

CONVERGENCE RATE IN MULTIDIMENSIONAL IRREGULAR SAMPLING RESTORATION

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Abstract. New magnitude of truncation error upper bound and convergence rate are obtained for Whittaker–Kotel’nikov–Shannon (WKS) sampling restoration sum for Bernstein function class $B_{\pi,d}^q$, $q \geq 1$, $d \in \mathbb{N}$, when the sampled functions decay rate is unknown. The case of multidimensional irregular sampling is discussed.

1. Introduction

The classical WKS sampling theorem has been extended to the case of nonuniform sampling by numerous authors. For detailed information on the theory and its various applications, we refer to [3, 6].

Most known irregular sampling results deal with Paley–Wiener functions which have $L^2(\mathbb{R})$ restrictions on the real line. It seems that the best known nonuniform WKS sampling results for entire functions in L^p -spaces were given in [4, 5, 19]. However, there are no explicit truncation error upper bounds in multidimensional WKS reconstructions in open literature. Recently the authors derived multidimensional L^p -WKS sampling theorems with precise truncation error estimates, see [11, 12] for more details and discussions.

In this paper we use methods developed in [11, 12] to investigate multidimensional irregular sampling in L^p -spaces. New magnitude of truncation error upper bound and convergence rate are obtained.

2. Multidimensional Plancherel–Pólya inequality

To prove the main theorem we need the multidimensional analog of the Plancherel–Pólya inequality, see [11].

Various multidimensional Plancherel–Pólya inequalities can be found e.g. in Triebel’s book [18]; also, during last years several additional very far going generalizations of the multidimensional Plancherel–Pólya inequality were obtained, among others in [2] and [13]. Unfortunately, no explicit estimates of the Plancherel–Pólya constant appeared neither in these articles, nor in articles referenced therein including [14].

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In course to expose our estimate upon the multidimensional Plancherel–Pólya constant we make the following few conventions: **(i)** denote $\|\cdot\|_p$ the L_p -norm in finite case (while $\|\cdot\|_\infty \equiv \text{ess sup}|\cdot|$), and **(ii)** hereinafter $B_{\boldsymbol{\sigma},d}^r$, $r > 0$ denotes the Bernstein class [7] of d -variable entire functions of exponential type at most $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_d)$ coordinatewise whose restriction to \mathbb{R}^d is in $L^r(\mathbb{R}^d)$.

THEOREM 1. [11, Theorem 1] *Let $\mathfrak{T} = \{t_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^d}$, $t_{\mathbf{n}} = (t_{n_1}, \dots, t_{n_d})$ be real separated sequence, i.e.*

$$\inf_{n_\ell \neq m_\ell} |t_{n_\ell} - t_{m_\ell}| \geq \delta_\ell > 0, \quad \ell = \overline{1, d}.$$

Let $f \in B_{\boldsymbol{\sigma},d}^r$, $r \geq 1$. Then

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} |f(t_{\mathbf{n}})|^r \leq \mathfrak{B}_{d,r} \|f\|_r^r,$$

where

$$\mathfrak{B}_{d,r} = \left(\frac{8}{r\pi}\right)^d \prod_{\ell=1}^d \frac{e^{r\delta_\ell \sigma_\ell/2} - 1}{\sigma_\ell \delta_\ell^2}.$$

3. Multidimensional irregular sampling

In this section we introduce the multidimensional sampling theorem for $B_{\boldsymbol{\sigma},d}^r$ functional class. The theorem was proven in [12].

Let $\mathfrak{T} := \{t_{\mathbf{n}} = \mathbf{n} + h_{\mathbf{n}}, \text{mathfrak{h}} := (h_1, \dots, h_n), \mathbf{N} := (N_1, \dots, N_d) \in \mathbb{N}^d$, while

$$\mathfrak{J}_{\mathbf{x}} := \left\{ \mathbf{n} : \bigwedge_{j=1}^d (|x_j - n_j| \leq N_j) \right\}$$

and

$$S(\mathbf{x}, t_{\mathbf{n}}) = \prod_{j=1}^d \frac{G_{N_j}(x_j, x_j)}{G'_{N_j}(x_j, t_{n_j})(x_j - t_{n_j})}, \tag{1}$$

where $G'_N(x, t)$ denotes a derivative with respect to t , being

$$G_N(x, t) = (t - h_0) \text{sinc}(t) \prod_{\substack{|x-k| \leq N \\ k \neq 0}} \left(1 - \frac{h_k}{t - k}\right) \frac{k}{t_k}, \tag{2}$$

$$\text{sinc}(t) := \begin{cases} \frac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}$$

