where* $-\infty<\alpha<\beta<\infty$ *and* $-\infty\leqslant a<b\leqslant\infty$*, and let function* $\lambda:[\alpha,\beta]\to\mathbb{R}$ *be either continuous or of bounded variation satisfying*

$$\lambda(\alpha)\leqslant\lambda(t)\leqslant\lambda(\beta)\ \text{ for all } t\in[\alpha,\beta],\quad \lambda(\beta)-\lambda(\alpha)>0. \tag{4.3}$$

If $f:(a,b)\to\mathbb{R}$ *is a convex function, then*

$$f\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta}x(t)\mathrm{d}\lambda(t)\right)\leqslant\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta}f(x(t))\mathrm{d}\lambda(t). \tag{4.4}$$

holds.

THEOREM 4. *Let* $x:[\alpha,\beta]\to[a,b]$ *be a continuous and monotonic function, where* $-\infty<\alpha<\beta<+\infty$ *and* $-\infty<a<b<+\infty$*. Let function* $\lambda:[\alpha,\beta]\to\mathbb{R}$ *be either continuous or of bounded variation satisfying* (4.3)*. Then for any continuous convex function* $f:[a,b]\to\mathbb{R}$ *inequality* (4.2) *holds.*

Proof. Suppose that x is continuous and nondecreasing i.e., $x(t_1)\leqslant x(t_2)$ for $t_1<t_2\in[\alpha,\beta]$. Let $\widetilde{\alpha},\widetilde{\beta}$ be any real numbers such that $\widetilde{\alpha}<\alpha$ and $\beta<\widetilde{\beta}$. We define function $\widetilde{x}:[\widetilde{\alpha},\widetilde{\beta}]\to[a,b]$ by

$$\widetilde{x}(t)=\begin{cases} a+\frac{x(\alpha)-a}{\alpha-\widetilde{\alpha}}(t-\widetilde{\alpha}), & t\in[\widetilde{\alpha},\alpha];\\ x(t), & t\in[\alpha,\beta];\\ b+\frac{b-x(\beta)}{\widetilde{\beta}-\beta}(t-\widetilde{\beta}), & t\in[\beta,\widetilde{\beta}].\end{cases}$$

$\widetilde{x}$ is also continuous and nondecreasing. Furthermore, $\widetilde{x}([\widetilde{\alpha},\alpha])=[a,x(\alpha)]$, $\widetilde{x}([\beta,\widetilde{\beta}])=[x(\beta),b]$ and $\widetilde{x}(t)=x(t)$ for all $t\in[\alpha,\beta]$.

Next, we define two functions $\widetilde{\lambda}_s:[\widetilde{\alpha},\widetilde{\beta}]\to\mathbb{R}$ and $\widetilde{\lambda}_c:[\widetilde{\alpha},\widetilde{\beta}]\to\mathbb{R}$ by

$$\widetilde{\lambda}_s(t)=\begin{cases}1, & t=\widetilde{\alpha};\\ 0, & t\in(\widetilde{\alpha},\widetilde{\beta});\\ -1, & t=\widetilde{\beta},\end{cases}\quad\text{and}\quad \widetilde{\lambda}_c(t)=\begin{cases}1, & t\in[\widetilde{\alpha},\alpha];\\ \frac{\lambda(\beta)-\lambda(t)}{\lambda(\beta)-\lambda(\alpha)}, & t\in[\alpha,\beta];\\ 0, & t\in[\beta,\widetilde{\beta}].\end{cases}$$

