Journal of Mathematical Inequalities
Volume 3, Number 4 (2009), 577–589

MAPPINGS CONNECTED WITH HERMITE–HADAMARD INEQUALITIES FOR SUPERQUADRATIC FUNCTIONS

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. Using some characterizations of superquadratic functions we obtain some results on two mappings H and F connected with Hermite-Hadamard inequality. Our results generalize corresponding results for convex functions. Especially, when the superquadratic function is convex at the same time, then our results represent also the refinements of those results.

1. Introduction

Let $\varphi : I \to \mathbb{R}$ be a convex function on interval $I \subseteq \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b] \subseteq \mathbb{R}$ such that $g([a,b]) \subseteq I$, and let $p : [a,b] \to \mathbb{R}$ be a positive integrable function.

In [7] Y. J. Cho, M. Matić and J. Pečarić considered the following two mappings $H,F : [0,1] \to \mathbb{R}$ defined as

$$H(t) = \frac{1}{P}\int_a^b p(x)\,\varphi(tg(x)+(1-t)\overline{g})\,dx \tag{1.1}$$

and

$$F(t) = \frac{1}{P^2}\int_a^b\int_a^b p(x)\,p(y)\,\varphi(tg(x)+(1-t)g(y))\,dxdy \tag{1.2}$$

where

$$P = \int_a^b p(x)\,dx, \qquad \overline{g} = \frac{1}{P}\int_a^b p(x)\,g(x)\,dx. \tag{1.3}$$

They established convexity and the following estimates for the values of these two mappings:

$$\inf_{t\in[0,1]} H(t) = H(0) \qquad \text{and} \qquad \sup_{t\in[0,1]} H(t) = H(1) \tag{a}$$

$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) \qquad \text{and} \qquad \sup_{t\in[0,1]} F(t) = F(0) = F(1) \tag{b}$$

$$F(t) \geqslant \max\{H(t), H(1-t)\}, \qquad \forall t \in [0,1]. \tag{c}$$

Mathematics subject classification (2000): 26D15.
Keywords and phrases: Hermite-Hadamard inequality, convex functions, superquadratic functions.

Paper JMI-03-56

In this paper we established analogies of the above results for superquadratic functions, a new class of functions recently introduced by S. Abramovich, G. Jameson and G. Sinnamon in their papers [2] and [3]. In the special case of nonnegative superquadratic function φ, these analogies represent the refinements of the above results.

First we quote definition of this class of functions, some characterizations and basic properties used to obtain the results in this paper.

DEFINITION 1. [2] A function $\varphi:[0,\infty)\to\mathbb{R}$ is superquadratic provided that for all $x\geqslant 0$ there exists a constant $C(x)\in\mathbb{R}$ such that

$$\varphi(y)\geqslant\varphi(x)+C(x)(y-x)+\varphi(|y-x|) \tag{1.4}$$

for all $y\geqslant 0$.

Jensen's inequality for superquadratic function was first proved in [2], and here we use its following version:

$$\varphi\left(\frac{1}{P}\int_a^b p(x)f(x)\,dx\right)\leqslant\frac{1}{P}\int_a^b p(x)\left[\varphi(f(x))-\varphi\left(\left|f(x)-\overline{f}\right|\right)\right]dx, \tag{1.5}$$

As it was shown in [6], the function φ is superquadratic if and only if the following inequality

$$\begin{aligned}\varphi(\lambda y_1+(1-\lambda)y_2)\leqslant&\lambda\varphi(y_1)+(1-\lambda)\varphi(y_2)\\&-\lambda\varphi((1-\lambda)|y_1-y_2|)-(1-\lambda)\varphi(\lambda|y_1-y_2|)\end{aligned} \tag{1.6}$$

holds for all $y_1,y_2\geqslant 0$ and $\lambda\in[0,1]$.

Superquadratic function φ has the following properties: $\varphi(0)\leqslant 0$ and
(S1) if $\varphi(0)=\varphi'(0)=0$, then $C(x)=\varphi'(x)$ whenever φ is differentiable at $x>0$;
(S2) if $\varphi\geqslant 0$, then φ is convex and $\varphi(0)=\varphi'(0)=0$.