Denote $\mathbf{M} = (M_1, \dots, M_d)$, $\boldsymbol{\delta} = (\delta_1, \dots, \delta_d)$, $\tilde{M} := \max_{j=\overline{1,d}} M_j$. Assume that $t_{n_j} = n_j + h_{n_j}$, $|h_{n_j}| \leq M_j$, $j = \overline{1, d}$; for all $\mathbf{n} \in \mathfrak{J}_{\mathbf{x}}$.

THEOREM 2. [12, Theorem 2] *Let $f \in B_{\sigma,d}^q$, $q \geq 1$, $\sigma_j \leq \pi$ for all j , $\mathfrak{T} = \{t_n\}_{n \in \mathbb{Z}^d}$ be real separated sequence with*

$$\tilde{M} \leq \frac{1}{4} \quad \text{for } q = 1 \quad \text{and} \quad \tilde{M} < \frac{1}{4q} \quad \text{for } 1 < q < \infty. \tag{3}$$

Then the sampling expansion

$$f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} f(t_n) \prod_{j=1}^d \frac{G_{N_j}(x_j, x_j)}{G'_{N_j}(x_j, t_{n_j})(x_j - t_{n_j})}, \tag{4}$$

holds uniformly on each bounded \mathbf{x} -subset of \mathbb{R}^d . Moreover, the series in (4) converges absolutely too.

In this framework the sampling restoration procedure becomes of Lagrange–Yen type [1, 6, 15, 21].

For the general case of multidimensional irregular sampling with window canonical product sampling function $S(\mathbf{x}, t_n)$ we have time-jittered nodes outside \mathfrak{J}_x . Nonvanishing time-jitter \mathfrak{h} outside \mathfrak{J}_x leads to functions $G_N(x, t)$ given by formulae different from (2), see [4]. However, in irregular sampling applications we would like to approximate $f(\mathbf{x})$ using only its values from sample nodes t_n indexed by \mathfrak{J}_x . Therefore we can try to use the truncated to \mathfrak{J}_x sampling approximation sum

$$Y_{\mathfrak{J}_x}(f; \mathbf{x}) = \sum_{\mathbf{n} \in \mathfrak{J}_x} f(t_n) \prod_{j=1}^d \frac{G_{N_j}(x_j, x_j)}{G'_{N_j}(x_j, t_{n_j})(x_j - t_{n_j})} \tag{5}$$

with $G_N(x, t)$ (such that is given by (2)) even for arbitrary sample nodes outside \mathfrak{J}_x . Under such assumptions the *truncation error*

$$\|T_{\mathbf{N},d}(f; \mathbf{x})\|_\infty = \|f(\mathbf{x}) - Y_{\mathfrak{J}_x}(f; \mathbf{x})\|_\infty$$

coincides with truncation error for the case given by (1)–(4). One can see that $T_{\mathbf{N},d}(f; \mathbf{x})$ depends on nonvanishing \mathfrak{h} in \mathfrak{J}_x due to multiplicative form of (4) and (5). Therefore, the multidimensional sampling problems are more difficult than the one-dimensional ones.

4. Truncation error upper bounds

In this section we obtain universal truncation bounds for multidimensional irregular sampling restoration procedure.

The most frequently appearing estimate of the truncation error is of the form

$$\|T_{\mathfrak{J}}(f; \mathbf{x})\|_\infty \leq \left(\sum_{\mathbf{n} \in \mathbb{Z}^d \setminus \mathfrak{J}} |S(\mathbf{x}, t_n)|^p \right)^{1/p} \left(\sum_{\mathbf{n} \in \mathbb{Z}^d \setminus \mathfrak{J}} |f(t_n)|^q \right)^{1/q} =: A_p B_q, \tag{6}$$

p, q being a conjugated Hölder pair, i.e. $1/p + 1/q = 1$.