Notice that $\widetilde{\lambda}_s$ is a step function with only jumps at end points of the interval $[\widetilde{\alpha},\widetilde{\beta}]$. Therefore, the integral $\int_{\tilde{\alpha}}^{\tilde{\beta}}y(t)\mathrm{d}\widetilde{\lambda}_s(t)$ is well defined for any function $y:[\tilde{\alpha},\tilde{\beta}]\to\mathbb{R}$ continuous at the points $\tilde{\alpha}$ and $\tilde{\beta}$, and we have

$$\begin{aligned}\int_{\tilde{\alpha}}^{\tilde{\beta}}y(t)\mathrm{d}\tilde{\lambda}_s(t)&=y(\tilde{\alpha})[\tilde{\lambda}_s(\tilde{\alpha}+0)-\tilde{\lambda}_s(\tilde{\alpha})]+y(\tilde{\beta})[\tilde{\lambda}_s(\tilde{\beta})-\tilde{\lambda}_s(\tilde{\beta}-0)]\\ &=-y(\tilde{\alpha})-y(\tilde{\beta}).\end{aligned} \tag{4.5}$$

Also, if λ is continuous on $[\alpha,\beta]$ then $\tilde{\lambda}_c$ is continuous on $[\tilde{\alpha},\tilde{\beta}]$, and if λ is of bounded variation on $[\alpha,\beta]$ then $\tilde{\lambda}_c$ is of bounded variation on $[\tilde{\alpha},\tilde{\beta}]$. Therefore the integral $\int_{\tilde{\alpha}}^{\tilde{\beta}} y(t)\mathrm{d}\tilde{\lambda}_c(t)$ is well defined for any continuous and piecewise monotonic function $y:[\tilde{\alpha},\tilde{\beta}]\to\mathbb{R}$, and we have

$$\begin{aligned}\int_{\tilde{\alpha}}^{\tilde{\beta}} y(t)\mathrm{d}\tilde{\lambda}_c(t) &= \int_{\tilde{\alpha}}^{\alpha} y(t)\mathrm{d}\tilde{\lambda}_c(t)+\int_{\alpha}^{\beta} y(t)\mathrm{d}\tilde{\lambda}_c(t)+\int_{\beta}^{\tilde{\beta}} y(t)\mathrm{d}\tilde{\lambda}_c(t)\\ &= \int_{\alpha}^{\beta} y(t)\mathrm{d}\tilde{\lambda}_c(t)=\int_{\alpha}^{\beta} y(t)\mathrm{d}\left[\frac{\lambda(\beta)-\lambda(t)}{\lambda(\beta)-\lambda(\alpha)}\right] \qquad (4.6)\\ &= -\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} y(t)\mathrm{d}\lambda(t).\end{aligned}$$

Now we define $\tilde{\lambda}:[\tilde{\alpha},\tilde{\beta}]\to\mathbb{R}$ by $\tilde{\lambda}(t)=\tilde{\lambda}_c(t)-\tilde{\lambda}_s(t)$, $t\in[\tilde{\alpha},\tilde{\beta}]$. Notice that

$$\tilde{\lambda}(\tilde{\beta})-\tilde{\lambda}(\tilde{\alpha})=\tilde{\lambda}_c(\tilde{\beta})-\tilde{\lambda}_c(\tilde{\alpha})-\tilde{\lambda}_s(\tilde{\beta})+\tilde{\lambda}_s(\tilde{\alpha})=0-1+1+1=1. \qquad (4.7)$$

From previous observations we conclude that the integral $\int_{\tilde{\alpha}}^{\tilde{\beta}} y(t)\mathrm{d}\tilde{\lambda}(t)$ is well defined for any continuous and piecewise monotonic function $y:[\tilde{\alpha},\tilde{\beta}]\to\mathbb{R}$, and from (4.5) and (4.6) we have

$$\begin{aligned}\int_{\tilde{\alpha}}^{\tilde{\beta}} y(t)\mathrm{d}\tilde{\lambda}(t) &= \int_{\tilde{\alpha}}^{\tilde{\beta}} y(t)\mathrm{d}\tilde{\lambda}_c(t)-\int_{\tilde{\alpha}}^{\tilde{\beta}} y(t)\mathrm{d}\tilde{\lambda}_s(t)\\ &= -\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} y(t)\mathrm{d}\lambda(t)+y(\tilde{\alpha})+y(\tilde{\beta}). \qquad (4.8)\end{aligned}$$