In [5] authors proved the following pair of inequalities, that is called Hermite-Hadamard inequalities for superquadratic functions:

THEOREM 1. *Let $\varphi:[0,\infty)\to\mathbb{R}$ be a continuous superquadratic function, and $0\leqslant a<b$, then*

$$\varphi\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b\varphi\left(\left|t-\frac{a+b}{2}\right|\right)dt\leqslant\frac{1}{b-a}\int_a^b\varphi(t)\,dt \tag{1.7}$$

$$\leqslant\frac{\varphi(a)+\varphi(b)}{2}-\frac{1}{(b-a)^2}\int_a^b[(b-t)\varphi(t-a)+(t-a)\varphi(b-t)]\,dt. \tag{1.8}$$

This pair of inequalities represents an analogue (and a refinement in the special case when $\varphi\geqslant 0$) of the well-known Hermite-Hadamard inequalities:

$$\varphi\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b\varphi(t)\,dt\leqslant\frac{\varphi(a)+\varphi(b)}{2}, \tag{1.9}$$

which hold for any convex function $\varphi : I \to \mathbb{R}$ and $a, b \in I$.

To see some examples of superquadratic functions and the construction of these functions in one and in several variables, readers are refered to [1] and [2].

In the second section we obtain the results for superquadratic function analogous to some results of Cho, Matić and Pečarić in [7] related to convex functions and its refinements in the case of nonnegative superquadratic function.

In the third section we give generalizations of results by M. Akkouchi [4] and obtain new inequalities for some special means. All our results in this section are established for differentiable superquadratic function φ and represent the refinements of Akkouchi's results in the special case when $\varphi \geqslant 0$ and therefore a convex function.

2. The results on the mappings H and F

Now we consider the case when $\varphi : [0,\infty) \to \mathbb{R}$ is a continuous superquadratic function, and g is a nonnegative continuous function on $[a,b]$. In this case we obtain the following results:

THEOREM 2. *Let $\varphi : [0,\infty) \to \mathbb{R}$ be a continuous superquadratic function, $g : [a,b] \to [0,\infty)$ be a continuous function and $p : [a,b] \to (0,\infty)$ be an integrable function. Then for the function H defined by* (1.1) *and for all $t \in [0,1]$ the following inequalities:*

$$H(0) + \frac{1}{P}\int_a^b p(x)\,\varphi(t\,|g(x)-\overline{g}|)\,dx \leqslant H(t) \tag{2.1}$$

$$\begin{aligned}&\leqslant H(1) - (1-t)\frac{1}{P}\int_a^b p(x)\,\varphi(|g(x)-\overline{g}|)\,dx \\ &\quad - t\frac{1}{P}\int_a^b p(x)\,\varphi((1-t)\,|g(x)-\overline{g}|)\,dx - (1-t)\frac{1}{P}\int_a^b p(x)\,\varphi(t\,|g(x)-\overline{g}|)\,dx.\end{aligned} \tag{2.2}$$

hold, where P and $\overline{g}$ are defined by (1.3).

Proof. Applying inequality (1.6) to the superquadratic function φ, for $\lambda = t \in [0,1]$, $y_1 = g(x)$ and $y_2 = \overline{g}$, then applying (1.5) for the function g instead of f we get

$$\begin{aligned}H(t) &\leqslant t\frac{1}{P}\int_a^b p(x)\,\varphi(g(x))\,dx + (1-t)\,\varphi\left(\frac{1}{P}\int_a^b p(x)\,g(x)\,dx\right) \\ &\quad - t\frac{1}{P}\int_a^b p(x)\,\varphi((1-t)\,|g(x)-\overline{g}|)\,dx - (1-t)\frac{1}{P}\int_a^b p(x)\,\varphi(t\,|g(x)-\overline{g}|)\,dx. \\ &\leqslant \frac{1}{P}\int_a^b p(x)\,\varphi(g(x))\,dx - (1-t)\frac{1}{P}\int_a^b p(x)\,\varphi(|g(x)-\overline{g}|)\,dx \\ &\quad - t\frac{1}{P}\int_a^b p(x)\,\varphi((1-t)\,|g(x)-\overline{g}|)\,dx - (1-t)\frac{1}{P}\int_a^b p(x)\,\varphi(t\,|g(x)-\overline{g}|)\,dx.\end{aligned}$$

Hence, inequality (2.2) holds for all $t \in [0,1]$.

On the other hand, applying (1.5) with the substitution

$$f(x) \to tg(x) + (1-t)\overline{g},$$

for each $t \in [0,1]$, we have

$$\begin{aligned} H(t) &\geqslant \varphi\left(t\frac{1}{P}\int_a^b p(x)g(x)\,dx + (1-t)\overline{g}\right) + \frac{1}{P}\int_a^b p(x)\varphi(t|g(x)-\overline{g}|)\,dx \\ &= \varphi(\overline{g}) + \frac{1}{P}\int_a^b p(x)\varphi(t|g(x)-\overline{g}|)\,dx. \end{aligned}$$

It means that (2.1) holds for all $t \in [0,1]$.