To obtain a class of truncation error upper bounds when the decay rate of the initial signal function is not known one operates with the straightforward $B_q \leq C_{f, \mathfrak{X}} \|f\|_q$ where $C_{f, \mathfrak{X}}$ is suitable absolute constant. Thus (6) becomes

$$\|T_{\mathfrak{J}}(f; \mathbf{x})\|_{\infty} \leq C_{f, \mathfrak{X}} A_p \|f\|_q.$$

We are interested in estimates for A_p such that vanish with $|\mathfrak{J}_{\mathbf{x}}| \rightarrow \infty$. Therefore, the obtained upper bounds really will be universal for wide classes of $f(\mathbf{x})$ and \mathfrak{X} .

We will use the following two-sided bounds for a ratio of gamma functions, see [16, 17, 20].

LEMMA 1. *Let $z > 0$, $b \in (0, 1)$. Then*

$$\frac{z}{(z+b)^{1-b}} \leq \frac{\Gamma(z+b)}{\Gamma(z)} \leq z^b. \tag{7}$$

THEOREM 3. *Let \tilde{M} satisfy (3), $f \in B_{\sigma, d}^q$, $q \geq 1$, $\sigma_j \leq \pi$ for all $j = \overline{1, d}$. Then we have*

$$\|T_{\mathbf{N}, d}(f, \mathbf{x})\|_{\infty} \leq K_{\delta}(\mathbf{N}, \mathbf{M}) \cdot \|f\|_q \tag{8}$$

where

$$K_{\delta}(\mathbf{N}, \mathbf{M}) = \left(\frac{8}{q\pi^2}\right)^{d/q} \prod_{j=1}^d \frac{(e^{q\pi\delta_j/2} - 1)^{1/q}}{\delta_j^{2/q}} \left(\sum_{k=1}^d \left\{C_1(N_k, M_k) \times \prod_{\substack{j=1 \\ j \neq k}}^d (C_1(N_j, M_j) + C_2(M_j, \delta)C_3(N_j, M_j))\right\}\right)^{1/p}, \tag{9}$$

and

$$C_1(N, M) := \frac{2^{Mp+2p+1}\pi^p(M+1/2)^p}{3^p\Gamma^{2p}(M+1/2)} \left(1 + \frac{N}{p-1}\right) \left(\frac{(1+N)^M(N-1/2)}{N(N-M-1/2)}\right)^p (N+3/2)^{2Mp};$$

$$C_2(M, \delta) := \left(\frac{3}{4M}\right)^p \left(\frac{\pi}{2}\right)^{2Mp-p} \left(\frac{e}{M\sqrt{\delta}}\right)^{4Mp};$$

$$C_3(N, M) := 2(M+3/2)^{4pM-p} + \frac{2(N+M+3/2)^{4pM-p+1} - 2(M+3/2)^{4pM-p+1}}{4pM-p+1}.$$

Proof. As we mentioned earlier, the function $Y_{\mathfrak{J}_{\mathbf{x}}}(f; \mathbf{x})$ does not depend on samples in $\mathfrak{X} \setminus \{t_{\mathbf{n}} : \mathbf{n} \in \mathfrak{J}_{\mathbf{x}}\}$ and we assume that outside $\mathfrak{J}_{\mathbf{x}}$ it is $\mathfrak{h} \equiv 0$. Therefore, the structure of $\{t_{n_j} : n_j \notin \mathfrak{J}_{x_j}\}$ becomes uniform $t_{n_j} \equiv n_j$. In this case Theorem 2 guarantees that $f(\mathbf{x})$ admits the representation (4), and the so evaluated model (6) gives us

$$|T_{\mathbf{N}, d}(f; \mathbf{x})| \leq \left(\sum_{\mathbf{n} \in \mathbb{Z}^d \setminus \mathfrak{J}_{\mathbf{x}}} \prod_{j=1}^d \left| \frac{G_{N_j}(x_j, x_j)}{G'_{N_j}(x_j, t_{n_j})(x_j - t_{n_j})} \right|^p \right)^{1/p} \left(\sum_{\mathbf{n} \in \mathbb{Z}^d \setminus \mathfrak{J}_{\mathbf{x}}} |f(\mathbf{n})|^q\right)^{1/q} =: A_p \cdot B_q.$$