If we apply Theorem D on the functions $\widetilde{x}$, f and $\tilde{\lambda}$ (we can do that even if the function $\tilde{\lambda}$ is neither continuous nor of bounded variation since all the integrals are well defined) we obtain

$$f\left(\frac{1}{\tilde{\lambda}(\tilde{\beta})-\tilde{\lambda}(\tilde{\alpha})}\int_{\tilde{\alpha}}^{\tilde{\beta}}\widetilde{x}(t)\mathrm{d}\tilde{\lambda}(t)\right)\leqslant\frac{1}{\tilde{\lambda}(\tilde{\beta})-\tilde{\lambda}(\tilde{\alpha})}\int_{\tilde{\alpha}}^{\tilde{\beta}} f(\widetilde{x}(t))\,\mathrm{d}\tilde{\lambda}(t). \qquad (4.9)$$

Using (4.7), (4.8) with $y=\widetilde{x}$ and (4.8) with $y=f(\widetilde{x})$, from (4.9) we have

$$f\left(a+b-\tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} x(t)\mathrm{d}\lambda(t)\right)\leqslant f(a)+f(b)-\tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} f(x(t))\mathrm{d}\lambda(t). \qquad (4.10)$$

In case when x is nonincreasing we define function $\widetilde{x}:[\widetilde{\alpha},\widetilde{\beta}]\to[a,b]$ by

$$\widetilde{x}(t)=\begin{cases} b+\frac{x(\alpha)-b}{\alpha-\widetilde{\alpha}}(t-\widetilde{\alpha}), & t\in[\widetilde{\alpha},\alpha];\\ x(t), & t\in[\alpha,\beta];\\ a+\frac{a-x(\beta)}{\widetilde{\beta}-\beta}(t-\widetilde{\beta}), & t\in[\beta,\widetilde{\beta}]\end{cases}$$

and obtain (4.10) in the same way. This completes the proof. □

There is no loss in generality if we assume $\lambda(\beta)-\lambda(\alpha)=1$ and consider the normalized Jensen-Mercer functional

$$\mathscr{M}(f,x,\lambda):=f(a)+f(b)-\int_{\alpha}^{\beta}f(x(t))\mathrm{d}\lambda(t)-f\left(a+b-\int_{\alpha}^{\beta}x(t)\mathrm{d}\lambda(t)\right). \tag{4.11}$$

Under the appropriate assumptions on f, x and λ, we always have $\mathscr{M}(f,x,\lambda)\geqslant 0$.

For $-\infty<\alpha<\beta<\infty$ let $\Lambda_{[\alpha,\beta]}$ denotes the class of all functions $\lambda:[\alpha,\beta]\to\mathbb{R}$ which are either continuous or of bounded variation and satisfy the condition

$$\lambda(\alpha)\leqslant\lambda(t)\leqslant\lambda(\beta)\ \text{ for all } t\in[\alpha,\beta],\quad \lambda(\beta)-\lambda(\alpha)=1. \tag{4.12}$$

Notice that any nondecreasing function $\lambda:[\alpha,\beta]\to\mathbb{R}$ with $\lambda(\beta)-\lambda(\alpha)=1$ belongs to $\Lambda_{[\alpha,\beta]}$.

Now we can state the integral analogues of the results from the previous section. First, we give the integral version of Theorem 1.

