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A PROPERTY OF A FUNCTIONAL INCLUSION CONNECTED WITH HYERS–ULAM STABILITY

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. We prove that a set-valued map $F : X \to \mathscr{P}_0(Y)$ satisfying the functional inclusion $F(x)\Diamond F(y) \subseteq F(x*y)$ admits, in appropriate conditions, a unique selection $f : X \to Y$ satisfying the functional equation $f(x) \diamond f(y) = f(x*y)$, where $(X,*)$, $(Y,\diamond)$ are square-symmetric grupoids and $\Diamond$ is the extension of $\diamond$ to the collection $\mathscr{P}_0(Y)$ of all nonempty parts of Y.

1. Introduction

In the theory of functional equations one of the main topics is Hyers-Ulam stability. The first result on this topic was given by D.H. Hyers who obtained the following result concerning the Cauchy functional equation [4]:

"Let X be a linear normed space, Y a Banach space, $\varepsilon > 0$ and $f : X \to Y$ a function satisfying the inequality

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon, \quad x,y \in X. \tag{1.1}$$

Then there exists a unique additive function $g : X \to Y$ such that

$$\|f(x) - g(x)\| \leqslant \varepsilon, \quad x \in X." \tag{1.2}$$

The previous result is an answer given to a problem proposed by S.M. Ulam in 1940 in a talk to a conference at Wisconsin University (see [5], [6]). W. Smajdor [16] and R. Ger, Z. Gajda [3] observed that if f is a solution of (1.1), then the set-valued map $F : X \to \mathscr{P}_0(Y)$ defined by the relation

$$F(x) = f(x) + \overline{B}(0,\varepsilon), \quad x \in X \tag{1.3}$$

is subadditive, i.e.

$$F(x+y) \subseteq F(x) + F(y), \quad x,y \in X \tag{1.4}$$

and the function g from the relation (1.2) satisfies the relation $g(x) \in F(x)$, i.e. is a **selection** of F. ($\mathscr{P}_0(Y)$ is the collection of all nonempty subsets of Y and $\overline{B}(0,\varepsilon)$ is the closed ball of center 0 and radius ε in Y).

Now one may ask under what conditions a subadditive set-valued map admits an additive selection. A first answer to this question was given by Z. Gajda and R. Ger [3].

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Furthermore the result of Gajda and Ger was extended to set-valued maps satisfying general linear inclusions by D. Popa [11]. A new step on this direction was made by D. Popa [12] and K. Nikodem, D. Popa [8] who considered set-valued maps F satisfying functional inclusions of the form

$$F(x*y) \subseteq F(x)\Diamond F(y), \quad x,y \in X, \tag{1.5}$$

where $(X,*)$, $(Y,\diamond)$ are square-symmetric grupoids and $\Diamond$ is a square-symmetric operation on $\mathscr{P}_0(Y)$ determined by $\diamond$. More precisely, D. Popa proved in [12] that in appropriate conditions a set-valued map satisfying (1.5) admits a selection f with the property

$$f(x*y) = f(x)\diamond f(y), \quad x,y \in X. \tag{1.6}$$

The purpose of this paper is to obtain an analogous result for set-valued maps satisfying the converse functional inclusion

$$F(x)\Diamond F(y) \subseteq F(x*y), \quad x,y \in X. \tag{1.7}$$

Some results on this direction, for particular cases of the functional inclusion (1.7), were obtained by W. Smajdor and A. Smajdor [15], [17].

Let us recall that J. Ratz [14] pointed out the role of square-symmetry for the stability of functional equations and Zs. Páles [9], Zs. Páles, P. Volkmann, R.D. Luce [12] considered the stability of the Cauchy functional equation on square-symmetric grupoids. In this paper we will use some ideas and terminology from [10] and [12].

A grupoid $(X,*)$ is called **square-symmetric** if

$$(x*y)*(x*y) = (x*x)*(y*y) \tag{1.8}$$

for all $x,y \in X$. An operation $*$ on X is square-symmetric if and only if the function $\sigma_* : X \to X$ given by

$$\sigma_*(x) = x*x, \quad x \in X \tag{1.9}$$

is an endomorphism of $(X,*)$. The grupoid $(X,*)$ is called **divisible** if σ_* is an automorphism of $(X,*)$. The triple $(Y,*,d)$ is called a **metric grupoid** if $(Y,*)$ is a grupoid, (Y,d) is a metric space and $*$ is a continuous operation with respect to the topology of (Y,d). For a nonempty set Y we denote by $\mathscr{P}_0(Y)$ the collection of all nonempty subsets of Y. If (Y,d) is a metric space by $cl(Y)$ we denote the collection of all nonempty closed subsets of Y. In a linear normed space $(Y,\|\cdot\|)$ we define the following families of sets:

$$\begin{aligned} c(Y) &:= \{A \mid A \in \mathscr{P}_0(Y),\ A \text{ is convex set}\} \\ ccl(Y) &:= \{A \mid A \in \mathscr{P}_0(Y),\ A \text{ is closed and convex set}\} \\ cc(Y) &:= \{A \mid A \in \mathscr{P}_0(Y),\ A \text{ is compact and convex set}\}. \end{aligned} \tag{1.10}$$

Let (Y,d) be a metric space. The **diameter** of a set $A \in \mathscr{P}_0(Y)$ is defined by

$$\delta(A) := \sup\{d(x,y) \mid x,y \in A\}. \tag{1.11}$$

The **Lipschitz modulus** of a function $f : Y \to Y$ is the smallest real extended number L with the property

$$d(f(x),f(y)) \leqslant Ld(x,y), \quad x,y \in Y. \tag{1.12}$$

The Lipschitz modulus of a function f is denoted by $Lipf$. Finally recall that a **selection** of a set-valued map $F: X \to \mathscr{P}_0(Y)$ is a single-valued map $f: X \to Y$ with the property $f(x) \in F(x)$ for all $x \in X$.

2. Main results

In this section we denote by $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ the set of all nonnegative integers.

First we recall some notions and results from [12].

Let $(Y, \diamond, d)$ be a metric grupoid. We extend the operation $\diamond$ to an operation $\Diamond$ on $\mathscr{P}_0(Y)$ as follows

$$A \Diamond B = \{x |\, x = a \diamond b,\ a \in A,\ b \in B\}, \quad A, B \in \mathscr{P}_0(Y). \tag{2.1}$$

If $\diamond$ is a square-symmetric operation on Y, then $\Diamond$ is not necessary square-symmetric on $\mathscr{P}_0(Y)$. For the square-symmetry of $\Diamond$ it suffices that $\diamond$ satisfies the condition of **bisymmetry** introduced by J. Aczél [1]. The following result holds.

LEMMA 2.1. [12] *Let $(Y, \diamond)$ be a a grupoid with a bisymmetric operation, i.e.*

$$(x_1 \diamond y_1) \diamond (x_2 \diamond y_2) = (x_1 \diamond x_2) \diamond (y_1 \diamond y_2) \tag{2.2}$$

for all $x_1, x_2, y_1, y_2 \in Y$. Then $\sigma_\Diamond$ is an increasing endomorphism of $(\mathscr{P}_0(Y), \Diamond, \subseteq)$.

The main results of this paper are contained in the next theorems.

THEOREM 2.2. *Let $(X, *)$ be a square-symmetric divisible grupoid, $(Y, \diamond, d)$ a complete metric bisymmetric divisible grupoid and $F: X \to \mathscr{P}_0(Y)$ a set-valued map with the property*

$$F(x) \Diamond F(y) \subseteq F(x * y), \quad x, y \in X. \tag{2.3}$$

If

$$\sigma_\Diamond^n \circ F \circ \sigma_*^{-n}(x) \in cl(Y), \quad x \in X,\ n \in \mathbb{N}_0 \tag{2.4}$$

and

$$\lim_{n \to \infty} \delta(F \circ \sigma_*^{-n}(x)) Lip(\sigma_\diamond^n) = 0, \quad x \in Y \tag{2.5}$$

then there exists a unique selection $f: X \to Y$ of F with the property

$$f(x) \diamond f(y) = f(x * y), \quad x, y \in X. \tag{2.6}$$

Proof. **Existence.** Replacing $y = x$ in (2.3) we get

$$\sigma_\Diamond(F(x)) \subseteq F(\sigma_*(x)), \quad x \in X \tag{2.7}$$

and applying $\sigma_\Diamond^n$ to (2.7) it follows, in view of Lemma 2.1

$$\sigma_\Diamond^{n+1} \circ F(x) \subseteq \sigma_\Diamond^n \circ F \circ \sigma_*(x), \quad x \in X,\ n \in \mathbb{N}_0. \tag{2.8}$$