REMARK 1. If the superquadratic function φ is also nonnegative, then by property $(S2)$ φ is also convex. In this case, since p is positive, all the integrals in (2.1) and (2.2), which are subtracted from $H(1)$ or added to $H(0)$, are nonnegative. Therefore these two inequalities represent refinements of the inequalities

$$H(0) \leqslant H(t) \leqslant H(1), \qquad t \in [0,1],$$

which hold for a convex function φ (see (a) above and [7]).

In the rest of the paper, we use the following notations and definitions:

The generalized logarithmic mean

$$L_q(a,b) = \left[\frac{b^{q+1} - a^{q+1}}{(b-a)(q+1)}\right]^{\frac{1}{q}}, \quad a \neq b, \quad a,b > 0, \quad q \in \mathbb{R} \setminus \{-1,0\}. \tag{2.3}$$

The arithmetic mean

$$A(a,b) = \frac{a+b}{2}, \quad a,b \geqslant 0. \tag{2.4}$$

Applying Theorem 2 and (1.8) we get the following result which refines one of Dragomir's results in [8].

EXAMPLE 1. The function $\varphi(x) = x^q$ $(x \geqslant 0)$ is superquadratic for $q \geqslant 2$ ([2]). Applying this function to Theorem 2 for the special case $p(x) \equiv 1$ and $g(x) = x$ $(x \in [a,b] \subseteq [0,\infty))$, then applying inequality (1.8) to the same function $\varphi(x) = x^q$, using (2.3) and (2.4), we get that

$$\begin{aligned} &A^q(a,b) + \frac{1}{q+1}\left[\frac{t(b-a)}{2}\right]^q \leqslant L_q^q\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right) \\ &\leqslant L_q^q(a,b) - \frac{1}{q+1}\left(\frac{b-a}{2}\right)^q (1-t)\left[1 + t(1-t)^{q-1} + t^q\right] \\ &\leqslant A(a^q,b^q) - \frac{2(b-a)^q}{(q+1)(q+2)} - \frac{1}{q+1}\left(\frac{b-a}{2}\right)^q (1-t)\left[1 + t(1-t)^{q-1} + t^q\right] \end{aligned} \tag{2.5}$$

holds for all $t \in (0,1]$. In particular, for $t = \frac{1}{2}$ we get

$$\begin{aligned} A^q(a,b) + \frac{1}{q+1}\left(\frac{b-a}{4}\right)^q &\leqslant L_q^q\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right) \\ &\leqslant L_q^q(a,b) - \frac{1}{2(q+1)}\left(\frac{b-a}{2}\right)^q\left[1+\frac{1}{2^{q-1}}\right] \\ &\leqslant A(a^q,b^q) - \frac{2(b-a)^q}{(q+1)(q+2)} - \frac{1}{2(q+1)}\left(\frac{b-a}{2}\right)^q\left[1+\frac{1}{2^{q-1}}\right]. \end{aligned} \tag{2.6}$$

REMARK 2. Since $q \geqslant 2$, $t \in (0,1]$ and $b > a \geqslant 0$, all the additional terms in (2.5) (added to or subtracted from means) are nonnegative, so inequalities (2.5) represent refinements of the inequalities

$$A^q(a,b) \leqslant L_q^q\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right) \leqslant L_q^q(a,b) \leqslant A(a^q,b^q),$$

proved by S. S. Dragomir in [8].

Now we establish results for the function F defined by (1.2) for a superquadratic and continuous function φ.

THEOREM 3. *Let $\varphi : [0,\infty) \to \mathbb{R}$ be a continuous superquadratic function, $g : [a,b] \to [0,\infty)$ be a continuous function and $p : [a,b] \to (0,\infty)$ be an integrable function. Then, for the function F defined by (1.2) and for all $t \in [0,1]$ the following inequalities:*

$$F\left(\frac{1}{2}\right) + \frac{1}{2P^2}\int_a^b\int_a^b p(x)p(y)\varphi\left(\frac{|(2t-1)(g(x)-g(y))|}{2}\right)dxdy \leqslant F(t) \tag{2.7}$$

$$\begin{aligned} &\leqslant F(1) - t\frac{1}{P^2}\int_a^b\int_a^b p(x)p(y)\varphi((1-t)|g(x)-g(y)|)\,dxdy \\ &\quad -(1-t)\frac{1}{P^2}\int_a^b\int_a^b p(x)p(y)\varphi(t|g(x)-g(y)|)\,dxdy, \end{aligned} \tag{2.8}$$

hold, where P and $\overline{g}$ are defined by (1.3).