THEOREM 5. *Let λ and μ be two functions from $\Lambda_{[\alpha,\beta]}$, $-\infty<\alpha<\beta<\infty$. Let $x:[\alpha,\beta]\to[a,b]$, $-\infty<a<b<\infty$ be a continuous function and let $f:[a,b]\to\mathbb{R}$ be a convex function.*

a) If μ is nondecreasing and if $m\geqslant 0$ is a constant such that the function $\rho:[\alpha,\beta]\to\mathbb{R}$ defined by

$$\rho(t):=\lambda(t)-m\mu(t),\quad t\in[\alpha,\beta] \tag{4.13}$$

is also nondecreasing, then

$$\mathscr{M}(f,x,\lambda)\geqslant m\mathscr{M}(f,x,\mu). \tag{4.14}$$

b) If λ is nondecreasing and if $M>0$ is a constant such that the function $\sigma:[\alpha,\beta]\to\mathbb{R}$ defined by

$$\sigma(t):=M\mu(t)-\lambda(t),\quad t\in[\alpha,\beta] \tag{4.15}$$

is also nondecreasing, then

$$M\mathscr{M}(f,x,\mu)\geqslant\mathscr{M}(f,x,\lambda). \tag{4.16}$$

Proof. $a)$ Since μ and $\rho=\lambda-m\mu$ are assumed to be nondecreasing and $m\geqslant 0$, the function $\lambda=\rho+m\mu$ is nondecreasing too. Hence, $\mathscr{M}(f,x,\lambda)$ and $\mathscr{M}(f,x,\mu)$ are well defined. Since ρ is nondecreasing it has to be $m\leqslant 1$. If $m=1$ then (4.14) holds with equality sign. Hence, it remains to consider the case when $m<1$. In this case $\rho(\beta)-\rho(\alpha)=1-m>0$ and ρ is nondecreasing by our assumption so that (4.2) can be applied to ρ. Now, using (4.2) and then the convexity of f we obtain (4.14).

$b)$ Since λ and $\sigma=M\mu-\lambda$ are assumed to be nondecreasing and $M>0$, the function $\mu=\frac{1}{M}(\sigma+\lambda)$ is nondecreasing too. Hence, $\mathscr{M}(f,x,\lambda)$ and $\mathscr{M}(f,x,\mu)$ are well defined. Since ρ is nondecreasing it has to be $M\geqslant 1$. If $M=1$ then (4.16) holds with equality sign. Hence, it remains to consider the case when $M>1$. In this case $\sigma(\beta)-\sigma(\alpha)=M-1>0$ and σ is nondecreasing by our assumption so that (4.2) can be applied to σ. Now, using (4.2) and then the convexity of f we obtain (4.16). This completes the proof. □

COROLLARY 4. *Let λ and μ be two functions from $\Lambda_{[\alpha,\beta]}$, $-\infty<\alpha<\beta<\infty$, $x:[\alpha,\beta]\to[a,b]$, $-\infty<a<b<\infty$ be a continuous function and $f:[a,b]\to\mathbb{R}$ be a convex function.*

$a)$ Let μ is strictly increasing and define

$$\tilde{m}=\tilde{m}(\lambda,\mu):=\inf_{\alpha<t<\beta}\left\{\inf\left\{\tfrac{\lambda(t)-\lambda(s)}{\mu(t)-\mu(s)}:\alpha\leqslant s\leqslant\beta,\ s\neq t\right\}\right\}.$$

If $\widetilde{m}\geqslant 0$, then

$$\mathscr{M}(f,x,\lambda)\geqslant\tilde{m}\mathscr{M}(f,x,\mu). \tag{4.17}$$

$b)$ Let λ is nondecreasing, μ is strictly increasing and define

$$\tilde{M}=\tilde{M}(\lambda,\mu):=\sup_{\alpha<t<\beta}\left\{\sup\left\{\tfrac{\lambda(t)-\lambda(s)}{\mu(t)-\mu(s)}:\alpha\leqslant s\leqslant\beta,\ s\neq t\right\}\right\}.$$

If $\widetilde{M}<\infty$, then

$$\tilde{M}\mathscr{M}(f,x,\mu)\geqslant\mathscr{M}(f,x,\lambda). \tag{4.18}$$

Proof. $a)$ Since μ is strictly increasing it is injective and $\tilde{m}$ is well defined real number in $[-\infty,\infty)$. If $\tilde{m}\geqslant 0$, then $\rho=\lambda-\tilde{m}\mu$ is well defined nondecreasing function on $[\alpha,\beta]$ and we can apply Theorem 5 $a)$ with $m=\tilde{m}$ to obtain (4.17).