Now we replace x from (2.8) with $\sigma_*^{-n-1}(x)$ to obtain:

$$\sigma_\Diamond^{n+1} \circ F \circ \sigma_*^{-n-1}(x) \subseteq \sigma_\Diamond^n \circ F \circ \sigma_*^{-n}(x), \quad x \in X,\ n \in \mathbb{N}. \tag{2.9}$$

Let $x \in X$ be fixed. Define the sequence $(F_n(x))_{n \geqslant 0}$ by

$$F_n(x) = \sigma_\Diamond^n \circ F \circ \sigma_*^{-n}(x), \quad n \in \mathbb{N}_0. \tag{2.10}$$

The sequence $(F_n(x))_{n \geqslant 0}$ is decreasing in view of (2.9). We prove that

$$\lim_{n\to\infty} \delta(F_n(x)) = 0. \tag{2.11}$$

Let $u, v \in F_n(x)$. It follows

$$\sigma_\diamond^{-n}(u) \in F \circ \sigma_*^n(x), \quad \sigma_\diamond^{-n}(v) \in F \circ \sigma_*^{-n}(x) \tag{2.12}$$

Denote $\sigma_\diamond^{-n}(u) = s$, $\sigma_\diamond^{-n}(v) = t$. We get:

$$\begin{aligned} d(u,v) &= d(\sigma_\diamond(s), \sigma_\diamond(t)) \leqslant Lip(\sigma_\diamond^n) \cdot d(s,t) \\ &= Lip(\sigma_\diamond^n) \delta(F \circ \sigma_*^{-n}(x)) \end{aligned} \tag{2.13}$$

which leads to

$$\delta(F_n(x)) \leqslant Lip(\sigma_\diamond^n) \cdot \delta(F \circ \sigma_*^{-n}(x)). \tag{2.14}$$

Now taking account of (2.5) it follows $\lim_{n\to\infty} \delta(F_n(x)) = 0$.

The sequence of sets $(F_n(x))_{n \geqslant 0}$ satisfies the conditions of the Cantor theorem in the complete metric space (Y, d), hence

$$\bigcap_{n=0}^{\infty} F_n(x) \tag{2.15}$$

is a singleton $f(x)$. The function $f : X \to Y$ is a selection of F since $f(x) \in F_0(x) = F(x)$ for every $x \in X$. Let us prove that f satisfies the equation (2.6). First we will prove that

$$F_n(x) \Diamond F_n(y) \subseteq F_n(x * y), \quad x, y \in X,\ n \in \mathbb{N}_0. \tag{2.16}$$

In (2.3) replace x by $\sigma_*^{-n}(x)$, y by $\sigma_*^{-n}(y)$ to get

$$(F \circ \sigma_*^{-n}(x)) \Diamond (F \circ \sigma_*^{-n}(y)) \subseteq F \circ \sigma_*^{-n}(x * y), \quad x, y \in X \tag{2.17}$$

and applying $\sigma_\Diamond^n$ to (2.17) follows (2.16).

Since $\{f(x)\} = \bigcap_{n=0}^{\infty} F_n(x)$, $x \in X$, we have $f(x) \diamond f(y) \in F_n(x) \cap F_n(y)$, for all $n \in \mathbb{N}_0$, $x, y \in X$, hence, in view of (2.16) we get

$$d(f(x) \diamond f(y), f(x * y)) \leqslant \delta(F_n(x * y)), \quad x, y \in X,\ n \in \mathbb{N}_0. \tag{2.18}$$

Taking account of (2.11) we get by (2.18)

$$f(x) \diamond f(y) = f(x * y), \quad x, y \in X. \tag{2.19}$$

The existence is proved.

Uniqueness. Suppose that F admits two selections f, g satisfying

$$\begin{array}{l} f(x) \diamond f(y) = f(x * y) \\ g(x) \diamond f(y) = g(x * y) \end{array}, \quad x, y \in X. \tag{2.20}$$

From (2.20) follows

$$\begin{array}{l} \sigma_\diamond^{-n} \circ f(x) = f \circ \sigma_*^{-n}(x), \\ \sigma_\diamond^{-n} \circ g(x) = g \circ \sigma_*^{-n}(x) \end{array}, \quad x \in X,\ n \in \mathbb{N}_0. \tag{2.21}$$