Proof. Analogously to the proof of the previous theorem, applying inequality (1.6) to a superquadratic function φ, for $\lambda = t$, $y_1 = g(x)$ and $y_2 = g(y)$ we get inequality (2.8).

Furthermore, applying inequality (1.6) to $\lambda = 1/2$, $y_1 = tg(x) + (1-t)g(y)$ and $y_2 = tg(y) + (1-t)g(x)$ to a superquadratic function φ and to a nonnegative function g which is continuous on $[a,b] \subseteq [0,\infty)$, we get

$$\begin{aligned} \varphi\left(\frac{g(x)+g(y)}{2}\right) \leqslant \frac{1}{2}\left[\varphi(tg(x)+(1-t)g(y)) + \varphi(g(y)+(1-t)g(x))\right] \\ - \varphi\left(\frac{|(2t-1)(g(x)-g(y))|}{2}\right). \end{aligned}$$

Multiplying the above inequality by $p(x)>0$ and $p(y)>0$, then integrating it over $[a,b]\times[a,b]$ and dividing it by P^2, we obtain (2.7).

It is important to remark that in (2.8) we can replace $F(0)$ with $F(1)$, because $F(0)=F(1)$.

REMARK 3. In particular in the case of nonnegative superquadratic function φ, all the integrals in (2.7) and (2.8) are nonnegative. Since in this case the function φ is convex, then for all $t\in[0,1]$ obtained inequalities represent refinements of the inequalities

$$F\left(\frac{1}{2}\right)\leqslant F(t)\leqslant F(1)=F(0)$$

from [7], which hold for convex function φ (see (b) in Introduction).

In the special case that $p(x)=1$ and $g(x)=x$ (for all $x\in[a,b]\subseteq[0,\infty)$) we get from (2.7)

$$\begin{aligned}\frac{1}{(b-a)^2}\int_a^b\int_a^b\varphi\left(\frac{x+y}{2}\right)dxdy &\leqslant \frac{1}{(b-a)^2}\int_a^b\int_a^b\varphi(tx+(1-t)y)\,dxdy\\ &\quad-\frac{1}{2(b-a)^2}\int_a^b\int_a^b\varphi\left(\frac{|(2t-1)(x-y)|}{2}\right)dxdy.\end{aligned}$$

If the function φ is nonnegative superquadratic, then the previous inequality refines the inequality

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b\varphi\left(\frac{x+y}{2}\right)dxdy\leqslant\frac{1}{(b-a)^2}\int_a^b\int_a^b\varphi(tx+(1-t)y)\,dxdy,$$

for a convex function φ as obtained by Dragomir [8, Theorem 2].

Relations between the functions F and H when φ is a continuous superquadratic function are given in the following theorem:

THEOREM 4. *Under the same assumptions as in the previous two theorems, the following inequality*

$$\begin{aligned}F(t)\geqslant\max\Big\{&H(t)+\frac{1}{P}\int_a^b p(x)\,\varphi((1-t)\,|(g(x)-\overline{g})|)\,dx,\\ &H(1-t)+\frac{1}{P}\int_a^b p(x)\,\varphi(t\,|(g(x)-\overline{g})|)\,dx\Big\}\end{aligned}\tag{2.9}$$

holds for all $t\in[0,1]$.

Proof. Applying (1.5) to the function $\widetilde{g}(y)=tg(x)+(1-t)g(y)$ (instead of $f(x)$) for an arbitrarily chosen and fixed $x\in[a,b]$ and $t\in[0,1]$, where φ is a superquadratic function, we get

$$
\begin{aligned}
F(t) &= \frac{1}{P}\int_a^b p(x)\left[\frac{1}{P}\int_a^b p(y)\varphi(tg(x)+(1-t)g(y))\,dy\right]dx \\
&\geqslant \frac{1}{P}\int_a^b p(x)\varphi\left(\frac{1}{P}\int_a^b p(y)\left[tg(x)+(1-t)g(y)\right]dy\right)dx \\
&\quad + \frac{1}{P}\int_a^b p(x)\left[\frac{1}{P}\int_a^b p(y)\varphi(|(1-t)(g(y)-\overline{g})|)\,dy\right]dx \\
&= \frac{1}{P}\int_a^b p(x)\varphi(tg(x)+(1-t)\overline{g})\,dx \\
&\quad + \frac{1}{P^2}\int_a^b\int_a^b p(x)p(y)\varphi((1-t)|(g(y)-\overline{g})|)\,dydx \\
&= H(t)+\frac{1}{P}\int_a^b p(y)\varphi((1-t)|(g(y)-\overline{g})|)\,dy
\end{aligned}
$$