$b)$ Since λ is nondecreasing and μ is strictly increasing $\tilde{M}$ is well defined real number in $(0,\infty]$. If $\tilde{M}<\infty$ then $\sigma=\tilde{M}\mu-\lambda$ is well defined nondecreasing function on $[\alpha,\beta]$ and we can apply Theorem 5 $b)$ with $M=\tilde{M}$ to obtain (4.18). □

REMARK 2. If $\tilde{m}=0$, then the inequality (4.17) is trivially fulfilled. Similarly, if $\tilde{M}=\infty$, then the inequality (4.18) is trivially fulfilled. Two simple examples which illustrate such cases are as follows.

For $[\alpha,\beta]=[0,1]$ let $\lambda(t)=t^3$, $\mu(t)=t$, $t\in[0,1]$. Then $\tilde{m}=0$, $\tilde{M}=3$.

For $[\alpha,\beta]=[0,1]$ let $\lambda(t)=t$, $\mu(t)=t^3$, $t\in[0,1]$. Then $\tilde{m}=\frac{1}{3}$, $\tilde{M}=\infty$.

As in the discrete case we can consider the uniform distribution i.e., the function $\upsilon \in \Lambda_{[\alpha,\beta]}$ defined by $\upsilon(t) := \frac{1}{\beta-\alpha}t$, $t \in [\alpha,\beta]$ and the corresponding nonweighted integral Jensen-Mercer functional

$$\mathscr{M}(f,x) := \mathscr{M}(f,x,\upsilon) = f(a)+f(b) - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(x(t))\mathrm{d}t - f\left(a+b-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} x(t)\mathrm{d}t\right).$$

Then we can state the following special case of Corollary 4.

COROLLARY 5. *Let λ be a function from $\Lambda_{[\alpha,\beta]}$, $-\infty<\alpha<\beta<\infty$, $x:[\alpha,\beta]\to[a,b]$, $-\infty<a<b<\infty$ be a continuous function and $f:[a,b]\to\mathbb{R}$ be a convex function.*

a) If $\tilde{m}_0 := (\beta-\alpha)\cdot\inf_{\alpha<t<\beta}\left\{\inf\left\{\frac{\lambda(t)-\lambda(s)}{t-s} : \alpha\leqslant s\leqslant\beta,\ s\neq t\right\}\right\}\geqslant 0$, then $\mathscr{M}(f,x,\lambda)\geqslant \tilde{m}_0\mathscr{M}(f,x)$.

b) If λ is nondecreasing and $\tilde{M}_0 := (\beta-\alpha)\cdot\sup_{\alpha<t<\beta}\left\{\sup\left\{\frac{\lambda(t)-\lambda(s)}{t-s} : \alpha\leqslant s\leqslant\beta,\ s\neq t\right\}\right\}<\infty$, then $\tilde{M}_0\mathscr{M}(f,x)\geqslant\mathscr{M}(f,x,\lambda)$.

Next, we give the integral version of Theorem 2.

