ESTIMATIONS OF THE ERROR FOR GENERAL SIMPSON TYPE FORMULAE VIA PRE–GRÜSS INEQUALITY

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. Generalizations of estimations of general Simpson type formulae are given, by using the pre-Grüss inequality.

1. Introduction

In the recent paper [8] N. Ujevic used the generalization of pre-Grüss inequality to derive some better estimations of the error for Simpson’s quadrature rule. In fact, he proved the next three theorems:

THEOREM 1. Let $I \subset \mathbb{R}$ be a closed interval and $a, b \in \text{Int} I$, $a < b$. If $f : I \rightarrow \mathbb{R}$ is an absolutely continuous function with $f' \in L_2(a, b)$ then we have

$$\left| \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{(b-a)^{3/2}}{6} K_1, \quad (1.1)$$

where

$$K_1^2 = \|f''\|_2^2 - \frac{1}{b-a} \left( \int_a^b f'(t) dt \right)^2 - \left( \int_a^b f'(t) \Psi_0(t) dt \right)^2, \quad (1.2)$$

and $\Psi(t) = t - \frac{a+b}{2}$, $\Psi_0(t) = \Psi(t)/\|\Psi\|_2$.

THEOREM 2. Let $I \subset \mathbb{R}$ be a closed interval and $a, b \in \text{Int} I$, $a < b$. If $f : I \rightarrow \mathbb{R}$ is such that $f'$ is an absolutely continuous function with $f'' \in L_2(a, b)$ then we have

$$\left| \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{(b-a)^{5/2}}{12\sqrt{30}} K_2, \quad (1.3)$$

where

$$K_2^2 = \|f''\|_2^2 - \frac{1}{b-a} \left( \int_a^b f''(t) dt \right)^2 - \left( \int_a^b f''(t) \Psi_0(t) dt \right)^2, \quad (1.4)$$


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\[ \Psi(t) = \begin{cases} 
1, & t \in \left[ a, \frac{a + b}{2} \right] \\
-1, & t \in \left( \frac{a + b}{2}, b \right] 
\end{cases} \]  
and \( \Psi_0(t) = \Psi(t)/\|\Psi\|_2 \).

**Theorem 3.** Let \( I \subset \mathbb{R} \) be a closed interval and \( a, b \in \text{Int}I, \ a < b \). If \( f : I \to \mathbb{R} \) is such that \( f'' \) is an absolutely continuous function with \( f''' \in L_2(a, b) \) then we have

\[ \left| b - a \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \int_a^b f(t)dt \right| \leq \frac{(b - a)^{7/2}}{48\sqrt{105}} K_3, \]  
where

\[ K_3^2 = \|f''\|_2^2 - \frac{1}{b - a} \left( \int_a^b f'''(t)dt \right)^2 - \left( \int_a^b f'''(t)\Psi_0(t)dt \right)^2, \]  
and \( \Psi_0(t) = \Psi(t)/\|\Psi\|_2 \).

In this paper we will unify and generalize these results so that we will give the results for general Euler-Simpson formula and for functions whose derivative of order \( n \), \( n \geq 1 \), is from \( L_2(0, 1) \) space. We will also give related results for the general dual Euler-Simpson formula. We will use interval \([0, 1]\) because of simplicity and since it involves no loss in generality.

2. Estimations of the error for general Euler-Simpson formula

In the recent paper [6] the following identity, named the general Euler-Simpson formula, has been proved. For \( n \geq 1 \) and every \( t \in [0, 1] \) we have

\[ \int_0^1 f(t)dt = D(u, v) - T_n(u, v) + S_n(f) \]  
where

\[ D(u, v) = \frac{1}{2u + v} \left[ uf(0) + vf \left( \frac{1}{2} \right) + uf(1) \right], \]

\( T_0(u, v) = 0 \) and

\[ T_m(u, v) = \frac{1}{2u + v} \sum_{k=1}^m \tilde{B}_k \frac{1}{k!} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right], \]  
for \( 1 \leq m \leq n \), while

\[ \tilde{B}_k = uB_k(0) + vB_k \left( \frac{1}{2} \right) + uB_k(1), \ k \geq 1, \]
Bernoulli polynomials as $u$ for example [1] or [2]. We have

$$S_n(x) = \frac{1}{(2u+v)(n!)} \int_0^1 G_n(t) f^{(n)}(t) dt$$

and

$$G_n(t) = 2uB_n^*(1-t) + vB_n^* \left( \frac{1}{2} - t \right), \quad t \in \mathbb{R}.$$  