for all $t\in[0,1]$. Arguing quite analogously as in the previous part of the proof, but applying (1.5) to the function $\widehat{g}(x)=tg(x)+(1-t)g(y)$ (instead of $f(x)$) for an arbitrarily chosen and fixed $y\in[a,b]$ and $t\in[0,1]$, and for a superquadratic function φ, we obtain that

$$
F(t)\geqslant H(1-t)+\frac{1}{P}\int_a^b p(x)\varphi(t|(g(x)-\overline{g})|)\,dx \tag{2.10}
$$

holds for all $t\in[0,1]$. Finally, from the first part of the proof and (2.10) we get (2.9).

REMARK 4. If the function φ in the previous theorem is nonnegative (and therefore also convex), both integrals on the right hand side of inequality (2.9) are nonnegative, so the above inequality represents refinement of inequality

$$
F(t)\geqslant \max\{H(t),H(1-t)\},\qquad t\in[0,1],
$$

obtained in [7] for a convex function φ (see (c) in Introduction).

3. Further results related to the mapping H

In the special case when $p(x)=1$ and $g(x)=x$ for all $x\in[a,b]$, the mapping H is

$$
H(t)=\frac{1}{b-a}\int_a^b \varphi\left(tx+(1-t)\frac{a+b}{2}\right)dx \tag{3.1}
$$

From this $H(t)$ and from the properties quoted in the previous section, a connection between the mapping H and Hermite-Hadamard inequalities (1.9) can be easily seen. Namely, those inequalities can be written as

$$
H(0)\leqslant H(1)\leqslant \frac{\varphi(a)+\varphi(b)}{2}.
$$

The mapping H defined by (3.1) was introduced by S.S. Dragomir in [8] where he used it to get some refinements of Hermite-Hadamard inequalities.

In [4], M. Akkouchi proved some inequalities for the mapping H, defined by (3.1), where φ is a convex and differentiable function. He proved the following theorem:

THEOREM 5. [4, Theorem 2.1] *Let $\varphi:[a,b]\to\mathbb{R}$ be a convex and differentiable function. Then for all $t\in[0,1]$, we have the following inequalities*

$$\begin{aligned} 0 &\leqslant (1-t)[H(1)-H(0)] \leqslant H(1)-H(t) \\ &\leqslant (1-t)[H(1)-H(0)]+t(1-t)\left[\frac{\varphi(a)+\varphi(b)}{2}-H(1)\right] \end{aligned} \tag{3.2}$$

where the mapping H is defined by (3.1).

To Akkouchi's result we add the following upper bound:

PROPOSITION 1. *Under the same conditions as in Theorem 5, inequality*

$$\begin{aligned} &(1-t)[H(1)-H(0)]+t(1-t)\left[\frac{\varphi(a)+\varphi(b)}{2}-H(1)\right] \\ &\leqslant \max_{t\in[0,1]}\left\{(1-t)[H(1)-H(0)]+t(1-t)\left[\frac{\varphi(a)+\varphi(b)}{2}-H(1)\right]\right\} \\ &= \frac{\left(\frac{\varphi(a)+\varphi(b)}{2}-H(0)\right)^2}{4\left(\frac{\varphi(a)+\varphi(b)}{2}-H(1)\right)}, \end{aligned} \tag{3.3}$$

holds for all $t\in[0,1]$.

Proof. Using the notations $A=H(1)-H(0)$ and $B=\frac{\varphi(a)+\varphi(b)}{2}-H(1)$, we are looking for a maximum of the function $h:[0,1]\to\mathbb{R}$,

$$h(t)=(1-t)A+t(1-t)B=-Bt^2+(B-A)t+A.$$

As we have $B\geqslant A\geqslant 0$, by a simple calculation, we get that $\max_{t\in[0,1]}h(t)=h\left(\frac{B-A}{2B}\right)=\frac{(A+B)^2}{4B}$, which is (3.3).