THEOREM 6. *Let λ and μ be two functions from $\Lambda_{[\alpha,\beta]}$, $-\infty<\alpha<\beta<\infty$, either both continuous or both of bounded variation. Let $x:[\alpha,\beta]\to[a,b]$, $-\infty<a<b<\infty$ be a monotonic function (either nondecreasing or nonincreasing) and $f:[a,b]\to\mathbb{R}$ be a convex function.*

a) If $m\geqslant 0$ is a constant such that for all $\alpha<t<\beta$

$$\lambda(t)-\lambda(\alpha)-m(\mu(t)-\mu(\alpha))\geqslant 0,\quad \lambda(\beta)-\lambda(t)-m(\mu(\beta)-\mu(t))\geqslant 0, \tag{4.19}$$

then

$$\mathscr{M}(f,x,\lambda)\geqslant m\mathscr{M}(f,x,\mu). \tag{4.20}$$

b) If $M>0$ is a constant such that for all $\alpha<t<\beta$

$$M(\mu(t)-\mu(\alpha))-(\lambda(t)-\lambda(\alpha))\geqslant 0,\ M(\mu(\beta)-\mu(t))-(\lambda(\beta)-\lambda(t))\geqslant 0, \tag{4.21}$$

then

$$M\mathscr{M}(f,x,\mu)\geqslant\mathscr{M}(f,x,\lambda). \tag{4.22}$$

Proof. First note that under the given assumptions on λ, μ, x and f, $\mathscr{M}(f,x,\lambda)$ and $\mathscr{M}(f,x,\mu)$ are well defined and nonnegative.

a) From (4.19) follows that it has to be $m\leqslant 1$. If $m=1$ then (4.20) obviously holds with equality sign. Hence, it remains to consider the case $0\leqslant m<1$. We define the function $\rho:[\alpha,\beta]\to\mathbb{R}$ by $\rho(t):=\lambda(t)-m\mu(t)$, $t\in[\alpha,\beta]$. If λ and μ are both continuous (both of bounded variation), then ρ is continuous (of bounded variation) too. Further, from (4.19) follows that for all $\alpha<t<\beta$

$$\rho(t)-\rho(\alpha)=\lambda(t)-\lambda(\alpha)-m(\mu(t)-\mu(\alpha))\geqslant 0,$$
$$\rho(\beta)-\rho(t)=\lambda(\beta)-\lambda(t)-m(\mu(\beta)-\mu(t))\geqslant 0.$$

Also, since $\lambda, \mu \in \Lambda_{[\alpha,\beta]}$

$$\rho(\beta)-\rho(\alpha)=\lambda(\beta)-\lambda(\alpha)-m(\mu(\beta)-\mu(\alpha))=1-m>0.$$

We conclude that the normalized function $\frac{1}{1-m}\rho$ belongs to the class $\Lambda_{[\alpha,\beta]}$. Now, we follow our proof of Theorem 5 $a)$, but instead of using (4.2) we apply Theorem 4. Thus we obtain (4.20).

$b)$ From (4.21) follows that it has to be $M \geqslant 1$. If $M=1$ then (4.22) holds with equality sign. Hence, it remains to consider the case $M>1$. We define the function $\sigma : [\alpha,\beta] \to \mathbb{R}$ by $\sigma(t) := M\mu(t)-\lambda(t)$, $t \in [\alpha,\beta]$. If λ and μ are both continuous (both of bounded variation), then σ is continuous (of bounded variation) too. Further, from (4.21) follows that for all $\alpha<t<\beta$

$$\sigma(t)-\sigma(\alpha)=M(\mu(t)-\mu(\alpha))-(\lambda(t)-\lambda(\alpha)) \geqslant 0,$$
$$\sigma(\beta)-\sigma(t)=M(\mu(\beta)-\mu(t))-(\lambda(\beta)-\lambda(t)) \geqslant 0.$$

Also, since $\lambda, \mu \in \Lambda_{[\alpha,\beta]}$

$$\sigma(\beta)-\sigma(\alpha)=M(\mu(\beta)-\mu(\alpha))-(\lambda(\beta)-\lambda(\alpha))=M-1>0.$$

We conclude that the normalized function $\frac{1}{M-1}\sigma$ belongs to the class $\Lambda_{[\alpha,\beta]}$. Now we follow our proof of Theorem 5 $b)$ but instead of using (4.2) we apply Theorem 4. Thus we obtain (4.22). □