The identity holds for every function $f : [0, 1] \rightarrow \mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0, 1]$. $u, v \in \mathbb{Z}^+$ and the greatest common divisor of $u$ and $v$ is 1. The functions $B_k(t)$ are the Bernoulli polynomials, $B_k = B_k(0)$ are the Bernoulli numbers, and $B_k^*(t)$, $k \geq 0$, are periodic functions of period 1, related to the Bernoulli polynomials as

$$B_k^*(t) = B_k(t), \quad 0 \leq t < 1 \quad \text{and} \quad B_k^*(t+1) = B_k^*(t), \quad t \in \mathbb{R}.$$  

The Bernoulli polynomials $B_k(t)$, $k \geq 0$ are uniquely determined by the following identities

$$B_k'(t) = kB_{k-1}(t), \quad k \geq 1; \quad B_0(t) = 1, \quad B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geq 0.$$  

For some further details on the Bernoulli polynomials and the Bernoulli numbers see for example [1] or [2]. We have $B_0^*(t) = 1$ and $B_1^*(t)$ is a discontinuous function with a jump of $-1$ at each integer. It follows that $B_k(1) = B_k(0) = B_k$ for $k \geq 2$, so that $B_k^*(t)$ are continuous functions for $k \geq 2$. We get

$$B_k^*(t) = kB_k^*_{k-1}(t), \quad k \geq 1$$

for every $t \in \mathbb{R}$ when $k \geq 3$, and for every $t \in \mathbb{R} \setminus \mathbb{Z}$ when $k = 1, 2$.

In the proof of our main result we shall use the following result of N. Ujević ([8]):

**Lemma 1.** If $g, h, \Psi \in L_2(0, 1)$ and $\int_0^1 \Psi(t) dt = 0$ then we have

$$|S_\Psi(g, h)| \leq S_\Psi(g, g)^{1/2}S_\Psi(h, h)^{1/2}, \quad (2.4)$$

where

$$S_\Psi(g, h) = \int_0^1 g(t)h(t) dt - \int_0^1 g(t) dt \int_0^1 h(t) dt - \int_0^1 g(t) \Psi_0(t) dt \int_0^1 h(t) \Psi_0(t) dt$$

and $\Psi_0(t) = \Psi(t)/\|\Psi\|_2$.

**Theorem 4.** If $f : [0, 1] \rightarrow \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous function with $f^{(n)} \in L_2(0, 1)$ then we have

$$\left| \int_0^1 f(t) dt - D(u, v) + T_n(u, v) \right| \leq \frac{1}{2u+v} \left[ \frac{(-1)^{n-1}}{(2n)!} \left[ 4u^2 + v^2 - 4uv(1 - 2^1 - 2^n) \right] B_{2n} \right]^{1/2} K,$$  

where $D(u, v)$ and $T_n(u, v)$ are the remainder terms in the approximation of $f$ by the Simpson formula and the Grüss inequality, respectively.
where
\[
K^2 = \|f^{(n)}\|_2^2 - \left( \int_0^1 f^{(n)}(t) dt \right)^2 - \left( \int_0^1 f^{(n)}(t) \Psi_0(t) dt \right)^2. \quad (2.6)
\]

For \( n \) even
\[
\Psi(t) = \begin{cases} 
1, & t \in \left[0, \frac{1}{2}\right], \\
-1, & t \in \left(\frac{1}{2}, 1\right],
\end{cases}
\]

while for \( n \) odd we have
\[
\Psi(t) = \begin{cases} 
t + \frac{2^{1-n}u-2u+v}{2^{1-n}v-2^{2-n}u+8u-4v}, & t \in \left[0, \frac{1}{2}\right], \\
t + \frac{2^{1-n}(u-v)+3v-6u}{2^{1-n}v-2^{2-n}u+8u-4v}, & t \in \left(\frac{1}{2}, 1\right].
\end{cases}
\]

**Proof.** It is not difficult to verify that
\[
\int_0^1 G_n(t) dt = 0, \quad (2.7)
\]
\[
\int_0^1 \Psi(t) dt = 0, \quad (2.8)
\]
\[
\int_0^1 G_n(t) \Psi(t) dt = 0. \quad (2.9)
\]

From (2.1), (2.7) and (2.9) it follows that
\[
\int_0^1 f(t) dt - D(u,v) + T_n(u,v)
\]
\[
= \frac{1}{(2u+v)(n!)} \int_0^1 G_n(t) f^{(n)}(t) dt - \frac{1}{(2u+v)(n!)} \int_0^1 G_n(t) dt \int_0^1 f^{(n)}(t) dt
\]
\[
- \frac{1}{(2u+v)(n!)} \int_0^1 G_n(t) \Psi_0(t) dt \int_0^1 f^{(n)}(t) \Psi_0(t) dt
\]
\[
= \frac{1}{(2u+v)(n!)} S_{\Psi}(G_n, f^{(n)}). \quad (2.10)
\]