EXAMPLE 2. Let $0<a<b$ and consider the function $\varphi:[a,b]\to\mathbb{R}$, $\varphi(x)=x^q$, $q>1$. For this function we have

$$H(1)=\frac{1}{b-a}\int_a^b x^q dx=\frac{b^{q+1}-a^{q+1}}{(b-a)(q+1)}=L_q^q(a,b) \tag{3.4}$$

and

$$H(0)=\left(\frac{a+b}{2}\right)^q=A^q(a,b). \tag{3.5}$$

Applying Proposition 1, we get

$$\begin{aligned}&(1-t)\left[L_q^q(a,b)-A^q(a,b)\right]+t(1-t)\left[A(a^q,b^q)-L_q^q(a,b)\right]\\&\quad\leqslant\frac{\left(A(a^q,b^q)-A^q(a,b)\right)^2}{4\left(A(a^q,b^q)-L_q^q(a,b)\right)}.\end{aligned}$$

So, the above inequality gives an upper bound for Akkouchi's inequalities [4]*:*

$$\begin{aligned}0\leqslant\ &(1-t)\left[L_q^q(a,b)-A^q(a,b)\right]\\\leqslant\ &L_q^q(a,b)-L_q^q\left(\frac{a+b}{2}-t\frac{b-a}{2},\frac{a+b}{2}+t\frac{b-a}{2}\right)\\\leqslant\ &(1-t)\left[L_q^q(a,b)-A^q(a,b)\right]+t(1-t)\left[A(a^q,b^q)-L_q^q(a,b)\right].\end{aligned}\tag{3.6}$$

In the case of weighted version of the function H (defined by (1.1)) we are able to prove only the first two inequalities of Theorem 5.

When the function φ is superquadratic, and H is as defined in (3.1), we obtain in the following Theorem 6, Corollary 1, Remark 5 and Example 3, analogous results to Theorem 5, and when the function φ is also positive we get refinements of Theorem 5 and its consequence ([4, Corollary 2.1]):

THEOREM 6. *Let $\varphi:[0,\infty)\to\mathbb{R}$ be a superquadratic and differentiable function with $\varphi(0)=\varphi'(0)=0$ and let $[a,b]\subseteq[0,\infty)$, $a<b$. Then for all $t\in[0,1]$, we have the following inequalities*

$$\begin{aligned}&(1-t)[H(1)-H(0)]+\Delta\leqslant H(1)-H(t)\\&\quad\leqslant(1-t)[H(1)-H(0)]+t(1-t)\left[\frac{\varphi(a)+\varphi(b)}{2}-H(1)\right]-\Delta,\end{aligned}\tag{3.7}$$

where

$$\Delta=\frac{1}{b-a}\int_a^b\left[t\varphi\left((1-t)\left|x-\tfrac{a+b}{2}\right|\right)+(1-t)\varphi\left(t\left|x-\tfrac{a+b}{2}\right|\right)\right]dx.\tag{3.8}$$

Proof. Since φ is superquadratic, using (1.6) we get

$$\begin{aligned}\varphi\left(tx+(1-t)\tfrac{a+b}{2}\right)&\leqslant t\varphi(x)+(1-t)\varphi\left(\tfrac{a+b}{2}\right)\\&\quad-t\varphi\left((1-t)\left|x-\tfrac{a+b}{2}\right|\right)-(1-t)\varphi\left(t\left|x-\tfrac{a+b}{2}\right|\right).\end{aligned}$$

Integrating both sides of this inequality over the segment $[a,b]$, and multiplying it by $1/(b-a)$ we obtain

$$\begin{aligned}H(t)&\leqslant tH(1)+(1-t)H(0)\\&\quad-\frac{1}{b-a}\int_a^b\left[t\varphi\left((1-t)\left|x-\tfrac{a+b}{2}\right|\right)+(1-t)\varphi\left(t\left|x-\tfrac{a+b}{2}\right|\right)\right]dx.\end{aligned}$$

Hence we have

$$\begin{aligned} H(t)-H(1) \leqslant\ & (1-t)[H(0)-H(1)] \\ & -\frac{1}{b-a}\int_a^b \left[t\varphi\left((1-t)\left|x-\tfrac{a+b}{2}\right|\right)+(1-t)\varphi\left(t\left|x-\tfrac{a+b}{2}\right|\right)\right]dx, \end{aligned}$$

from which we get the first inequality in (3.7).