The multiplicative structure of $S(\mathbf{x}, t_n)$ enables to estimate A_p in the following way

$$\begin{aligned}
 A_p^p &\leq \sum_{k=1}^d \sum_{n_k \in \mathbb{Z} \setminus \mathfrak{J}_{x_k}} \left| \frac{G_{N_k}(x_k, x_k)}{G'_{N_k}(x_k, n_k)(x_k - n_k)} \right|^p \prod_{\substack{j=1 \\ j \neq k}}^d \sum_{n_j \in \mathbb{Z}} \left| \frac{G_{N_j}(x_j, x_j)}{G'_{N_j}(x_j, t_{n_j})(x_j - t_{n_j})} \right|^p \\
 &= \sum_{k=1}^d \sum_{n_k \in \mathbb{Z} \setminus \mathfrak{J}_{x_k}} \left| \frac{G_{N_k}(x_k, x_k)}{G'_{N_k}(x_k, n_k)(x_k - n_k)} \right|^p \prod_{j=1}^d \left(\sum_{\substack{n_j \in \mathbb{Z} \setminus \mathfrak{J}_{x_j} \\ j \neq k}} \left| \frac{G_{N_j}(x_j, x_j)}{G'_{N_j}(x_j, n_j)(x_j - n_j)} \right|^p \right. \\
 &\quad \left. + \sum_{n_j \in \mathfrak{J}_{x_j}} \left| \frac{G_{N_j}(x_j, x_j)}{G'_{N_j}(x_j, t_{n_j})(x_j - t_{n_j})} \right|^p \right). \tag{10}
 \end{aligned}$$

Let us estimate $\sum_{n \in \mathbb{Z} \setminus \mathfrak{J}_x} \left| \frac{G_N(x, x)}{G'_N(x, n)(x - n)} \right|^p$. Note, that due to our assumptions

$$|\psi_N(n, x)| := \left| \frac{G_N(x, x)}{G'_N(x, t_n)(x - t_n)} \right| = \left| \frac{\sin(\pi x)}{\pi(x - n)} \prod_{j \in \mathfrak{J}_x} \frac{(t_j - x)(j - n)}{(t_j - n)(j - x)} \right| \quad n \in \mathbb{Z} \setminus \mathfrak{J}_x.$$

Hence

$$|\psi_N(n, x)| = \left| \operatorname{sinc}(x - j_x) \frac{(t_{j_x} - x)(j_x - n)}{(x - n)(t_{j_x} - n)} \prod_{\substack{|j-x| \leq N \\ j \neq j_x}} \frac{(t_j - x)(j - n)}{(t_j - n)(j - x)} \right|$$

where j_x denotes the index closest to x , i.e. $j_x - 0.5 \leq x < j_x + 0.5$.

Due to $|h_{j_x}| \leq M$ we have

$$\left| \operatorname{sinc}(x - j_x) \frac{t_{j_x} - x}{x - n} \right| \leq \frac{M + 1/2}{|x - n|} \tag{11}$$

and

$$\left| \frac{j_x - n}{t_{j_x} - n} \right| \leq 1 + \frac{M}{|x - n| - M - 1/2} \leq 1 + \frac{M}{N - M - 1/2}. \tag{12}$$

Due to $|h_{j_x}| \leq M$ and lemma 1 one concludes

$$\begin{aligned}
 \prod_{\substack{|j-x| \leq N \\ j \neq j_x}} \left| \frac{t_j - x}{j - x} \right| &\leq \prod_{\substack{|j-x| \leq N \\ j \neq j_x}} \frac{|j - x| + M}{|j - x|} \leq \left(\prod_{j=0}^N \frac{j + 1/2 + M}{j + 1/2} \right)^2 \\
 &= \left(\frac{\Gamma(1/2)}{\Gamma(M + 1/2)} \right)^2 \left(\frac{\Gamma(N + 1 + 1/2 + M)}{\Gamma(N + 1 + 1/2)} \right)^2 \\
 &\leq \frac{\pi}{\Gamma^2(M + 1/2)} (N + 3/2)^{2M}. \tag{13}
 \end{aligned}$$