Using (2.10) and (2.4) we get
\[
\left| \int_0^1 f(t) dt - D(u,v) + T_n(u,v) \right| \leq \frac{1}{(2u+v)(n!)} S_{\Psi}(G_n, G_n)^{1/2} S_{\Psi}(f^{(n)}, f^{(n)})^{1/2}. \quad (2.11)
\]

We also have (see [6])
\[
S_{\Psi}(G_n, G_n) = \|G_n\|_2^2 - \left( \int_0^1 G_n(t) dt \right)^2 - \left( \int_0^1 G_n(t) \Psi_0(t) dt \right)^2
\]
\[
= (-1)^{n-1} \frac{(n!)^2}{(2n)!} \left[ 4u^2 + v^2 - 4uv(1 - 2^{1-2n}) \right] B_{2n}. \quad (2.12)
\]
and
\[ S_\Psi(f^{(n)}, f^{(n)}) = \|f^{(n)}\|^2_2 - \left( \int_0^1 f^{(n)}(t) dt \right)^2 - \left( \int_0^1 f^{(n)}(t) \Psi(t) dt \right)^2 = K^2. \] (2.13)

From (2.11)–(2.13) we easily get (2.5). □

REMARK 1. Function \( \Psi(t) \) can be any function which satisfies conditions
\[ \int_0^1 \Psi(t) dt = 0 \text{ and } \int_0^1 G_n(t) \Psi(t) dt = 0. \]
Since \( G_n(1-t) = (-1)^n G_n(t) \) (see [6]), for \( n \) we can even take function \( \Psi(t) \) such that \( \Psi(1-t) = -\Psi(t) \). For \( n \) odd we have to calculate \( \Psi(t) \) and without lost of generality in our theorem we take the form
\[ \Psi(t) = \begin{cases} t + a, & t \in \left[0, \frac{1}{2}\right], \\ t + b, & t \in \left(\frac{1}{2}, 1\right]. \end{cases} \]

REMARK 2. The inequality (2.5) achieves minimum of \( \left[\frac{(-1)^{n-1}}{(2n)!} 2^{-2n} B_{2n}\right]^{1/2} \) for \( u = 1 \) and \( v = 2 \) which is bitrapezoid formula (see [4]). For \( n = 1 \) it is \( 1/4\sqrt{3} \).

REMARK 3. For \( u = 1 \) and \( v = 4 \) in Theorem 4 we get Euler-Simpson formula (see [3]) and then we have
\[ \left| \int_0^1 f(t) dt - D(1,4) + T_n(1,4) \right| \leq \frac{1}{3} \left[ \frac{(-1)^{n-1}}{(2n)!} \left(1 + 2^{3-2n}\right) B_{2n} \right]^{1/2} K, \] (2.14)
where
\[ D(1,4) = \frac{1}{6} \left[ f(0) + 4f\left(\frac{1}{2}\right) + f(1) \right], \]
and
\[ T_n(1,4) = \sum_{k=2}^{\lfloor n/2 \rfloor} \frac{1}{3(2k)!} \left(1 - 2^{-2} 2^{k}\right) B_{2k} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right]. \]

For \( n \) even
\[ \Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2}, 1\right], \end{cases} \]
while for \( n \) odd we have
\[ \Psi(t) = \begin{cases} t + \frac{2^{-n+1}}{4(2^{-n-1})}, & t \in \left[0, \frac{1}{2}\right], \\ t + \frac{3(1-2^{-n})}{4(2^{-n-1})}, & t \in \left(\frac{1}{2}, 1\right]. \]

For \( n = 1, 2 \) and \( 3 \) in the inequality (2.14) we get inequalities (1.1), (1.3) and (1.6) respectively.
3. Estimations of the error for general dual Euler-Simpson formula

In the recent paper [7] the following identity, named the general dual Euler-Simpson formula, has been proved. For \( n \geq 1 \) and every \( t \in [0, 1] \) we have

\[
\int_0^1 f(t) dt = F(u, v) - T^D_n(u, v) + R_n(f) \tag{3.1}
\]

where

\[
F(u, v) = \frac{1}{2u - v} \left[ uf\left(\frac{1}{4}\right) - vf\left(\frac{1}{2}\right) + uf\left(\frac{3}{4}\right)\right],
\]

\( T^D_0(u, v) = 0 \) and

\[
T^D_m(u, v) = \frac{1}{2u - v} \sum_{k=1}^{m} \frac{\tilde{B}_k}{k!} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right],
\tag{3.2}
\]

for \( 1 \leq m \leq n \), while

\[
\tilde{B}_k = uB_k\left(\frac{1}{4}\right) - vB_k\left(\frac{1}{2}\right) + uB_k\left(\frac{3}{4}\right), \quad k \geq 1,
\]

\[
R_n(x) = \frac{1}{(2u - v)(n!)} \int_0^1 G^D_n(t) f^{(n)}(t) dt
\]

and

\[
G^D_n(t) = uB^*_n\left(\frac{1}{4} - t\right) - vB^*_n\left(\frac{1}{2} - t\right) + uB^*_n\left(\frac{3}{4} - t\right), \quad t \in \mathbb{R}.
\]

The identity holds for every function \( f : [0, 1] \to \mathbb{R} \) such that \( f^{(n-1)} \) is a continuous function of bounded variation on \([0, 1]\). \( u, v \in \mathbb{Z}^+ \), \( v < 2u \) and the greatest common divisor of \( u \) and \( v \) is 1.