Since the function φ is superquadratic and differentiable with $\varphi(0)=\varphi'(0)=0$, then by Definition 1 and property $(\mathrm{S1})$ of superquadratic function, for all $x \geqslant 0$ and $t \in [0,1]$ we have

$$\varphi\left(tx+(1-t)\tfrac{a+b}{2}\right)-\varphi(x) \geqslant (1-t)\left(\tfrac{a+b}{2}-x\right)\varphi'(x)+\varphi\left((1-t)\left|x-\tfrac{a+b}{2}\right|\right) \tag{3.9}$$

and

$$\varphi\left(tx+(1-t)\tfrac{a+b}{2}\right)-\varphi\left(\tfrac{a+b}{2}\right) \geqslant t\left(x-\tfrac{a+b}{2}\right)\varphi'\left(\tfrac{a+b}{2}\right)+\varphi\left(t\left|x-\tfrac{a+b}{2}\right|\right). \tag{3.10}$$

Multiplying (3.9) by t and (3.10) by $(1-t)$ and summing, we get

$$\begin{aligned} & \varphi\left(tx+(1-t)\tfrac{a+b}{2}\right)-t\varphi(x)-(1-t)\varphi\left(\tfrac{a+b}{2}\right) \\ & \quad \geqslant t(1-t)\left[\left(\tfrac{a+b}{2}-x\right)\varphi'(x)+\left(x-\tfrac{a+b}{2}\right)\varphi'\left(\tfrac{a+b}{2}\right)\right] \\ & \qquad +t\varphi\left((1-t)\left|x-\tfrac{a+b}{2}\right|\right)+(1-t)\varphi\left(t\left|x-\tfrac{a+b}{2}\right|\right). \end{aligned}$$

Integrating both sides of this inequality over the segment $[a,b]$, and multiplying it by $1/(b-a)$ we obtain

$$\begin{aligned} & H(t)-tH(1)-(1-t)H(0) \\ & \quad \geqslant \frac{t(1-t)}{b-a}\left[\int_a^b \left(\tfrac{a+b}{2}-x\right)\varphi'(x)\,dx+\varphi'\left(\tfrac{a+b}{2}\right)\int_a^b \left(x-\tfrac{a+b}{2}\right)dx\right] \\ & \qquad +\frac{1}{b-a}\left[\int_a^b \left[t\varphi\left((1-t)\left|x-\tfrac{a+b}{2}\right|\right)+(1-t)\varphi\left(t\left|x-\tfrac{a+b}{2}\right|\right)\right]dx\right]. \end{aligned}$$

Integrating by parts, we have

$$\int_a^b \left(\tfrac{a+b}{2}-x\right)\varphi'(x)\,dx=\int_a^b \varphi(x)\,dx-\frac{(b-a)(\varphi(a)+\varphi(b))}{2},$$

and since $\int_a^b \left(x-\frac{a+b}{2}\right)dx=0$, we get

$$\begin{aligned} & H(t)-H(1)+(1-t)[H(1)-H(0)] \\ & \quad \geqslant \frac{t(1-t)}{b-a}\left[\int_a^b \varphi(x)\,dx-\frac{(b-a)(\varphi(a)+\varphi(b))}{2}\right] \\ & \qquad +\frac{1}{b-a}\int_a^b \left[t\varphi\left((1-t)\left|x-\tfrac{a+b}{2}\right|\right)+(1-t)\varphi\left(t\left|x-\tfrac{a+b}{2}\right|\right)\right]dx. \end{aligned}$$

The last inequality implies the second inequality in (3.7).

The left hand side of Hermite-Hadamard inequality for superquadratic functions (1.7) multiplied by $(1-t)$, $0 \leqslant t \leqslant 1$ is

$$(1-t)[H(1)-H(0)] \geqslant \frac{1}{b-a}\int_a^b (1-t)\varphi\left(\left|x-\frac{a+b}{2}\right|\right)dx, \tag{3.11}$$

which is analogous to the first inequality in Theorem 5.

As seen before, for a nonnegative superquadratic function φ, the above inequality represents a refinement of the first inequality in (3.2).

In the case of weighted version of the function H (defined by (1.1)) we are able to prove only the first inequality of Theorem 6.

By setting $t = 1/2$ in Theorem 6 we get the following corollary:

COROLLARY 1. *Let* $\varphi : [0,\infty) \to \mathbb{R}$ *be a superquadratic and differentiable function with* $\varphi(0) = \varphi'(0) = 0$ *and let* $[a,b] \subseteq [0,\infty)$, $a < b$. *Then we have*

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b \varphi(x)\,dx - \varphi\left(\frac{a+b}{2}\right)\right] + \Delta_C \\
&\quad \leqslant \frac{1}{b-a}\int_a^b \varphi(x)\,dx - \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \varphi(x)\,dx \\
&\quad \leqslant \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \varphi(x)\,dx - \varphi\left(\frac{a+b}{2}\right)\right] + \frac{1}{4}\left[\frac{\varphi(a)+\varphi(b)}{2} - \frac{1}{b-a}\int_a^b \varphi(x)\,dx\right] - \Delta_C.
\end{aligned} \tag{3.12}$$

where

$$\Delta_C = \frac{1}{b-a}\int_a^b \varphi\left(\frac{1}{2}\left|x-\tfrac{a+b}{2}\right|\right)dx \tag{3.13}$$