Let $k(n, x) := \max(|j_x - n| - N, 1)$. Then by lemma 1 and $|h_{j_x}| \leq M \leq 1/4$ we have

$$\begin{aligned}
 \prod_{\substack{|j-x| \leq N \\ j \neq j_x}} \left| \frac{j-n}{t_j-n} \right| &\leq \prod_{\substack{|j-x| \leq N \\ j \neq j_x}} \frac{|j-n|}{|j-n|-M} = \prod_{k=k(n,x)}^{k(n,x)+2N} \frac{k}{k-M} \\
 &= \frac{\Gamma(k(n,x) + 2N + 1)}{\Gamma(k(n,x))} \frac{\Gamma(k(n,x) - M)}{\Gamma(k(n,x) + 2N + 1 - M)} \\
 &\leq (k(n,x) + 2N + 1 - M)^M \frac{k(n,x)}{(k(n,x) - M)(k(n,x))^M} \\
 &\leq \frac{4}{3} \left(\frac{k(n,x) + 2N + 1}{k(n,x)} \right)^M. \tag{14}
 \end{aligned}$$

Collecting all estimates (11)–(14), we deduce

$$|\psi_N(n, x)| \leq \frac{1}{|x-n|} \left(\frac{k(n,x) + 2N + 1}{k(n,x)} \right)^M \frac{4\pi(M+1/2)}{3\Gamma^2(M+1/2)} \frac{N-1/2}{N-M-1/2} (N+3/2)^{2M}$$

and hence

$$\begin{aligned}
 \sum_{n \in \mathbb{Z} \setminus \mathfrak{J}_x} \left| \frac{G_N(x, x)}{G'_N(x, n)(x-n)} \right|^p &= \sum_{n \in \mathbb{Z} \setminus \mathfrak{J}_x} |\psi_N(n, x)|^p \leq \sum_{n \in \mathbb{Z} \setminus \mathfrak{J}_x} \frac{1}{|x-n|^p} \left(\frac{k(n,x) + 2N + 1}{k(n,x)} \right)^{Mp} \\
 &\quad \times \left(\frac{4\pi(M+1/2)}{3\Gamma^2(M+1/2)} \frac{N-1/2}{N-M-1/2} \right)^p (N+3/2)^{2Mp}.
 \end{aligned}$$

Thus, we proceed evaluating

$$\begin{aligned}
 \sum_{n \in \mathbb{Z} \setminus \mathfrak{J}_x} \frac{1}{|x-n|^p} \left(\frac{k(n,x) + 2N + 1}{k(n,x)} \right)^{Mp} &= \sum_{n \in \mathbb{Z} \setminus \mathfrak{J}_x} \frac{1}{|x-n|^p} \left(1 + \frac{2N+1}{k(n,x)} \right)^{Mp} \\
 &\leq \frac{2(2+2N)^{Mp}}{N^p} + \int_N^\infty \frac{2}{t^p} \left(1 + \frac{2N+1}{t-N+1} \right)^{Mp} dt \\
 &= \frac{2(2+2N)^{Mp}}{N^p} + 2(2N+1)^{Mp} \int_N^\infty \frac{1}{t^p} \left(\frac{1}{2N+1} + \frac{1}{t-N+1} \right)^{Mp} dt \\
 &\leq \frac{2(2+2N)^{Mp}}{N^p} + \frac{2(2N+1)^{Mp}}{(p-1)N^{p-1}} \left(1 + \frac{1}{2N+1} \right)^{Mp} \\
 &= \frac{2^{Mp+1}(1+N)^{Mp}}{N^p} \left(1 + \frac{N}{p-1} \right),
 \end{aligned}$$

such that gives

$$\sum_{n \in \mathbb{Z} \setminus \mathfrak{J}_x} \left| \frac{G_N(x, x)}{G'_N(x, n)(x-n)} \right|^p \leq C_1(N, M).$$

Now, let us evaluate $\sum_{n \in \mathfrak{J}_x} |\psi_N(n, x)|^p$, the second addend in (10).