COROLLARY 6. *Let λ and μ be two functions from $\Lambda_{[\alpha,\beta]}$, $-\infty<\alpha<\beta<\infty$, either both continuous or both of bounded variation. Let $x:[\alpha,\beta]\to[a,b]$, $-\infty<a<b<\infty$ be a monotonic function (either nondecreasing or nonincreasing) and $f:[a,b]\to\mathbb{R}$ be a convex function. Assume that $\mu(\alpha)<\mu(t)<\mu(\beta)$ for all $\alpha<t<\beta$, and define*

$$\tilde{m}=\tilde{m}(\lambda,\mu) := \inf\left\{\frac{\lambda(t)-\lambda(\alpha)}{\mu(t)-\mu(\alpha)}, \frac{\lambda(\beta)-\lambda(t)}{\mu(\beta)-\mu(t)} : \alpha<t<\beta\right\}, \tag{4.23}$$

$$\tilde{M}=\tilde{M}(\lambda,\mu) := \sup\left\{\frac{\lambda(t)-\lambda(\alpha)}{\mu(t)-\mu(\alpha)}, \frac{\lambda(\beta)-\lambda(t)}{\mu(\beta)-\mu(t)} : \alpha<t<\beta\right\}. \tag{4.24}$$

Then

$$\tilde{M}\mathscr{M}(f,x,\mu) \geqslant \mathscr{M}(f,x,\lambda) \geqslant \tilde{m}\mathscr{M}(f,x,\mu). \tag{4.25}$$

Proof. Since $\mu(\alpha)<\mu(t)<\mu(\beta)$ for all $\alpha<t<\beta$, $\tilde{m}$ and $\tilde{M}$ are well defined, and obviously $\tilde{m}\in[0,\infty)$ and $\tilde{M}\in(0,\infty]$. Therefore, the right inequality in (4.25) follows from Theorem 6 $a)$ with $m=\tilde{m}$. If $\tilde{M}=\infty$, then the left inequality in (4.25) holds trivially, while for $\tilde{M}<\infty$ it follows from Theorem 6 $b)$ with $M=\tilde{M}$. □

As in the previous case we can consider the uniform distribution υ and the corresponding nonweighted integral Jensen-Mercer functional $\mathscr{M}(f,x)$ and state the following special case of Corollary 6:

COROLLARY 7. *Let* λ *be a function from* $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$. *Let* $x : [\alpha,\beta] \to [a,b]$, $-\infty < a < b < \infty$ *be a monotonic function (either nondecreasing or nonincreasing) and* $f : [a,b] \to \mathbb{R}$ *be a convex function. If* $\tilde{m}_0$ *and* $\tilde{M}_0$ *are defined by*

$$\tilde{m}_0 := (\beta-\alpha)\cdot\inf\left\{\frac{\lambda(t)-\lambda(\alpha)}{t-\alpha}, \frac{\lambda(\beta)-\lambda(t)}{\beta-t} : \alpha < t < \beta\right\},$$
$$\tilde{M}_0 := (\beta-\alpha)\cdot\sup\left\{\frac{\lambda(t)-\lambda(\alpha)}{t-\alpha}, \frac{\lambda(\beta)-\lambda(t)}{\beta-t} : \alpha < t < \beta\right\},$$

then $\tilde{M}_0\mathscr{M}(f,x) \geqslant \mathscr{M}(f,x,\lambda) \geqslant \tilde{m}_0\mathscr{M}(f,x)$.

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J. Barić
Faculty of Electrical Engineering
Mechanical Engineering and Naval Architecture
University of Split
Ruđera Boškovića b.b.
21000 Split
Croatia *e-mail:* jbaric@fesb.hr

A. Matković
Faculty of Electrical Engineering
Mechanical Engineering and Naval Architecture
University of Split
Ruđera Boškovića b.b.
21000 Split
Croatia
e-mail: anita@pmfst.hr