REMARK 5. Consider inequalities (3.7), proved in Theorem 6. When φ, the superquadratic and differentiable function is also nonnegative, the term Δ defined by (3.8) is nonnegative. As by property $(S2)$ φ is also convex and $\varphi(0) = \varphi'(0) = 0$, we get that inequalities (3.2) also hold, so the first inequality in (3.7) refines the second inequality in (3.2). Similarly, in this case the second inequality in (3.7) refines the third inequality in (3.2).

In the same way, we consider (3.12). Because for nonnegative φ we have $\Delta_C \geqslant 0$, the results of Corollary 1 refine the results of the corresponding Akkouchi's corollary in [4, Corollary 2.1] (inequalities (3.12) without terms Δ_C).

EXAMPLE 3. Consider again inequalities (3.7) in Theorem 6, this time for the superquadratic function $\varphi : [0,\infty) \to \mathbb{R}$, $\varphi(x) = x^q$ with $q \geqslant 2$. For this function and $0 \leqslant a < b$ we calculated the additional term Δ (now denoted by Δ_E) and obtain:

$$\Delta_E = \frac{1}{q+1}\left[\frac{b-a}{2}\right]^q \left[t(1-t)^q + (1-t)t^q\right]. \tag{3.14}$$

Using (3.4), (3.5) and

$$H(t)=L_q^q\left(\frac{a+b}{2}-t\frac{b-a}{2},\frac{a+b}{2}+t\frac{b-a}{2}\right),$$

and applying Theorem 6 we get the following inequalities

$$\begin{aligned}
&(1-t)\left[L_q^q(a,b)-A^q(a,b)\right]+\Delta_E\\
&\quad\leqslant L_q^q(a,b)-L_q^q\left(\frac{a+b}{2}-t\frac{b-a}{2},\frac{a+b}{2}+t\frac{b-a}{2}\right)\\
&\quad\leqslant(1-t)\left[L_q^q(a,b)-A^q(a,b)\right]+t(1-t)\left[A\left(a^q,b^q\right)-L_q^q(a,b)\right]-\Delta_E\,. \qquad (3.15)
\end{aligned}$$

Since in this case we have $\Delta_E\geqslant 0$, the inequalities in (3.15) are refinements of the second and third inequality in (3.6).

EXAMPLE 4. Consider inequality (3.11) for superquadratic function $\varphi(x)=x^q$, $q\geqslant 2$. Using the notations (2.3) and (2.4) we obtain:

$$(1-t)\left[L_q^q(a,b)-A^q(a,b)\right]\geqslant\frac{1-t}{q+1}\left[\frac{b-a}{2}\right]^q. \qquad (3.16)$$

Now, from the first inequality in (3.15) using (3.16) we get:

$$\begin{aligned}
&L_q^q\left(\frac{a+b}{2}-t\frac{b-a}{2},\frac{a+b}{2}+t\frac{b-a}{2}\right)\\
&\quad\leqslant L_q^q(a,b)-(1-t)\left[L_q^q(a,b)-A^q(a,b)\right] \qquad (3.17)\\
&\qquad-\frac{1}{q+1}\left(\frac{b-a}{2}\right)^q t(1-t)\left[(1-t)^{q-1}+t^{q-1}\right]\\
&\quad\leqslant L_q^q(a,b)-\frac{1}{q+1}\left(\frac{b-a}{2}\right)^q(1-t)\left[1+t(1-t)^{q-1}+t^q\right], \qquad (3.18)
\end{aligned}$$

Comparing the above inequalities with the inequalities in (2.5) we conclude that the first inequality in (3.15) (which is equivalent to inequality (3.17)) represents a refinement of the second inequality in (2.5).

REMARK 6. For $t=1/2$ we obtain from this example that:

$$\begin{aligned}
L_q^q\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)&\leqslant\frac{1}{2}\left[L_q^q(a,b)+A^q(a,b)\right]-\frac{1}{q+1}\left(\frac{b-a}{4}\right)^q\\
&\leqslant L_q^q(a,b)-\frac{1}{2(q+1)}\left(\frac{b-a}{2}\right)^q\left[1+\frac{1}{2^{q-1}}\right],
\end{aligned}$$

which is a refinement of the second inequality in (2.6).

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(Received October 31, 2008)

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