We can rewrite the function $G_N(x, t)$ into

$$G_N(x, t) = (t - h_0) \prod_{n=1}^{\infty} \left(1 - \frac{t}{\tilde{t}_n}\right) \left(1 - \frac{t}{\tilde{t}_{-n}}\right),$$

where

$$\tilde{t}_n = \begin{cases} t_n, & \text{if } |x - n| \leq N \\ n, & \text{else.} \end{cases}$$

Hence all results derived in [19] are valid for our function $G_N(x, t)$. By [19, Lemma 1.4.4 (a)] and [19, inequality (3.7)] we obtain

$$|\psi_N(n, x)| \leq \frac{3}{4M} \left(\frac{\pi}{2}\right)^{2M-1} \left(\frac{e}{M\sqrt{\delta}}\right)^{4M} (|x - t_n| + M + 3/2)^{4M-1}.$$

Therefore

$$\begin{aligned} \sum_{n \in \mathfrak{J}_x} |\psi_N(n, x)|^p &\leq \left(\frac{3}{4M}\right)^p \left(\frac{\pi}{2}\right)^{2Mp-p} \left(\frac{e}{M\sqrt{\delta}}\right)^{4Mp} \sum_{n \in \mathfrak{J}_x} (|x - t_n| + M + 3/2)^{4pM-p} \\ &= C_2(M, \delta) \sum_{n \in \mathfrak{J}_x} (|x - t_n| + M + 3/2)^{4pM-p}. \end{aligned}$$

Finally for M satisfying (3) we have the estimate

$$\begin{aligned} \sum_{n \in \mathfrak{J}_x} (|x - t_n| + M + 3/2)^{4pM-p} &\leq 2 \left((M + 3/2)^{4pM-p} + \int_0^N (t + M + 3/2)^{4pM-p} dt \right) \\ &= 2 \left((M + 3/2)^{4pM-p} + \frac{(N + M + 3/2)^{4pM-p+1} - (M + 3/2)^{4pM-p+1}}{4pM - p + 1} \right) = C_3(N, M), \end{aligned}$$

such that gives

$$\sum_{n \in \mathfrak{J}_x} \left| \frac{G_N(x, x)}{G'_N(x, t_n)(x - t_n)} \right|^p \leq C_2(M, \delta) C_3(N, M).$$

Therefore, by (10)

$$A_p^d \leq \sum_{k=1}^d \left(C_1(N_k, M_k) \prod_{\substack{j=1 \\ j \neq k}}^d \left(C_1(N_j, M_j) + C_2(M_j, \delta) C_3(N_j, M_j) \right) \right). \quad (15)$$

To estimate B_q we use Theorem 1 with $\sigma_1 = \dots = \sigma_d = \pi$. This results in

$$B_q^d \leq \mathfrak{B}_{d,q} = \left(\frac{8}{q\pi^2}\right)^d \prod_{j=1}^d \frac{e^{q\pi\delta_j/2} - 1}{\delta_j^2}.$$

Now, collecting all these involved estimates, we arrive at (8) and (9).

REMARK 1. Theorem 3 gives new truncation error upper bounds and the method to obtain them, compare with results in [12]. However, it has to be pointed out that there are many different bounds on the so-called Gautschi–Kershaw ratio of two gamma functions [17]. Applying instead of the Lemma 1 other estimates we obtain a set of truncation error upper bounds. We used (7) since its simplicity and elegance.

Denote here, and in what follows $\underline{\delta} := \min_{j=\overline{1,d}} \delta_j$, $\overline{\delta} := \max_{j=\overline{1,d}} \delta_j$.

COROLLARY 3.1. *Suppose that the conditions of Theorem 3 are satisfied. Then, we have*

$$\|T_{N,d}(f; \mathbf{x})\|_\infty \leq \tilde{K}_{\underline{\delta}, \overline{\delta}}(\mathbf{N}, \tilde{M}) \|f\|_q$$

where

$$\begin{aligned} \tilde{K}_{\underline{\delta}, \overline{\delta}}(\mathbf{N}, \tilde{M}) &= \left(\frac{8}{q\pi^2}\right)^{d/q} \left(\sum_{k=1}^d \left\{ C_1(N_k, \tilde{M}) \prod_{\substack{j=1 \\ j \neq k}}^d \left(C_1(N_j, \tilde{M}) \right. \right. \right. \\ &\quad \left. \left. \left. + C_2(\tilde{M}, \underline{\delta}) C_3(N_j, \tilde{M}) \right) \right\} \right)^{1/p} \left(\max \left\{ \frac{e^{q\pi\underline{\delta}/2} - 1}{\underline{\delta}^2}, \frac{e^{q\pi\overline{\delta}/2} - 1}{\overline{\delta}^2} \right\} \right)^{d/q}. \end{aligned} \tag{16}$$

Proof. If we make use of the estimate (15) for A_p with \tilde{M} instead of all M_j and $\underline{\delta}$ instead of all other δ_j , \mathfrak{T} could contain some additional \mathbf{t}_n which might causes only increasing A_p .