Let $x \in X$ be fixed. Then

$$\begin{aligned} d(\sigma_\diamond^{-n} \circ f(x), \sigma_\diamond^{-n} \circ g(x)) &= d(f \circ \sigma_*^{-n}(x), g \circ \sigma_*^{-n}(x)) \\ &\leqslant \delta(F \circ \sigma_*^{-n}(x)),\ x \in X,\ n \in \mathbb{N}_0. \end{aligned}$$

Put $\sigma_\diamond^{-n} \circ f(x) = s$, $\sigma_\diamond^{-n} \circ g(x) = t$. Then $f(x) = \sigma_\diamond^{n}(s)$, $g(x) = \sigma_\diamond^{n}(t)$ and

$$\begin{aligned} d(f(x), g(x)) = d(\sigma_\diamond^{n}(s), \sigma_\diamond^{n}(t)) &\leqslant Lip(\sigma_\diamond^{-n}) d(s,t) \\ &\leqslant Lip(\sigma_\diamond^{-n}) \delta(F \circ \sigma_*^{n}(x)). \end{aligned} \tag{2.22}$$

Letting n tends to infinity in (2.22) we get $f(x) = g(x)$, in view of (2.5). The uniqueness is proved.

THEOREM 2.3. *Let $(X, *)$ be a square-symmetric divisible grupoid, $(Y, \diamond, d)$ a metric bisymmetric divisible grupoid and A a divisible subgrupoid of $(\mathscr{P}_0(Y), \Diamond)$. Suppose that $F : X \to A$ is a set-valued map with the property*

$$F(x) \Diamond F(y) \subseteq F(x * y), \quad x, y \in X. \tag{2.23}$$

If

$$\lim_{n \to \infty} \delta(F \circ \sigma_*^{n}(x)) Lip(\sigma_\diamond^{-n}) = 0 \tag{2.24}$$

for every $x \in X$, then F is single valued and

$$F(x) \diamond F(y) = F(x * y), \quad x, y \in X. \tag{2.25}$$

Proof. For $y = x$ in (2.23) we get

$$\sigma_\Diamond(F(x)) \subseteq F(\sigma_*(x)), \quad x \in X. \tag{2.26}$$

Replacing in (2.26) x by $\sigma_*^{n}(x)$, $n \in \mathbb{N}_0$, and applying $\sigma_\Diamond^{-n-1}$ to both sides of (2.26) we obtain

$$\sigma_\Diamond^{-n} \circ F \circ \sigma_*^{n}(x) \subseteq \sigma_\Diamond^{-n-1} \circ F \circ \sigma_*^{n+1}(x) \tag{2.27}$$

taking account that $\sigma_\Diamond$ is increasing.

Let $x \in X$ be fixed. The sequence of sets $(F_n(x))_{n\geqslant 0}$

$$F_n(x) = \sigma_\Diamond^{-n} \circ F \circ \sigma_*^n(x), \quad n \geqslant 0, \tag{2.28}$$

is increasing. Then $(\delta(F_n(x))_{n\geqslant 0}$ is an increasing sequence of nonnegative numbers. As in the proof of Theorem 2.2 one obtains

$$\delta(F_n(x)) \leqslant \delta(F \circ \sigma_*^n(x)) Lip(\sigma_\diamond^{-n}), \quad n \geqslant 0. \tag{2.29}$$

Then $\delta(F_n(x)) = 0$ for every $n \in \mathbb{N}_0$, in view of (2.24) and the monotonicity of $(F_n(x))_{n\geqslant 0}$. It follows that $F_n(x)$ is single-valued for all $n \in \mathbb{N}_0$ and $F_0(x) = F(x)$ satisfies the relation $F(x) \diamond F(y) = F(x * y)$, for all $x, y \in X$. The theorem is proved.

Now we will give some consequences of the previous theorems, concerning linear inclusions, analogous to the results obtained in [12].

Suppose that Y is a Banach space over $\mathbb{R}$ and $\diamond$ is defined by

$$x \diamond y = px + qy, \quad x, y \in Y, \tag{2.30}$$

where $p, q \in \mathbb{R}$ are given numbers. The triple $(Y, \diamond, \|\cdot\|)$ is a metric grupoid with a bisymmetric operation.