**Theorem 5.** If \( f : [0, 1] \to \mathbb{R} \) is such that \( f^{(n-1)} \) is absolutely continuous function with \( f^{(n)} \in L^2(0, 1) \) then we have

\[
\left| \int_0^1 f(t) dt - F(u, v) + T^D_n(u, v) \right| \leq \frac{1}{2u - v} \left[ \frac{(-1)^{n-1}}{(2n)!} \left[ 2u^2 + v^2 - (2u^2 - uv \cdot 2^{2-2n}) (1 - 2^{1-2n}) \right] B_{2n} \right]^{1/2} K,
\tag{3.3}
\]

where

\[
K^2 = \|f^{(n)}\|^2_2 - \left( \int_0^1 f^{(n)}(t) dt \right)^2 - \left( \int_0^1 f^{(n)}(t) \Psi_0(t) dt \right)^2.
\tag{3.4}
\]

For \( n \) even

\[
\Psi(t) = \begin{cases} 
1, & t \in [0, \frac{1}{2}], \\
-1, & t \in (\frac{1}{2}, 1],
\end{cases}
\]
while for \( n \) odd we have

\[
\Psi(t) = \begin{cases} 
  t + \frac{2^{-n}u(1-2^{-n})+v}{4v(2^{-n-1}-1)}, & t \in [0, \frac{1}{2}], \\
  t + \frac{v(3-2^{-n+1})-2^{-n}u(1-2^{-n})}{4v(2^{-n-1}-1)}, & t \in (\frac{1}{2}, 1]. 
\end{cases}
\]

**Proof.** Similar as in Theorem 4. \( \square \)

**Remark 4.** For \( u = 2 \) and \( v = 1 \) in Theorem 5 we get the dual Euler-Simpson formula (see [5]) and then we have

\[
\left| \int_0^1 f(t)dt - F(2, 1) + T_n^D(2, 1) \right| \leq \frac{1}{3} \left[ \frac{(-1)^{n-1}}{(2n)!} \left[ 9 - (8 - 2^{3-2n})(1 - 2^{1-2n}) \right] B_{2n} \right]^{1/2} K, 
\]

(3.5)

where

\[
F(2, 1) = \frac{1}{3} \left[ 2f \left( \frac{1}{4} \right) - f \left( \frac{1}{2} \right) + 2f \left( \frac{3}{4} \right) \right],
\]

and

\[
T_n^D(2, 1) = \sum_{k=2}^{[n/2]} \frac{1}{3(2k)!} \left( 8 \cdot 2^{-4k} - 6 \cdot 2^{-2k} + 1 \right) B_{2k} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right].
\]

For \( n \) even

\[
\Psi(t) = \begin{cases} 
  1, & t \in [0, \frac{1}{2}], \\
  -1, & t \in (\frac{1}{2}, 1]. 
\end{cases}
\]

while for \( n \) odd we have

\[
\Psi(t) = \begin{cases} 
  t + \frac{2^{1-n}(1-2^{-n})+1}{4(2^{1-n}-1)}, & t \in [0, \frac{1}{2}], \\
  t + \frac{3-2^{2-2n}+2^{1-2n}}{4(2^{1-n}-1)}, & t \in (\frac{1}{2}, 1]. 
\end{cases}
\]

For \( n = 1, 2 \) and 3 we get inequalities

\[
\left| \frac{1}{3} \left[ 2f \left( \frac{1}{4} \right) - f \left( \frac{1}{2} \right) + 2f \left( \frac{3}{4} \right) \right] - \int_0^1 f(t)dt \right| \leq \frac{1}{3\sqrt{2}} K_1,
\]

\[
\left| \frac{1}{3} \left[ 2f \left( \frac{1}{4} \right) - f \left( \frac{1}{2} \right) + 2f \left( \frac{3}{4} \right) \right] - \int_0^1 f(t)dt \right| \leq \frac{\sqrt{13}}{48\sqrt{15}} K_2
\]

and

\[
\left| \frac{1}{3} \left[ 2f \left( \frac{1}{4} \right) - f \left( \frac{1}{2} \right) + 2f \left( \frac{3}{4} \right) \right] - \int_0^1 f(t)dt \right| \leq \frac{\sqrt{13}}{192\sqrt{70}} K_3
\]

respectively.
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