It was shown in [12] that

$$\mathfrak{B}_d^{1/d} \leq \max \left\{ \frac{8(e^{q\pi\underline{\delta}/2} - 1)}{q\pi^2 \underline{\delta}^2}, \frac{8(e^{q\pi\overline{\delta}/2} - 1)}{q\pi^2 \overline{\delta}^2} \right\}.$$

Collecting the involved estimates, we arrive at (16).

THEOREM 4. *Suppose that the conditions of Theorem 3 are satisfied. Let $\tilde{N} = \min_{j=\overline{1,d}} N_j \rightarrow \infty$ in such way that $\max_{k,j=\overline{1,d}} N_j/N_k = \mathcal{O}(1)$. Then it holds*

$$\|T_{N,d}(f; \mathbf{x})\|_\infty = \mathcal{O}(\tilde{N}^{3\tilde{M}p-p+1}) \rightarrow 0.$$

Proof. According to definitions of C_j , $j = \overline{1,3}$ from Theorem 3, letting $\tilde{N} \rightarrow \infty$, we get

$$C_1(N, M) = \mathcal{O}(N^{3Mp-p+1}); \quad C_2(M, \delta) = \mathcal{O}(1); \quad C_3(N, M) = \mathcal{O}(1 + N^{4Mp-p+1}).$$

Having in mind these facts by (16) we deduce

$$\tilde{K}_{\underline{\delta}, \overline{\delta}}^p(\mathbf{N}, \tilde{M}) \sim \sum_{k=1}^d \left\{ \mathcal{O}(N_k^{3\tilde{M}p-p+1}) \prod_{\substack{j=1 \\ j \neq k}}^d \left(\mathcal{O}(N_j^{3\tilde{M}p-p+1}) + \mathcal{O}(1 + N_j^{4\tilde{M}p-p+1}) \right) \right\}$$

$$\sim \mathcal{O}(\tilde{N}^{3\tilde{M}p-p+1}) \prod_{\substack{j=1 \\ N_j \neq \tilde{N}}}^d \left(\mathcal{O}(N_j^{3\tilde{M}p-p+1}) + \mathcal{O}(1) \right) \sim \mathcal{O}(\tilde{N}^{3\tilde{M}p-p+1}),$$

when $\tilde{N} \rightarrow \infty$ if \tilde{M} satisfies (3). The proof is complete.

REMARK 2. Theorem 4 improves the results on convergence rates derived in [12]. Namely, by specifying the sampling size numbers \tilde{N} and the irregular sampling deviation bound \tilde{M} , involving the Gautschi–Kershaw inequality for the ratio of Gamma functions (Lemma 1), and Voss’s upper bound on $\psi_N(n, x)$, we

- relax the conditions upon \tilde{M} , with respect to the earlier condition

$$\tilde{M} < \frac{1}{q(4d - 1)};$$

- obtain a better rate of convergence, with respect to the earlier one

$$\mathcal{O}(\tilde{N}^{3\tilde{M}p-p+1} \cdot N^{4\tilde{M}(d-1)})$$

appearing in [12, Corollary 3.2].

5. Final remarks

Methods proposed by authors give an opportunity to obtain approximation error estimates for wide functional classes without strong assumptions on functions/signals decay rate behaviour.

In this article new magnitude of truncation error upper bounds and rates of convergence were obtained in $\|\cdot\|_\infty$ -norm sampling theorem; the case of multidimensional irregular sampling was considered.

All presented results and the numerical simulations such that illustrate the achieved results open few new interesting and important problems:

1. To obtain sharp estimates in Theorem 3 (for uniform sampling and $p = 2$ such sharp estimates were derived in [10]);
2. To obtain the best possible rate of convergence in Theorem 4;
3. To apply the obtained results to irregular sampling restoration for random fields, see [8, 9].

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