For all $U, V \in \mathscr{P}_0(Y)$ the operation $\Diamond$ is defined by

$$U \Diamond V = pU + qV. \tag{2.31}$$

COROLLARY 2.4. *Let $(X, *)$ be a square-symmetric divisible grupoid, $(Y, \|\cdot\|)$ a Banach space over $\mathbb{R}$, $p, q \in \mathbb{R}$, $p + q \neq 0$, $p + q \neq 1$, and $F : X \to c(Y)$ a set-valued map with the property*

$$pF(x) + qF(y) \subseteq F(x * y), \quad x, y \in X. \tag{2.32}$$

Suppose that the following conditions are satisfied:

$$1) \quad F \circ \sigma_*^{-n}(x) \in cl(Y), \quad x \in X,\ n \in \mathbb{N}_0. \tag{2.33}$$

2) *There exists $M > 0$ such that $\delta(F(x)) \leqslant M$, $x \in X$.*

Then there exists a unique selection $f : X \to Y$ of F such that

$$pf(x) + qf(y) = f(x * y), \quad x, y \in X. \tag{2.34}$$

Proof. We have $\sigma_\Diamond(U) = (p+q)U$, $U \in c(Y)$, $\sigma_\Diamond$ is an automorphism of $(c(Y), \Diamond)$, $\sigma_\diamond^n(x) = (p+q)^n x$, $x \in X$, $n \in \mathbb{Z}$ and

$$Lip(\sigma_\diamond^n) = |p+q|^n, \quad n \in \mathbb{Z}.$$

i) If $|p+q| < 1$ we have

$$\sigma_\Diamond^n \circ F \circ \sigma_*^{-n}(x) = (p+q)^n F \circ \sigma_*^{-n}(x) \in cl(Y)$$

and

$$\delta(F\circ\sigma_*^{-n}(x))Lip(\sigma_\diamond^n)\leqslant M|p+q|^n,\quad x\in X,\ n\in\mathbb{N}_0,$$

hence there exists a unique selection of F satisfying (2.34), in view of Theorem 2.2.

ii) If $|p+q|>1$ we have

$$\delta(F\circ\sigma_*^n)Lip(\sigma_\diamond^{-n})\leqslant\frac{M}{|p+q|^n},\quad x\in X,\ n\in\mathbb{N}_0,$$

thus F is single-valued, according with Theorem 2.3.

Corollary 2.4 leads to the following stability result for the general linear equation.

COROLLARY 2.5. *Let $(X,*)$ be a square-symmetric divisible grupoid, $(Y,\|\cdot\|)$ a Banach space over $\mathbb{R}$, $p,q,\varepsilon>0$, $p+q<1$, and $b\in Y$. Suppose that $f:X\to Y$ is a function satisfying*

$$\|f(x*y)-pf(x)-qf(y)-b\|\leqslant\varepsilon,\quad x,y\in X. \tag{2.35}$$

Then there exists a unique function $g:X\to Y$ satisfying

$$g(x*y)=pg(x)+qg(y)+b,\quad x,y\in X \tag{2.36}$$

and

$$\|f(x)-g(x)\|\leqslant\frac{\varepsilon}{1-p-q},\quad x\in X. \tag{2.37}$$

Proof. Define the set-valued map $F:X\to ccl(Y)$ by

$$F(x)=f(x)+\frac{1}{1-p-q}(\overline{B}(0,\varepsilon)-b),\quad x\in X.$$

We have

$$\begin{aligned}pF(x)+qF(y)&=pf(x)+\frac{p}{1-p-q}(\overline{B}(0,\varepsilon)-b)\\&\quad+qf(y)+\frac{q}{1-p-q}(\overline{B}(0,\varepsilon)-b)\\&\subseteq f(x*y)+(\overline{B}(0,\varepsilon)-b)+\frac{p+q}{1-p-q}(\overline{B}(0,\varepsilon)-b)\\&=f(x*y)+\frac{1}{1-p-q}(\overline{B}(0,\varepsilon)-b)=F(x*y)\end{aligned}$$

for every $x,y\in X$. On the other hand it is obvious that

$$\delta(F(x))\leqslant\frac{2\varepsilon}{1-p-q}$$

therefore the conditions of Corollary 2.4 are satisfied. It follows that F has a unique selection h satisfying

$$h(x*y)=ph(x)+qh(y),\quad x,y\in X.$$

The function $g: X \to Y$, $g(x) = h(x) + \dfrac{b}{1-p-q}$ satisfies the equation (2.36) and (2.37).

The result obtained in Corollary 2.5 crosses with a more general result, obtained by Zs. Páles in [9], on the stability of the general linear equation and a result obtained by Z. Brzdek and A. Pietrzyk in [2] (see also [8]). It is also connected with a problem in [13] and corresponds also to some results in [7].

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